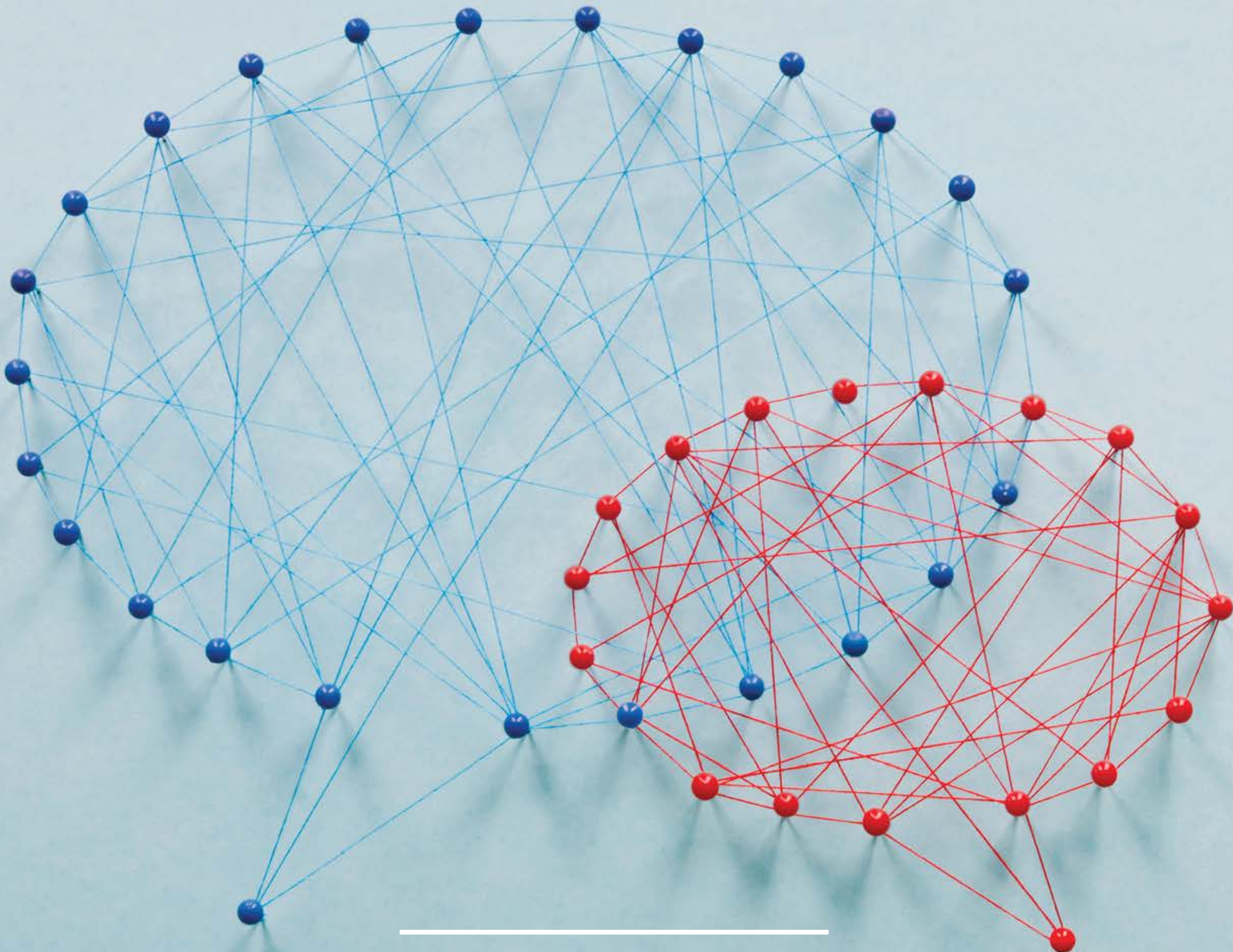
TEACHING MATH ONLINE



Increased Participation and Conversation Using Networked Devices

Online lesson design principles help teachers and curriculum developers imagine new possibilities for classrooms.

Christopher Danielson and Dan Meyer

For many of us, the phrase "teaching math online" evokes a vision of teaching and learning that is not based in physical classrooms. Perhaps teachers and students are even interacting asynchronously. In math classrooms in the United States, the increasing availability of devices (e.g. laptops, Chromebooks™, smartphones, and tablets) and networks allows students to access the Internet quickly and reliably. Imagining the possibilities for classrooms is an important responsibility of curriculum developers, district and school-level curriculum supervisors, and classroom teachers.

The authors of this article are on the teaching faculty at Desmos®, which offers a free, online graphing calculator that runs in the window of any modern web browser. In recent years, we have been extending this technology—and merging it with our pedagogical vision—by developing a suite of online classroom activities for use in secondary classrooms, with a goal of helping teachers and students maximize mathematics learning with digital tools. We currently have six dedicated activities (mostly for algebra) and two tools—Polygraph and Activity Builder—that teachers can configure to meet their curricular needs in a variety of topic areas. All of this is free to teachers for individual classroom use at teacher.desmos.com.

We begin this article by describing our vision through the principles our team has articulated in our curriculum development work. We then describe two activities we have developed that make novel use of classroom-based Internet access, including examples of the kinds of discourse and learning that these activities elicit.

PRINCIPLES

The principles that guide our lesson development work include the following:

- Use technology to provide students with feedback as they work.
- Use the existing network to connect students, supporting collaboration and discourse.
- Provide information to teachers in real time during class.

Feedback

Students often receive feedback from their teachers in the form of answers marked right or wrong, or points either added or docked. When students receive written descriptive feedback, it comes at the expense both of a waiting period and of the teacher's time.

Computers can mark answers right and wrong much faster than a teacher can. As a consequence, this quick assessment—together with hints—is the typical experience many students have of doing school math online.

We think electronic feedback should be more than hints and corrected answers. Computers can provide students with an understanding of the implications of their thinking. At a basic level, teachers who show students a graph and have them infer its symbolic form can then use computer feedback to have students check their work by using a graphing calculator. When students graph their equations, the computer shows them the implications of their thinking. In such situations, students naturally want to make changes in response to computer feedback. Maybe the parabola needs to shift left, or open a bit more slowly, or open downward. Students respond to the feedback and quickly get more feedback on their next attempt. For this reason, we refer to this process as *iterative feedback*.

Iterative feedback is low risk because students know they can revise their attempts—not just their first attempt, but their subsequent attempts as well. Important goals of iterative feedback are supporting students in taking intellectual risks and encouraging them to persist. When a student says or thinks, "I'll try again; I can make this better!" iterative feedback is doing its job well. Furthermore, when an online lesson is constructed to give good iterative feedback, students can respond to a prompt as simple as "Just draw (or try) anything." The teacher can trust that this entry will make the task accessible to all students while also moving students' mathematical understanding forward.

Collaboration

The increasing availability and quality of Internet-enabled devices in classrooms—and of Internet connections in those classrooms—is something we seek to harness in creative ways in our work. Rather than using the network for the purpose of connecting individual students to the teacher or to a central server, we use it to connect students with one another. In our lessons, students can share ideas, ask questions of one another, and challenge one another in rich and interesting ways. The network facilitates showing students the solutions their classmates have shared, challenging students with new tasks their classmates have designed, and sharing comments and solutions for these shared tasks.

Information for Teachers

Classroom-based online instructional platforms typically come with dashboards for teachers. These dashboards give teachers at-a-glance information about the progress of individual students in their classes. In many situations, this information is broken down by content strand and by proficiency

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level—e.g., if José is proficient at adding fractions with common denominators but struggling to write the decimal form of fractions, then a green square and a yellow square, indicating José's proficiencies, show up on the teacher dashboard's grid.

This type of dashboard provides a view of what students have mastered, but it does not give teachers insight into how students are thinking as they work. A richer dashboard can show a teacher what students are doing as they work and allow her to move quickly between views of an individual's work and of the whole class.

When we design a teacher dashboard for a Desmos lesson, we ask ourselves these questions:

- What information will a teacher find useful while the lesson is going on?
- What information will a teacher find useful after the lesson is over, as he prepares for the next day's instruction?

We then design the dashboard to capture this information and organize it in ways that help teachers do their work.

Information that is useful for teachers during the flow of the lesson, and that we strive to make quickly accessible in our dashboards, addresses the following questions: Who seems to be guessing rather than thinking carefully? Who has one of several common wrong answers? What different correct forms of an algebraic expression has my class generated so far?

Answers to these questions help teachers decide which students to speak with, when to pause the lesson for whole-class discussion, and how to structure a summary discussion at lesson's end.

Information that is useful in planning follow-up instruction might include the full text of student responses that a teacher can skim for the big picture of the class's work or search for use of vocabulary.

(We have not incorporated search as a feature, but we do display the responses of all students in a class on a single page of text, which a browser can search easily.)

TWO ACTIVITIES

What do these principles look like when they come into being in the form of classroom lessons? In this section, we describe two lessons and how each one relates to our principles.

Central Park

In this activity, students move from estimation to calculation to abstraction as they decide how to place dividers in a virtual parking lot so that each parking space in the lot is the same width. If one space is too wide, another will be too narrow—resulting in too few spaces and angry drivers. Students receive feedback as they watch the cars attempt to park. At each phase, students can try again by adjusting their answers and letting the cars park again.

Students begin by dragging dividers into place—no numbers, no computation, just estimation. In the example in **figure 1**, the rightmost space is too small. The student has received feedback by seeing the leftmost car park at an awkward angle to fill the large space and by seeing another car unable to park at all. Students can try again either by clicking "reset" or by moving the dividers.

In later phases of the activity, students calculate the width of the parking spaces (see **fig. 2**) and then use variables to describe these widths in multiple lots of different sizes and with varying divider sizes (see **fig. 3**). The activity is designed so that students use increasingly sophisticated tools in pursuit of expressing an algebraic relationship—and to validate the use of algebraic symbols as timesavers in doing repeated computations.

The online delivery of this activity allows students to receive feedback that goes beyond "right" or "wrong" and pre-loaded hints. Students can interpret their mistakes for themselves and adjust accordingly. Students usually will not use all necessary variables (i.e., w and p; see **fig. 4**). In such a case, they may successfully park cars in one scenario but fail in the others.

An important concern with iterative feedback of this nature is that students will sometimes guess-and-check their way to a solution rather than use the feedback to provoke rethinking. In the Central Park activity, a student may guess 10 feet for the width of the parking spaces in **figure 2,** then get feedback that this space is too wide. Such a student can guess his way to the correct value without noticing the important relationships between the given quantities. In designing this activity, we opted to

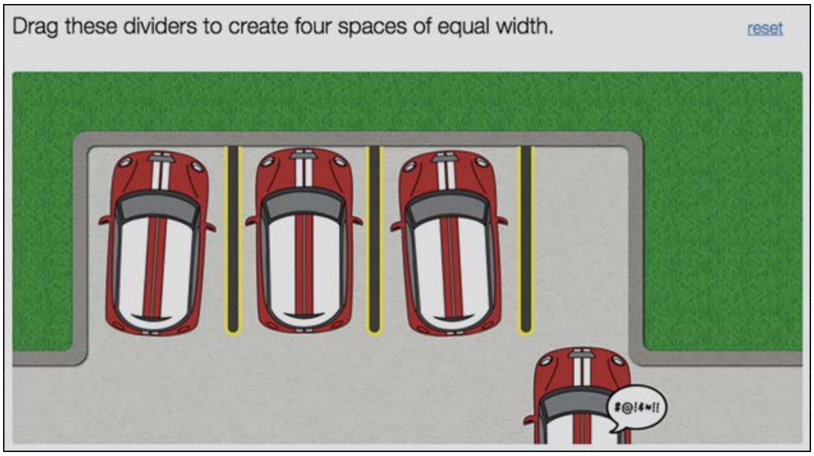

Fig. 1 When the spaces are correctly apportioned, all the cars can park; when there is trouble, as shown here, the drivers are aggravated.

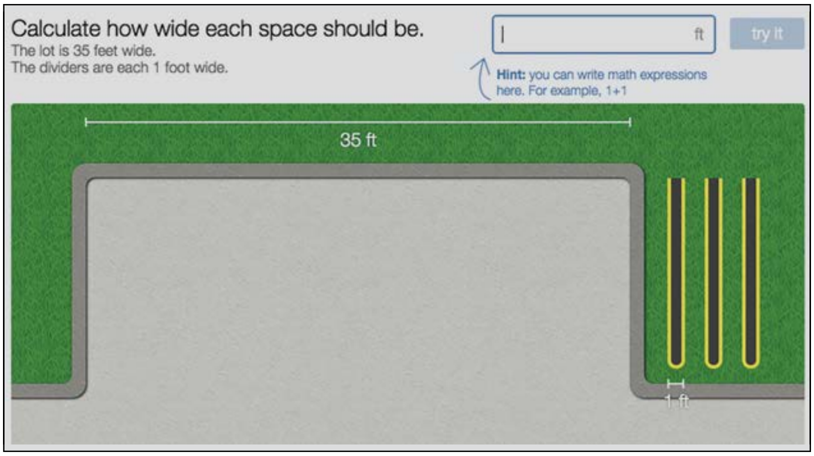

Fig. 2 The computation phase follows initial estimation.

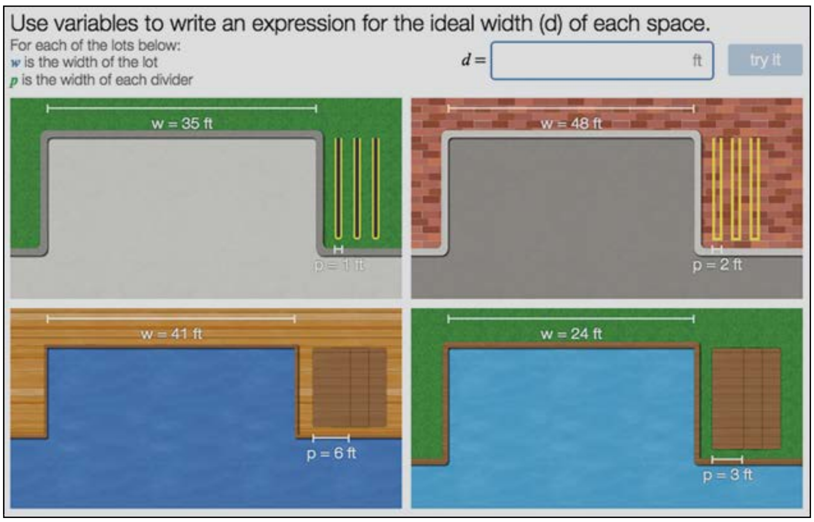

Fig. 3 Students generalize by writing algebraic expressions.

leave this necessary refocusing in the hands of the classroom teacher. We give the teacher the information she needs to make an instructional decision. In this case, the guess-and-check behavior will usually result in a student being unable to write an expression using variables at the appropriate phase. With

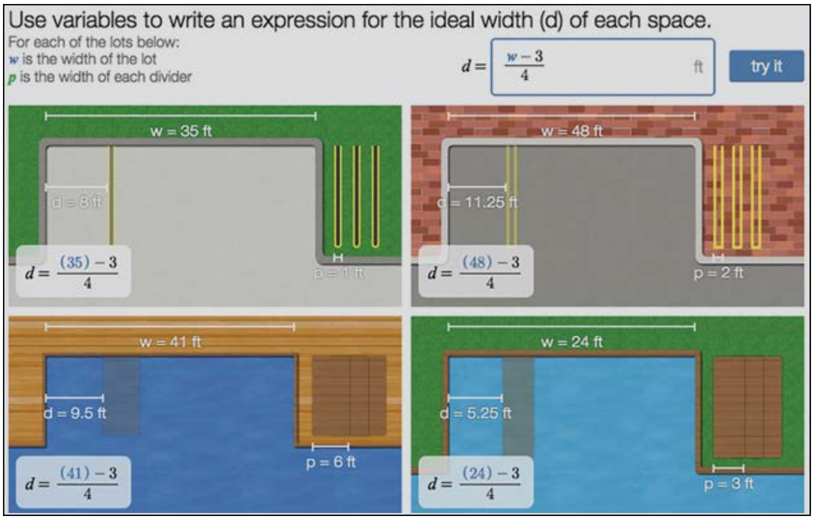

Fig. 4 The expression shown will result in an error because it does not account for the varying width of the dividers.

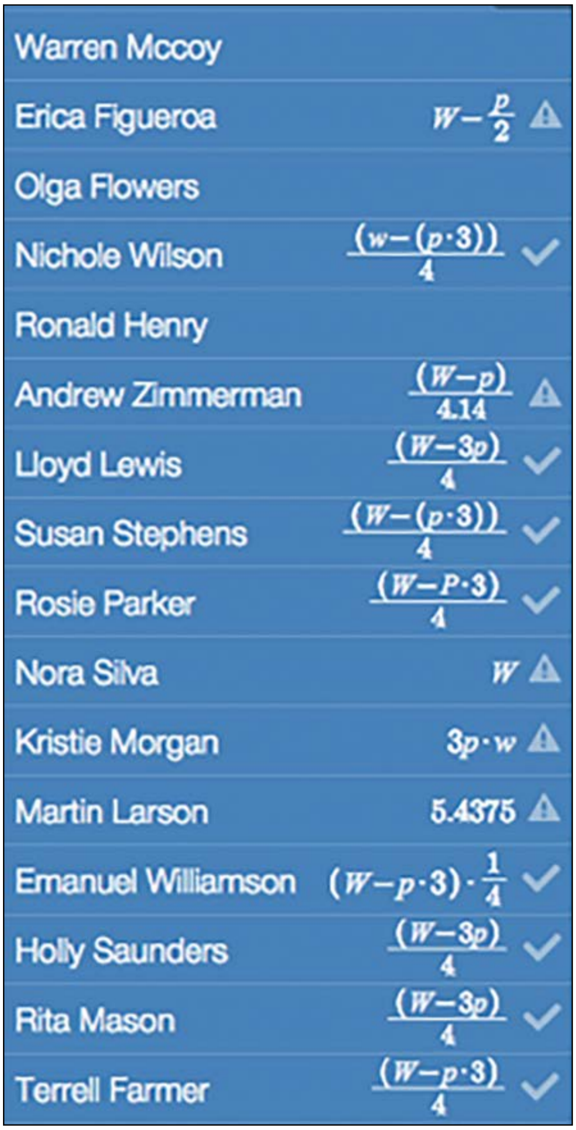

Fig. 5 Student names in this example are fictitious; the work is representative of the variety seen in actual classrooms.

a click, the teacher can see all of the expressions her students have written, along with an icon that indicates correct and incorrect expressions (see **fig. 5**). We also alert teachers in our planning materials that unproductive guessing behavior is something to look for in this phase of the lesson (see "The Student Experience" at https://teacher.desmos.com/centralpark/).

Polygraph

Another activity illustrates our use of networked devices to connect students with each other. Polygraph is a question-and-answer game played in pairs. One student—the picker—selects one object out of sixteen that are displayed in a 4 × 4 array. **Figure 6** shows the challenging rational functions version of the game, but there several versions, including lines, parabolas, quadrilaterals, and hexagons. The second student—the guesser—asks one question at a time, which the picker must answer by clicking yes, no, or I don't know. The goal is for the guesser to identify the correct object by asking about distinguishing features. The graphs are shuffled at the beginning of each round and appear in different places on each player's screen, in order to eliminate location in the array as a distinguishing feature.

The activity Polygraph emphasizes collaboration. Partners work together to determine the correct object. If the guesser makes an error by deleting the object chosen by the picker, the partners are instructed to review their questions and answers and to discuss (face to face) where they went wrong.

Working together in this way creates a need for students to talk about properties of objects—properties for which they may not yet have names. This environment allows students to develop rich informal language, which is captured for teachers to use later in discussion and formalizing. Herbel-Eisenmann (2002) describes how informal "bridging languages" support students in preparing to understand and use "official mathematical language" in more meaningful ways than approaches that begin with formal mathematical vocabulary.

For example, we find students asking questions such as these:

- Is your hexagon dented?
- Does your graph have two pieces?
- Is the bottom of your graph on the *x*-axis?

These are informal ways of talking about concavity (of hexagons), branches (of rational functions), and vertices (of parabolas) that come from the features that students notice. The terms *dented, pieces*, and *bottom* are examples of bridging

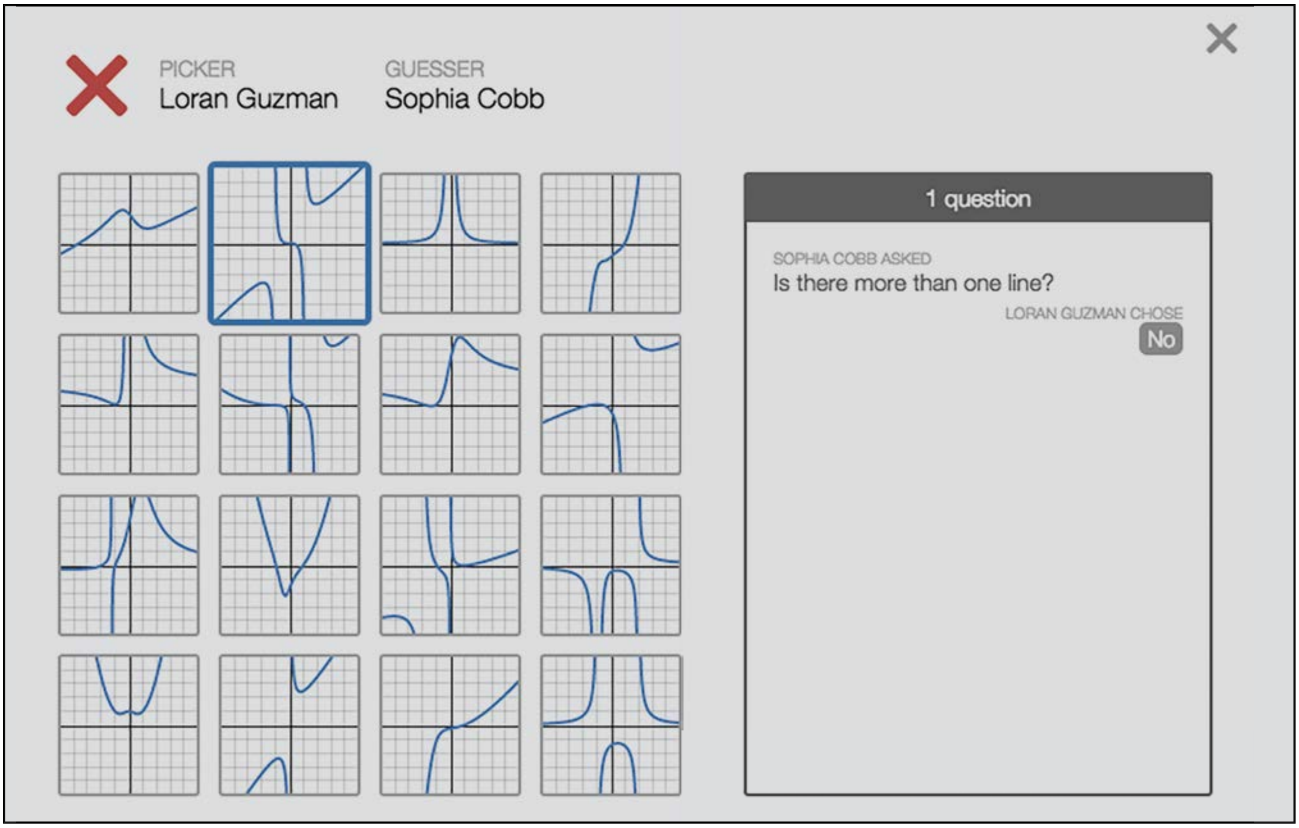

Fig. 6 In a game played in pairs, one student picks a graph and the other asks yes or no questions in an effort to figure out which graph was selected. Here, Loran's answer caused Sophia to eliminate the graph Loran had picked, and the game ended.

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language that supports students in describing and formalizing features of these mathematical objects prior to learning the official mathematical terms for them. The question-and-answer collaborative interface built into Polygraph elicits these ideas and terms—and captures them for the teacher—from many more students than could participate in a linear, whole-class discussion.

It is worth noting also that this approach is foreign to print textbooks because they lack a built-in mechanism for progressive disclosure. Printed pages tend to put the most formal level of math thinking together with the least formal (often omitting informal ways of thinking altogether). In the cases of those print textbooks that do explicitly recognize and value student-generated informal vocabulary—such as the *Connected Mathematics* curriculum (Lappan et al. 1998) that Herbel-Eisenmann studied—the work of noticing, capturing, and capitalizing on this language is left up to the teacher. A carefully designed online lesson can do some of the work of capturing this language, making it easier for the teacher to capitalize on it. Further, online activities can be structured to disclose progressively. In Polygraph, we ask students to describe parabolas informally and later offer the formal vocabulary to describe the properties students have noticed.

This process can offer new insights to teachers and open new mathematical avenues for study. When a College Algebra class played the rational functions version of Polygraph, the teacher noticed students asking questions such as these:

- Does your graph have more than one piece?
- Does it have more than one line?
- Is your graph broken?
- Are there any holes in your graph?

Standard approaches to rational functions in College Algebra and similar courses focus on identifying the existence and location of vertical asymptotes. For students in this particular class, the vertical asymptotes were less immediate features of the graph than the number of branches the graph comprised. The student identified as Loran in **figure 6** has chosen a rational function with three branches (names have been changed). Loran's partner Sophia has asked whether the graph has "more than one line." When Loran answered "no," Sophia eliminated that graph and the game came to an end. This miscommunication led to a face-to-face conversation in which Loran and Sophia worked out what each understood by Sophia's question about lines. In turn, this conversation prepared these students for the teacher to introduce the term *branch*.

To further contribute to the collaborative nature of the concept and vocabulary development in this lesson, students see questions, between rounds, that other students asked as they played (see **fig. 7**). This feature helps both informal and formal vocabulary be spread among students as the rounds proceed.

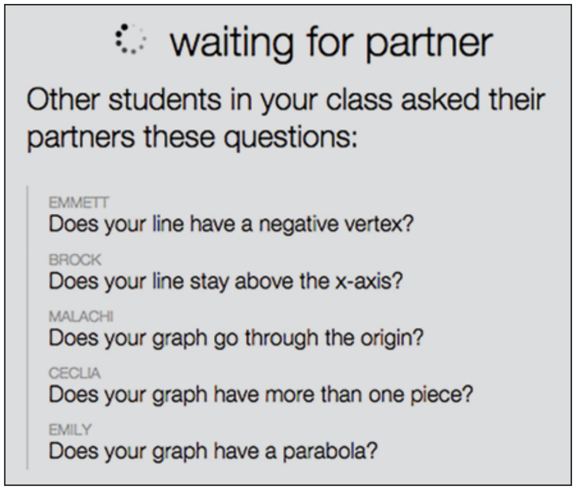

Fig. 7 While students wait for a partner between rounds in "Polygraph," they see questions that their classmates have asked while playing the game.

SUMMARY

Our principles for online instruction may be quite different from current orthodoxy, which calls for individualization and atomization of skills. We believe in the power of combining quality provocations, robust tools to connect students, and skilled teachers to help students build mathematical understanding, vocabulary, and skill.

We hope that these ideas are infectious and inspirational and that they help to improve the conversation about the possibilities of educational technology in mathematics classrooms. Our development work continues, with information necessary for getting started at teacher.desmos.com and learn.desmos.com.

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