MATHEMATICAL MODELING IN THE HIGH SCHOOL CURRICULUM

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Ideas from the GAIMME report illustrate how teachers can engage students in the modeling process.

Mathematical modeling—using mathematical approaches to understand and make decisions about real-world phenomena—"can be used to motivate curricular requirements and can highlight the importance and relevance of mathematics in answering important questions. It can also help students gain transferable skills, such as habits of mind that are pervasive across subject matter" (GAIMME 2016, p. 8). Although teachers recognize the value of engaging their students in mathematical modeling, few have had opportunities to experience modeling, and many teachers feel unsure of how to teach it.

In 2015, mathematics leaders and instructors from the Society for Industrial and Applied Mathematics (SIAM) and the Consortium for Mathematics and Its Applications (COMAP), with input from NCTM, came together to write the *Guidelines for Assessment and Instruction in Mathematical Modeling Education* (GAIMME) report as a resource for teachers who want to incorporate the practice of mathematical modeling in their classrooms. The GAIMME report, which can be downloaded for free (http://www.siam.org/reports/gaimme.php), provides insight into the modeling process, what mathematical modeling looks like across the grades, the role of the teacher, and how to assess students' modeling processes.

The GAIMME report provides a sense of what mathematical modeling is and is not, as well as practical advice on how to teach modeling through the grade levels. It is consistent with but not limited to how the Common Core (CCSSI 2010) includes modeling as one of the Standards for Mathematical Practice that are meant to span kindergarten through grade 12. The GAIMME authors demonstrate the importance of mathematical modeling and how and why it should be an essential part of every student's mathematics experience throughout their education. This article, written by members of a joint committee between NCTM and SIAM, serves as an introduction to the GAIMME report and offers information on how to incorporate modeling into the high school mathematics classroom.

WHAT IS MATHEMATICAL MODELING?

Mathematical modeling has been defined in many ways. The authors of the GAIMME report define it as "a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena" (GAIMME 2016, p. 8). Of particular importance is the emphasis on modeling as a process. Modeling is iterative and involves frequent revision. Moreover, modeling involves messy, open-ended problems that require students to make genuine choices about how to approach problems mathematically, what assumptions to make, and how to determine the effectiveness of the approach used.

The GAIMME report breaks the modeling process into six components, described in **figure 1**. Most articles represent these steps (or some variation of them) as a cycle, but the GAIMME report intentionally uses the representation shown in **figure 2**, which "reflects the fact that in practice a modeler often bounces back and forth through the various stages" (GAIMME 2016, p. 13).

In the high school curriculum, mathematical modeling can be implemented in any course and the nature of this implementation can vary depending on the

Components of a Modeling Process

Identify the problem: We identify something in the real world we want to know, do, or understand. The result is a question in the real world.
Make assumptions and identify variables: We select "objects" that seem important in the real-world question and identify relations between them. We decide what we will keep and what we will ignore about the objects and their interrelations. The result is an idealized version of the original question.
Do the math: We translate the idealized version into mathematical terms and obtain a mathematical formulation of the idealized questions. This formulation is the model. We do the math to see what insights and results we get.
Analyze and assess the solution: We consider: Does it address the problem? Does it make sense when translated back into the real world? Are the results practical, the answers reasonable, and the consequences acceptable?
Iterate: We iterate the process as needed to refine and extend our model.
Implement the model: For real world, practical applications, we report our results to others and implement the solution.

Fig. 1 The figure shows the components of the modeling process. (Excerpts from GAIMME 2016, pp. 12–13.)

learning goals. Embedded into a subject-centered course, modeling problems provide opportunities for students to develop mathematical approaches that apply tools from that course to problems that matter to the students. They can also form the basis of a stand-alone course in modeling, which could reinforce mathematical concepts from current or previous courses. High school students also have opportunities to engage in mathematical modeling competitions, such as Moody's Mega Math Challenge (m3challenge.siam.org) or High School Mathematical Contest in Modeling (HIMCM)® (comap.com/highschool/contests/himcm/). Preparing for these can provide focus for math clubs and summer programs.

AN EXAMPLE: DRIVING FOR GAS

In the Driving for Gas example from the GAIMME report, we demonstrate how high school students can engage in the modeling process through a problem situated in a context that is familiar to them. The problem can be used as a classroom activity or as a practice problem for students preparing for a modeling competition. This problem can be a good first step for students who are not familiar with solving open-ended modeling problems. **Figure 3** illustrates how students engage in the six components of the modeling process for this task.

> Most drivers have a 'usual' region in which they do most of their driving. However, gas prices may vary widely so that gas may be substantially cheaper somewhere other than within that usual region. Would it be more economical to go to a station outside the usual region to buy gas? Thus, the general question we wish to address is, "How might we determine which gas station is the most cost-efficient?" (GAIMME 2016, p. 181)

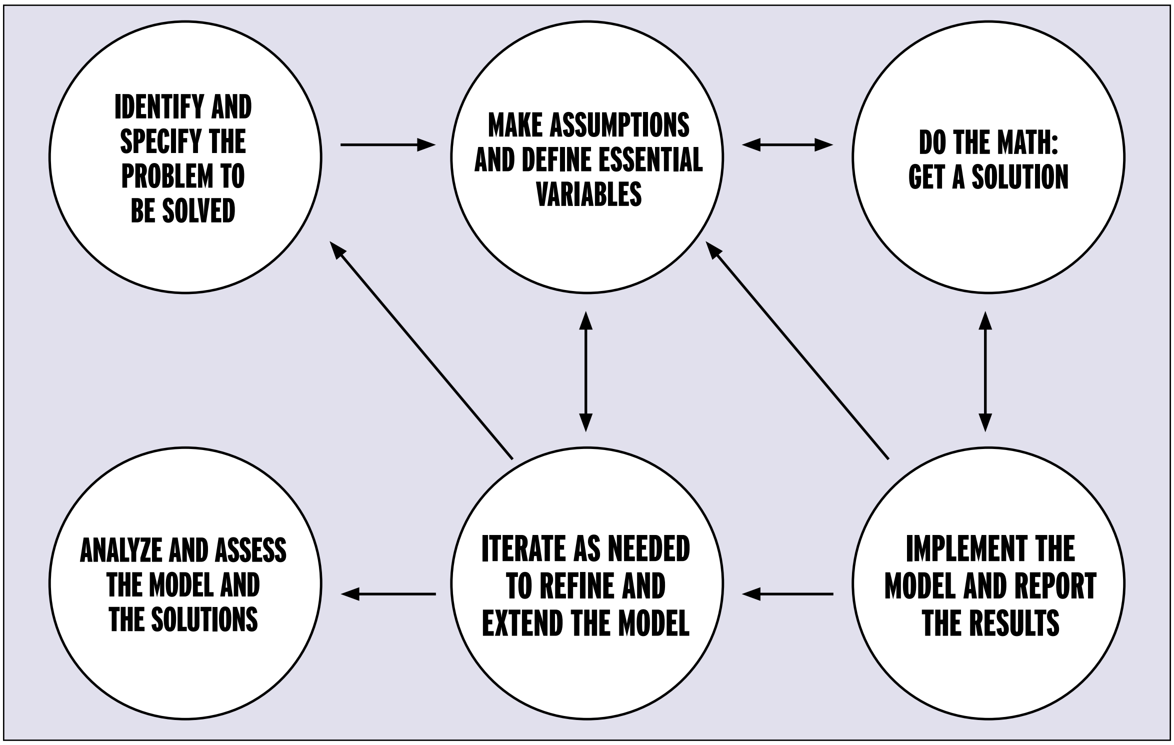

Fig. 2 The diagram illustrates the mathematical modeling process (GAIMME 2016, p. 31). Copyright © 2016 Consortium for Mathematics and Its Applications and Society for Industrial and Applied Mathematics. Reprinted with permission. All rights reserved.

The GAIMME report provides a comprehensive narrative explaining how to enact the Driving for Gas problem (GAIMME 2016, Appendix C). The narrative includes various levels of scaffolding, ideas for formative assessment, and extensions, with each component of the modeling process addressed explicitly.

THE TEACHERS' ROLE

Teaching modeling is challenging, especially for teachers who are new to the process. **Figure 4** illustrates a framework for thinking about the role of the teacher in facilitating student modeling (Carlson et al. 2016). The dark squares in the center of the figure show a simplified version of the modeling process that students engage in: posing questions, building solutions, and validating conclusions. Surrounding this center are teachers' actions.

Teaching modeling begins by selecting or developing a task. There is a growing body of task resources for teachers to consider, such as the GAIMME report, NCTM's *Mathematical Modeling and Modeling Mathematics* (Hirsch 2016), COMAP's Math Models (www.mathmodels.org), and Moody's Mega Math Challenge (www.m3challenge.siam.org). In creating or selecting a task, the most important question to consider is whether the task requires students to make decisions about how to approach the problem mathematically. Teachers also consider how familiar students are with the context and the mathematical concepts they hope students will use on the task. They anticipate how the task might play out. Teachers might ask themselves the following questions:

- What kinds of questions will the students have about the context?
- What additional information will they need or want?
- How will they get that additional information?
- What assumptions will they make as they begin to build their models?
- How can I help students feel comfortable about making assumptions?
- What kinds of problem-solving strategies are students likely to use?
- How do I want to balance small-group and whole-class discussions?
- At what point in the modeling process are students likely to get stuck?
- What kinds of strategies can I use to intervene without taking over the modeling process?
- What tools will students use to analyze their solutions and assess their models?

Identify the problem	The teacher selects the real-world problem. The Driving for Gas problem is situated in a context that many students are familiar with or have an interest in. It is open (at the beginning, middle, and end) and requires students to make decisions before they can tackle it mathematically.
Make assumptions and identify variables	Students consider a number of real-world factors. They may need to narrow the focus of the question to make the problem more tractable. For example, they might focus on a specific route that they travel regularly and create a solution based on that specific case. Some questions to consider might be these: What's my car's gas mileage? They may want to limit their focus to two cars with different gas mileage. How far off the route is the gas station with the cheaper gas? How much gas will we be purchasing?
Do the math	Students research the cost of gas at the gas station on the route and one location (with substantially cheaper gas) off the route. They make calculations for the specific type(s) of cars they decided to consider. They calculate the total cost of buying gas at each station. The choice of mathematical approach is ideally left to the student but may be constrained by the teacher to the techniques in a course or unit.
Analyze and assess the solution	Throughout the process, students consider if their assumptions and strategy make sense in the real-life context of the problem. Students give mathematical arguments explaining why some answers are unreasonable or not useful and support their own answer as reasonable or useful. For example, if the distance to the cheaper gas is 25 miles away, is it reasonable to think you would drive that far to get cheaper gas? They attend to the accuracy that is appropriate for the answer, such as rounding to the nearest mile or gallon of gas.
Iterate	Depending on how satisfied students are with their final product, they may go back and modify their approach. In the refinement of their model, students may consider what is meant by "cost efficient" in the original problem statement. They may have taken into account only the dollar cost of the gas and may not have considered such factors as the cost of their time or the cost to the environment of driving farther for gas.
Implement the model	One of the goals of creating mathematical models is to answer a question posed in a real-life context. After students have engaged in the process of creating a model, the model would be used to decide whether or not they should drive the extra miles to buy the cheaper gas.

Fig. 3 The figure shows the components of mathematical modeling in the Driving for Gas problem, as discussed in the GAIMME (2016) report.

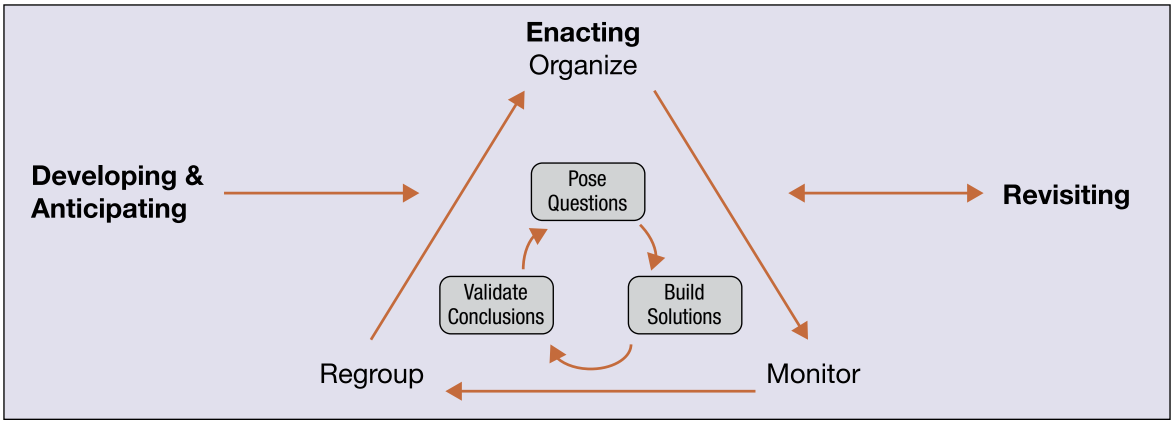

Fig. 4 The diagram illustrates the framework for teachers' roles in modeling (Carlson et al. 2016, p. 122).

After developing and anticipating, the teacher moves into enacting the modeling task. Just as modeling is an iterative process for the students, supporting student modeling is an iterative process for the teacher. Teachers must first organize the presentation of the task. This involves introducing the context and the problem and allowing students time to ask clarifying questions to ensure they understand what they are being asked to do. Depending on their students' familiarity with modeling and the teachers' instructional goals, teachers may ask students to brainstorm initial ideas on how to approach the task and to discuss the need to use mathematics to analyze the problem.

Next, as students begin working, teachers monitor their work by taking note of the strategies used, the assumptions students are making, the mathematical opportunities that arise, and the places where students are getting stuck. As students progress, the teacher can occasionally regroup by bringing the whole class together. This regrouping can be used for a variety of purposes, depending on where students are in the modeling process. A teacher can use regrouping to address misconceptions, answer clarifying questions, provide small suggestions, and have students share their progress and receive peer feedback. More frequent regrouping may lead to greater similarity in approaches across groups, while less whole-class communication can lead to a greater range of mathematical models.

As students draw closer to their final solutions, teachers should encourage them to analyze their solutions and assess their models. This stage can include asking students to make sense of the answers in the context of the problem, to find ways to measure the accuracy of their solutions, or to explore how their solutions change if they vary their assumptions.

The teacher's role in revisiting can serve to support students in several ways. The teacher might summarize the major mathematical ideas that students used in their solutions. This step can also serve as an opportunity to unpack and discuss the modeling process itself. Teachers might ask students to reflect on the modeling process and comment on strategies that helped them succeed in finding their solutions. Revisiting is an ideal opportunity for discussing how the problem could be changed or extended and whether the students' solutions are still viable in these new situations.

The authors of the GAIMME report suggest the following guiding principles to aid teachers as they consider how to introduce mathematical modeling into their classrooms:

- "Start small
- Scaffold initial experiences with leading questions and class discussion
- Use common, everyday experiences to motivate the use of mathematics
- Use bite-sized modeling scenarios that require only one or two components of a full modeling cycle
- Share your goals and instructional practices with parents and administrators" (GAIMME 2016, p. 59)

Developing tasks that draw on student interests gives teachers an ideal opportunity for integrating students' lives into the curriculum. For example, some high school students may be interested in social justice or sustainability issues. By having a say in choosing their topics of study in a math classroom, students can become more invested in their own learning while seeing the relevance of mathematics in their lives.

ASSESSMENT QUESTIONS

Before thinking about assessing students as modelers, teachers may consider the learning goals for their students. These goals can be focused on how students implement the modeling process (or components of the process), how they communicate their findings, or how they work as team members. The GAIMME report includes suggestions for learning goals and means for assessing each of those goals. **Figure 5** offers some guiding questions for teachers

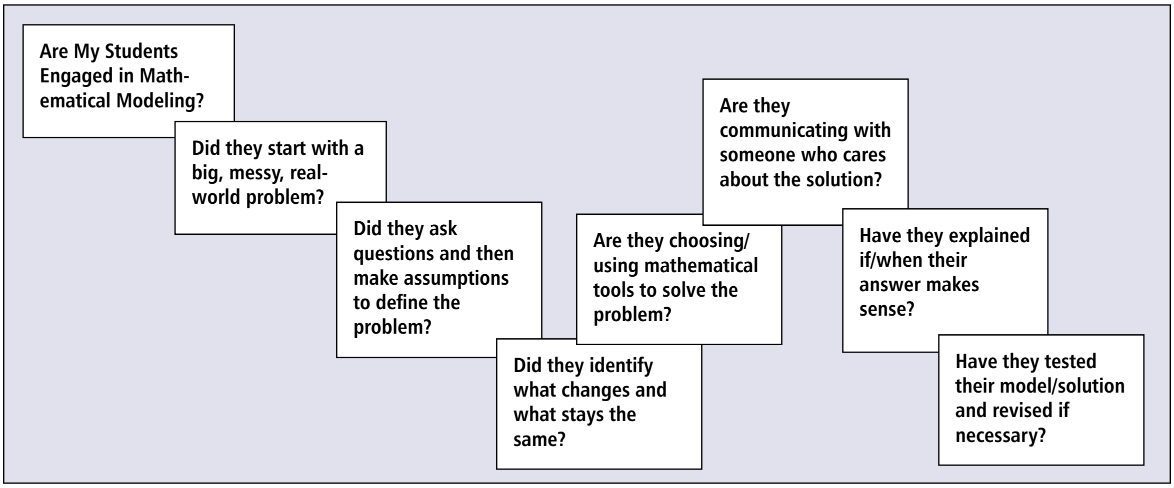

Fig. 5 The figure provides questions to assess whether students are modeling (NCTM 2016, p. 256).

who wonder, "How can I tell if my students are engaged in the modeling process?"

As students are given more opportunities to engage in the math modeling process, we hope that they become better mathematical modelers. The open-ended nature of the tasks can present a significant challenge when grading student work. The solutions can vary because students are free to make different assumptions and choose from a variety of tools to create models and solve problems. The main idea to keep in mind is that "assessment should focus on the process and not on the product or pieces only" (GAIMME 2016, p. 47).

The GAIMME report provides questions designed for formative assessment of student engagement in math modeling for each component of the modeling process. (See **fig. 6** for samples.)

Students engaged in mathematical modeling tasks can share their work in a variety of ways. Rubrics offered in the GAIMME report are designed to articulate expectations and standards for success on products that range from informal presentations of approaches and solutions to posters, or more formal written reports. It is important to remember that the goal of the sharing is for students to communicate their mathematical thinking in a clear fashion, making sense of both the mathematics and the context. As mathematician Henry O. Pollak notes,

> Mathematicians are in the habit of dividing the world into two parts: mathematics and everything else, sometimes called the 'real world'. People often tend to see these two as independent of one another—nothing could be further from the truth. When you use mathematics to understand a situation in the real world, and then perhaps use it to take action or even to predict the future, both the real world situation and the ensuing mathematics are taken seriously. (GAIMME 2016, p. 95)

MODELING COMPONENT	QUESTIONS ABOUT YOUR MODEL AND HOW YOU MADE IT	MODELING-RELATED VOCABULARY TO BUILD
DEFINING THE PROBLEM	What is the specific problem your model is going to solve? (My model will tell you . . .)	specific, focus
MAKING ASSUMPTIONS	What have you assumed in order to solve the problem? Why did you make these choices?	assumption, assumed
DEFINING VARIABLES	Where did you find the numbers that you used in your model?	resources, citations
GETTING A SOLUTION	What pictures, diagrams or graphs might help people understand your information, model, and results?	diagram, graph, labels
ANALYSIS AND MODEL ASSESSSMENT	How do you know you have a good/useful model? Why does your model make sense?	testing, validation
REPORTING RESULTS	What are the 5 most important things for your audience/client to understand about your model and/or solution?	client, audience

Fig. 6 The table shows a portion of GAIMME's "Modeling Assessment Rubric." (Excerpt from GAIMME 2016, p. 197; adapted from Rachel Levy, IMMERSION program.)

STUDENTS "REMEMBER THE MATH"

The ideas in this article and in the GAIMME report help to answer the questions "What is mathematical modeling?" and "How can we begin to incorporate mathematical modeling in high school classrooms?" Students who have engaged in the modeling process appreciate the opportunity to use their own ideas in creating a mathematical solution to a real-world problem and have experiences that help them regardless of what college or career path they follow. When asked to reflect on her experience during a math-modeling task, one precalculus student wrote,

> It [modeling] helps me remember the math, because then I have some kind of example that can help me think through a problem logically and relate it to something that I know about outside of the classroom. I feel like I can apply this method to a lot of things outside of math, like sciences and literature and history.

Modeling is not for science only—it transcends disciplines and affords tools for students to engage with real problems in their community and in society. If you have not yet tried modeling with your students, we hope the GAIMME report will help you "Start big. Start small. Just start" (GAIMME 2016, p. 92).

Let's chat about modeling.

MT **has a new way for our readers to interact and connect with authors and with one another.**

On Wednesday, January 25, 2017 at 9:00 p.m. EST,

we will talk about "Mathematical Modeling in the High School Curriculum," by Maria L. Hernandez, Rachel Levy, Mathew D. Felton-Koestler, and Rose Mary Zbiek.

Join the conversation at #MTchat.

We will also Storify the conversation for those who cannot join us live.

NOTE: There will be NO Twitter chat in December.

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