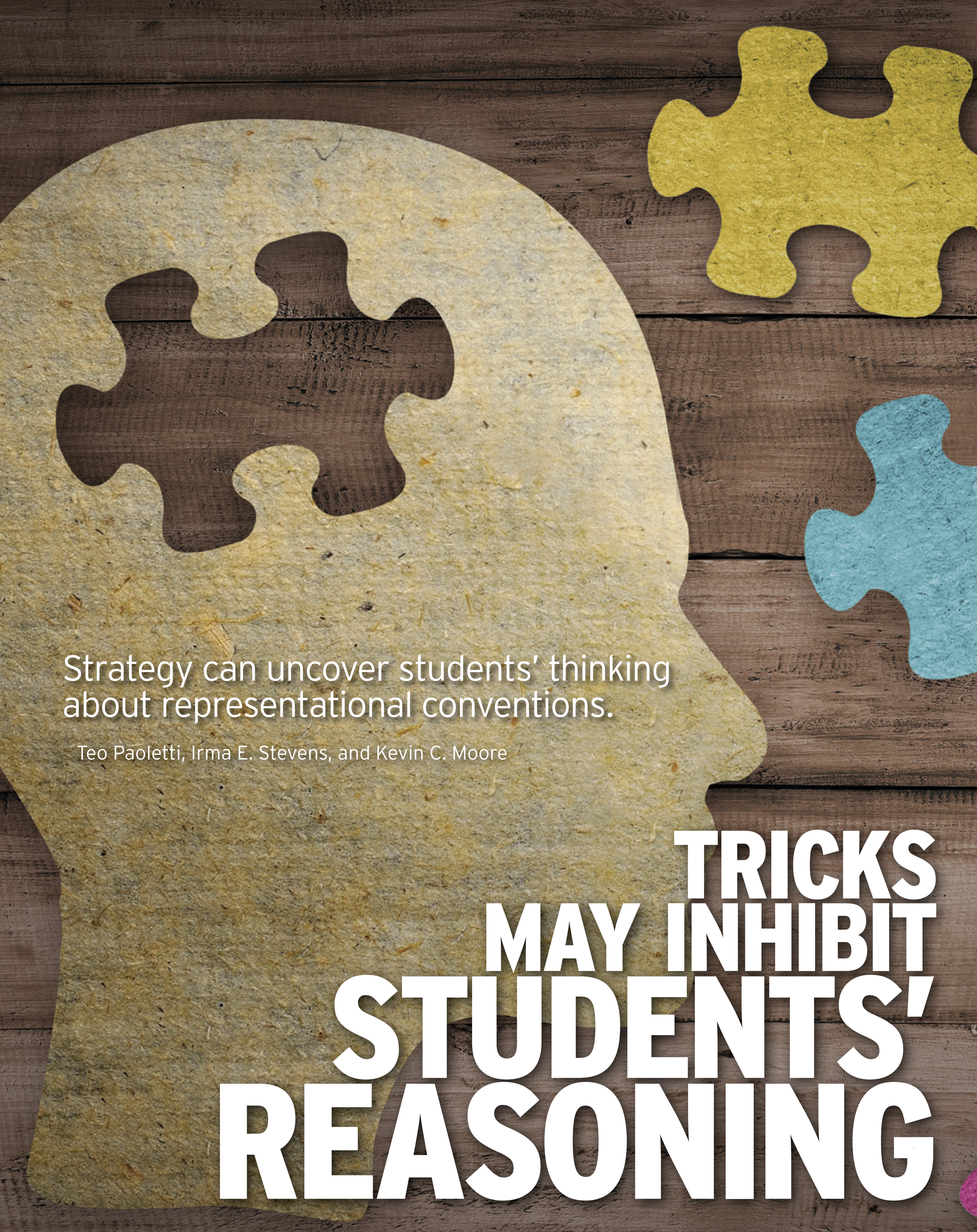


TRICKS MAY INHIBIT STUDENTS' REASONING

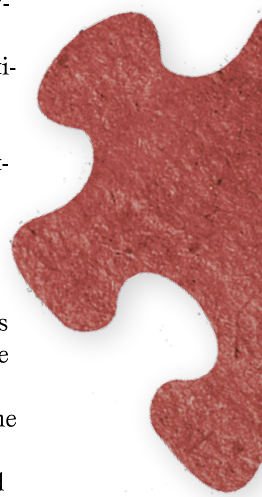
Strategy can uncover students' thinking about representational conventions.

Teo Paoletti, Irma E. Stevens, and Kevin C. Moore

LEOLINTANG/SHUTTERSTOCK

Have you ever held a class discussion you thought went well until your students made a claim that had you question how they were interpreting the mathematics at hand? When such a situation happened in our classroom, we used the interaction to change our teaching and research to address preemptively the particular inconsistencies we had noticed.

The classroom situation occurred during an activity based on the Power Tower (**fig. 1a**), an amusement ride at Cedar Point in Sandusky, Ohio, that shoots riders up the tower, then lets them fall toward the ground before shooting them up again, repeating this process two more times. After showing a video of the ride (see http://www.youtube.com/watch?v=HrGpGMrkUrM), we asked our students to sketch a graph of a rider's total distance traveled and the vertical distance from the ground, with the intention of having them investigate changes in the vertical distance from the ground along with changes in the total distance traveled (see Moore et al. [2014] for more on this task). The lead instructor presented two student-generated graphs (re-created in **fig. 1b** and **c**), and the class discussed how each graph accurately represented the quantities changing together. When the instructor concluded that the two graphs were "the same," however, several students objected, saying that one "graph is a function" and the other graph "is not." Although these students understood that the two graphs accurately represented some aspects of how the quantities covaried, the fact that one graph passed the vertical line test and the other did not meant to them that the graphs represented different relationships.



This interaction piqued our interest. We conjectured that in their previous school experiences, our students lacked opportunities to distinguish which features are mathematically critical to representing relationships (and functions) and which are merely a choice or convention (e.g., orientation of axes). Hence, we began to investigate the extent to which our students' understandings of mathematical concepts either were rooted in reasoning about relationships between quantities or relied on what we perceived to be curricular conventions.

In this article, we present students' ways of thinking about rate of change and functions that are grounded in quantitative and covariational reasoning. Many researchers (e.g., Carlson et al. 2002; Ellis et al. 2012; Johnson 2013; Thompson et al. 1994; Thompson 2011, 2013) have argued that such reasoning supports students' engagement in several of the Process Standards described in *Principles and Standards for School Mathematics* (NCTM 2000) and Standards for Mathematical Practice described in the Common Core State Standards (CCSSI 2010). We conclude with suggestions that teachers can use to help their students develop productive ways of thinking to differentiate aspects that are critical to a concept from those only intended to be curricular conventions.

PHOTOGRAPH BY ANDREW BORGAN

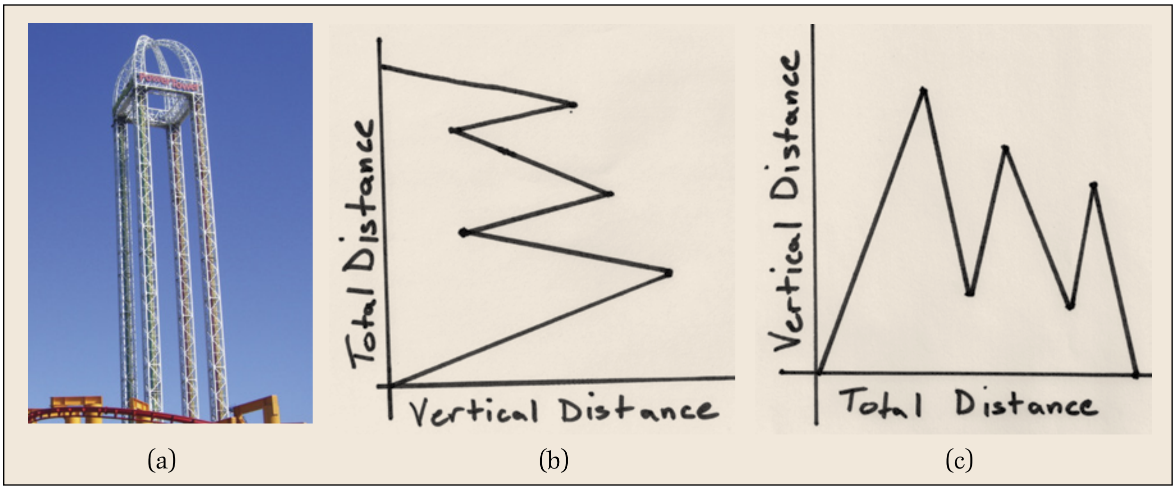


Fig. 1 The Power Tower (a) elicits two correct graphs (b and c) for related distances.

RATE OF CHANGE

Rate of change is an idea that unifies middle school through postsecondary mathematics courses, including algebra, calculus, and differential equations. Understanding that two quantities A and B covary at a constant rate of change involves understanding that if quantity A's value changes by any but no particular amount a, quantity B's value changes by an amount equal to some constant, m, times a (i.e., $m \bullet a$). Further, if quantity B's value changes by any but no particular amount b, then quantity A's value changes by an amount equal to the constant $1/m$ times b (i.e., $(1/m) \bullet b$). A student with such an understanding can interpret a linear relationship displayed in the Cartesian coordinate system by imagining variations in the quantity on either the horizontal (**fig. 2a**) or the vertical (**fig. 2b**) axis first and then imagining how the other quantity changes in relation to those variations; the student understands that the constant rate of change associated with a linear relationship can be thought of as either the measure m or the measure $1/m$, depending on which quantity he or she chooses first. Such an understanding of rate of change can be generalized to any representational system (e.g., polar graphs as in **fig. 2c** and **d**, tables, or equations).

To illustrate the flexibility and productivity of this way of thinking, consider the following illustration. We asked Student 1 to determine the rate of change of the relationship represented in the graph shown in **figure 3**. After she claimed that the rate of change of the relationship is 4, we posed that a hypothetical student identified a rate of change of one-fourth; this allowed us to investigate if Student 1 would consider an unconventional interpretation of the rate of change, specifically the rate of change of x with respect to y. This query is one example of a type of question we often pose: the hypothetical student response is mathematically viable but breaks from curricular conventions.

Student 1 understood that the rate of change of a relationship could be conveyed in more than one way depending on which quantity she chose to coordinate first. She could interpret the relationship as having a rate of change of 4 if she

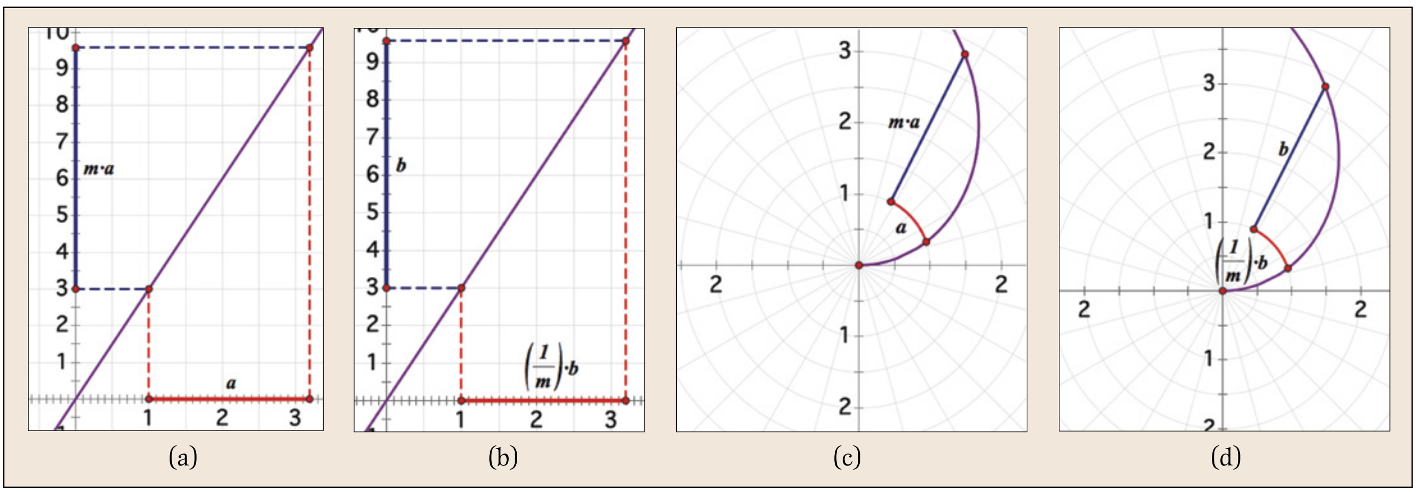


Fig. 2 Two quantities covary at a constant rate of change.

coordinated x first (y varies by an amount 4 times as large as the amount x varies) or 1/4 if she coordinated y first (x varies by an amount 1/4 times as large as the amount y varies).

As a point of contrast, we present Student 2's response when we provided the labeled graph shown in **figure 4a**. We stated that a hypothetical student, Xavier, constructed the graph when asked to graph the equation $y = 3x$, and we asked Student 2 to evaluate Xavier's graph. We wanted to know if Student 2 would interpret the graph as accurately representing the relationship defined by $y = 3x$.

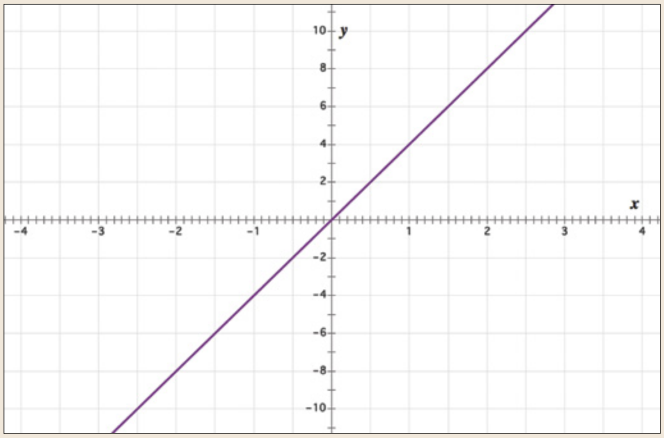
Whereas Student 1 thought about rate of change in terms of coordinating quantities' values, Student 2's understanding of rate of change was dominated by perceptual features of the graph (see **fig. 4**). Student 2 rotated the graph to obtain x on the horizontal axis (**fig. 4b**) and associated the rotated graph with a "negative slope." To Student 2, slope (or rate of change) was indicated by the perceived tilt of a line, rather than a relationship between the values of two quantities on the graph. Such tilt-slope associations enable a student to respond correctly when common curricular conventions of graphing are maintained, but the associations are not productive in unconventional situations, such as describing the rate of change of the relationship represented in **figure 4b**.

FUNCTION

A function is a particular type of relation in which each element of a defined set, called an input, is mapped to a unique element of a second defined set, called an output. "Function" is not an inherent property of a relationship between two sets, but instead depends on how one decides to define a relation between those sets. To illustrate a productive understanding of functions, consider Student 3, who was discussing the graph in **figure 1b**. (Note: [. . .] indicates a break in the transcript.)

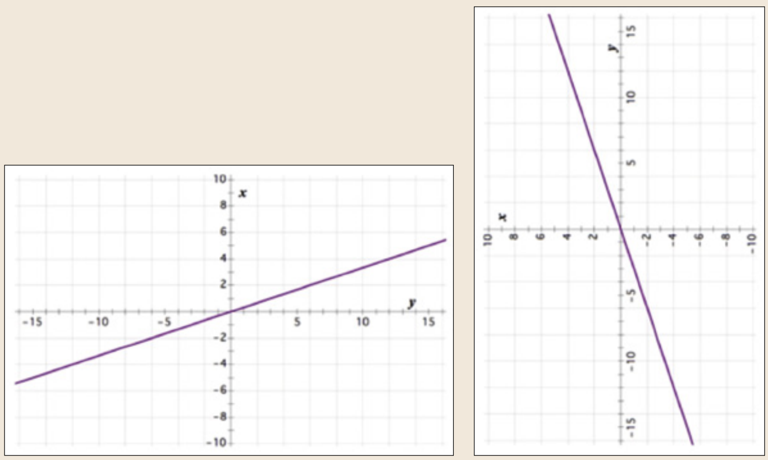
> *Student 3*: Well, it depends which axis is your input and which axis is your output. [. . .] If your inputs were actually your vertical axis, the outputs were your horizontal axis, then that would be a function 'cause you would have a [. . .] unique output for every input. [. . .] [*Student 3 continued by discussing how figure 1b may not represent a function*] if the x-axis is your input and y-axis is your output.

In contrast to the students who argued that the graphs in **figure 1b** and **c** were different because one "graph is a function" and the other graph "is not," Student 3 implied that it did not make sense to claim the graph "is" or "is not" a function. Rather, Student 3 understood that discussing whether the



Student 1: So, my rate of change is, as x increases by 1, y increases by 4. [. . .] But I could also say, as y increases by 1, x increases by [. . .] one-fourth. So we have a one to four relationship between the quantities on these axes. But [. . .] I don't know which is my input and which is my output. [. . .] So there's a one to four relationship between these no matter how you look at it. [. . .] So, if I'm calling my vertical [axis] my input, and my horizontal [axis] being my output, it's not going to be rise over run. It'd be run over rise. So this would be change in x over change in y. And then that would give you one-fourth.

Fig. 3 Student 1 addresses the graph with a response that considers an unconventional interpretation. Note: [. . .] indicates a break in the transcript.



(a) (b)

Student 2 rotated the graph in ***figure 4a*** *as shown in* ***figure 4b*** *and laid the marker on the line that is sloping downward, left to right.*

Student 2: This is negative slope. So I would [. . .] show them like the difference between positive and negative slopes. [. . .] Because that's something that, like, when I was in middle school we, like, learned kind of like a trick to remember positive, negative, no slope, and zero [making tilted hand motions to indicate a direction of line for each]. Like where the slopes were. And it's stuck with me until now, so it's important to know which direction they're going.

Fig. 4 The graph (a) of a hypothetical student, Xavier, prompts Student 2 to interpret the relationship using a horizontal y-axis (b). Note: [. . .] indicates a break in the transcript.

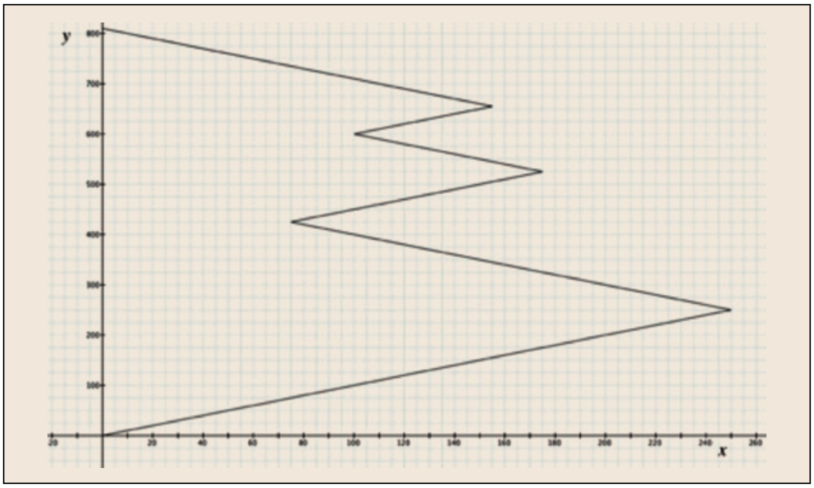


Fig. 5 Student 4 addressed the claim that this graph represents *x* as a function of *y*.

graphed relationship represents a function requires first defining input and output quantities and then determining uniqueness (e.g., total distance is not a function of vertical distance, but vertical distance is a function of total distance).

As did the students in the opening vignette, Student 4 initially claimed that the graph in **figure 5** (a graph similar to **fig. 1b** but without explicitly defined quantitative referents on each axis) did not represent a function. We then posed that a hypothetical student, Yolanda, argued that the graph represents "x is a function of y" and asked Student 4 to respond to Yolanda's claim.

WHEN WE IMPOSE AND MAINTAIN CONVENTIONS, STUDENTS CONSTRUCT UNDERSTANDINGS RESTRICTED TO SPECIFIC REPRESENTATIONS AND CONSTRAINTS.

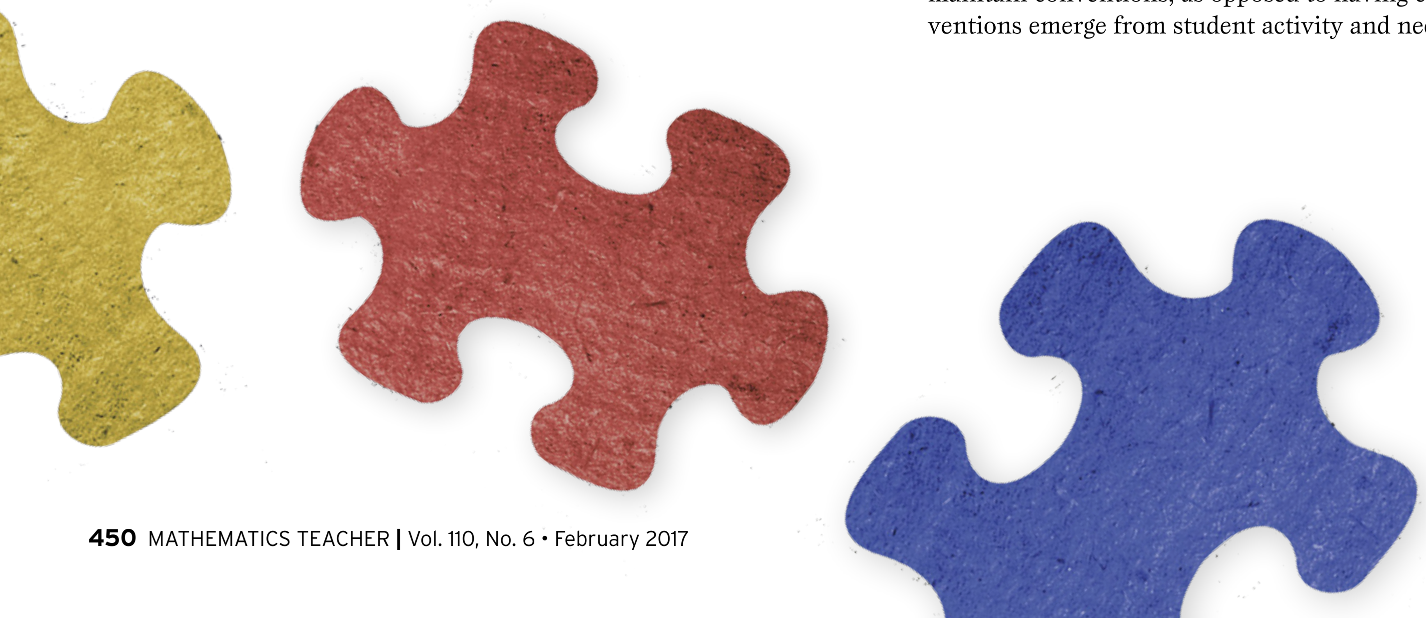
> *Student 4*: That x is a function of y? I mean x and y definitely correlate, there's definitely a relationship between them. But, I've always been taught a function, it has an x-value it can only correspond to one y-value. The same x-value cannot have a different y-value and still be a function.

In contrast to Student 3, Student 4's understanding entailed maintaining the input quantity of a function along the x-axis; representing the input quantity of a function along the x-axis was not merely a convention but inherent to Student 4's way of thinking. Thus, Student 4 did not interpret Yolanda's statement that the graph represents "x is a function of y" as different than the statement "y is a function of x."

STUDENTS' UNDERSTANDINGS

A notable difference exists between the students' responses above with respect to what we perceive to be curricular conventions. Students 1 and 3 held ways of thinking that entailed understanding particular conventions of representing certain ideas as exactly that, conventions: standard ways of doing things chosen from equally viable ways of doing things. For example, Student 1 understood that the rate of change of a relationship could be measured in multiple ways. Although she usually considers the rate of change of y with respect to x, it was equally viable to describe the rate of change of x with respect to y. In contrast, what we perceive to be conventions were instead inherent to how Students 2 and 4 thought about rate of change and function.

We are not surprised by Student 2's and Student 4's understandings, because these understandings likely worked in addressing questions posed by textbooks and teachers throughout their K–grade 14 schooling experiences. When we as educators (implicitly or explicitly) impose and maintain conventions, as opposed to having conventions emerge from student activity and need,



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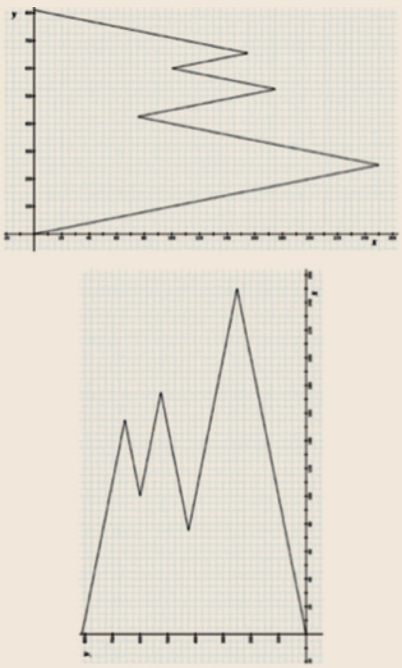
Representation	Questions for Students	Goal
	1. Do the two graphs represent the same relationship? Why or why not? 2. Does each graph represent a function? Why or why not? 3. What are you considering as the input?	A graph represents a relationship that can be defined in different ways. Under certain definitions, the graph represents a function, and under other definitions, the graph does not represent a function. Key to determining if the graph represents a function is whether each value of the defined *input* quantity corresponds to one unique value of the defined *output* quantity.

Fig. 6 Sample questions and goals relate to the tasks presented.

students can construct mathematical understandings dependent upon those conventions. In turn, students' understandings are restricted to specific representations and constraints, often obscuring features critical to a mathematical concept (e.g., rate of change as a coordination of covarying quantities). For example, associations between rate of change and the direction of a line are helpful only as long as graphs are represented in the Cartesian coordinate system with conventionally defined axes. Similarly, the vertical line test works if the input of a function is represented on the horizontal axis. However, these ways of thinking become problematic when conventions are not maintained (e.g., as in **fig. 4b**) or when students are asked to work in a different coordinate system (e.g., polar coordinates). When encountering graphs that depart from these conventions, students who hold such ways of thinking are not prepared to notice or interpret these departures as viable representations of an idea.

STUDENTS ADDRESS THEIR OWN ASSUMPTIONS

We provide sample questions (**fig. 6**) to focus classroom discussions. Rather than merely asking students to identify perceptual changes in the graphs, we encourage students to address their own assumptions about their graphical representations and to consider other students' ways of representing the same situation.

Tasks involving a collection of relationships between covarying quantities in various graphical orientations have helped students distinguish between what is mathematically critical versus what is merely a convention for representation. The key to orchestrating a discussion around such tasks is to focus on the mathematical ideas underlying the different relationships and representations as well as to discuss

> Have you ever taken a dive to the bottom of a pool and felt pain or pressure in your ears? This is caused by an increase in water pressure as you descend.
>
> Although we do not feel it, 14.5 pounds per square inch (psi) of pressure pushes down on our bodies at sea level. For every 33 feet down into water an individual travels, he experiences an additional 14.5 psi (NOAA 2016). For example, at 33 feet down, the diver experiences 2 • 14.5 psi, and at 66 feet down, the driver experiences 3 • 14.5 psi.
>
> A deep-sea diver starts on a boat 5 feet above sea level, enters the water, and descends to an unknown depth. Assuming there is no discernible difference in pressure between sea level and 5 feet above sea level, create a graph representing the relationship between the diver's height above sea level and the pressure he feels.

Fig. 7 The Deep Sea Diver problem was given to students.

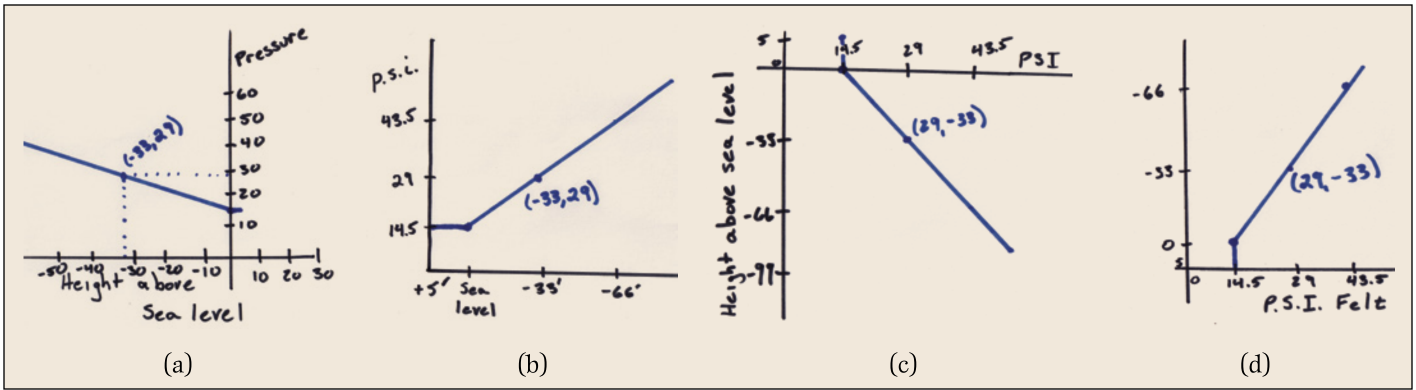


Fig. 8 Students produced these sample graphs for the Deep Sea Diver problem when they were permitted to choose their own axes and labeling.

what aspects of representational options are not critical. We present the Deep Sea Diver problem (**fig. 7**) as another example of such a task and describe some common student solutions (**fig. 8**).

In the Deep Sea Diver problem, we allow students to choose their own axes orientation and labeling, which often leads to four or more different, yet mathematically correct, ways to represent the relationship (see **fig. 8** for sample graphs). Some students represent the diver's distance above sea level on the horizontal axis (**fig. 8a, b**), whereas other students represent this quantity's values along the vertical axis (**fig. 8c, d**). Because the diver's height *above* sea level has a negative value throughout most of the situation, some students represent negative values in non-canonical orientations along the axes (**fig. 8b, d**), whereas other students maintain the canonical orientations of directed values (**fig. 8a, c**). Students might also change the point of intersection of the referent axes from the canonical (0, 0) (**fig. 8a, c**), since the diver starts by experiencing 14.5 psi 5 ft. above sea level (**fig. 8b, d**).

After students construct their graphs, we highlight the work of students who used axes with different orientations and labels. As our students consider their classmates' representations, we give them opportunities to discuss similarities and differences among their graphs; we provide hypothetical student solutions if additional variety is needed. We make sure to raise questions such as these:

- Are the different graphs representing the same change in height for a certain change in pressure? How can you tell?
- Does it matter in which direction the negative values are represented?
- Does it matter if the intersection of the horizontal and vertical axis is at (0, 0) or at (14.5, 5)?

We also prompt the students to discuss whether the graphs represent functions, leading to conversations regarding the necessity of defining input and output quantities.

The conversations that ensue allow students to discuss whether each graph accurately represents the intended relationship and if differences among the representations affect the "correctness" of each graph. These discussions afford students the opportunity to negotiate and establish representational conventions. After such conversations, students can conceive of conventions as useful tools for communicative purposes, often easing the burden of attending to different considerations involved when creating, interpreting, or comparing graphs.

CONSEQUENTIAL DISTINCTIONS

Providing our students with repeated opportunities to work in what we perceive to be conventional and unconventional representations—often

TEACHERS CAN HELP THEIR STUDENTS DEVELOP PRODUCTIVE WAYS OF THINKING TO DIFFERENTIATE CRITICAL ASPECTS FROM CURRICULAR CONVENTIONS.

through hypothetical student work that could be interpreted as creative sense-making—has supported their using notation and representations purposefully and meaningfully. Such situations encourage our students to establish and differentiate aspects specific to representational conventions from those aspects that are critical to mathematical concepts. Moreover, we have found that our students can then critically reflect on their own ways of thinking and establish representational conventions within their class. Teachers who determine that their students are not distinguishing between what is essential to a concept and what is to be considered a representational convention can give the students a chance to develop richer ways of thinking for these concepts by engaging them in a multitude of representations with attention to differences and similarities in those representations.

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