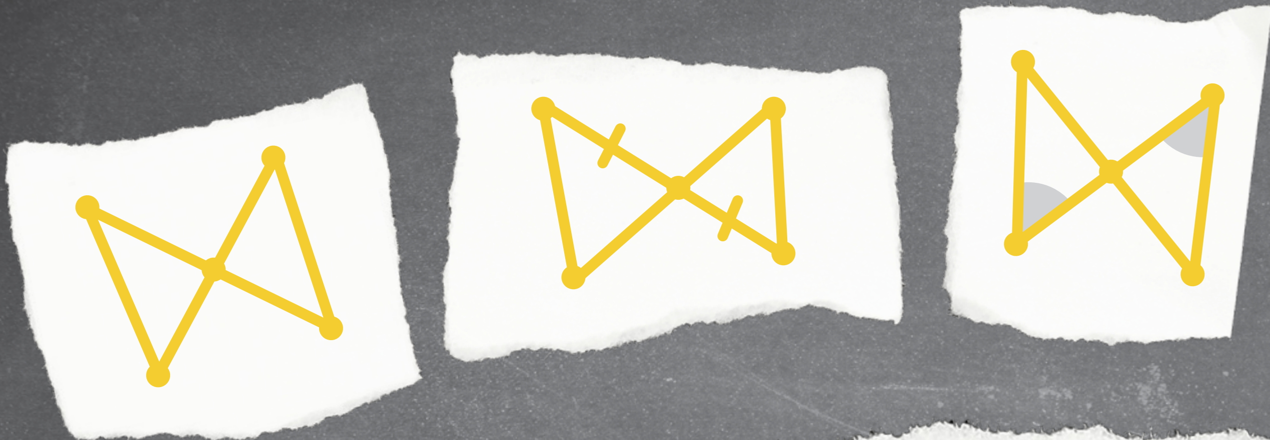

Proofs without Words

Pictures and diagrams help high school geometry students develop reasoning and proof-writing skills.

Since the 1970s, the Mathematical Association of America's (MAA) journals *Mathematics Magazine* and *College Mathematics Journal* have published "Proofs without Words" (PWWs) (Nelsen 1993). "PWWs are pictures or diagrams that help the reader see why a particular mathematical statement may be true and how one might begin to go about proving it true" (Nelsen 2000, p. i). This article explains how I created and implemented a variation of using diagrams without words to facilitate the development of high school geometry students' reasoning and proof skills.

The Common Core (CCSSI 2010) lists approximately twenty geometric theorems for students to prove (e.g., standards G.CO.9, G.CO.10, and G.CO.11) as well as the more general standard of using congruence and similarity criteria for triangles to prove relationships in geometric figures

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(i.e., standard G.SRT.5). Furthermore, Appendix A of the Common Core states that teachers should “encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words” (p. 29).

I used the following PWWs, adapted from Nelson (see **fig. 1**), when my Honors Geometry students studied right triangle trigonometry. Using the figure, students prove that $\sin(2\theta) = 2\cos\theta\sin\theta$ and $\cos(2\theta) = 2(\cos\theta)^2 - 1$, given that O is a semicircle with a radius of 1. (For further suggestions on how to use PWWs with students see Bell 2011).

Even though the Common Core’s Appendix A states that students should use diagrams without words, I was initially unsure of how I could use them for high school geometry theorems. As with **figure 1**, most PWWs that I was familiar with are a single diagram and seemed to fit the traditional norm of providing students with the diagram along with the “given” and the “prove” statements. After some thinking, I realized that a group of cards, each with the same base diagram but with different labels or markings on each card, could be arranged in a flowchart to “prove” a theorem. For example, consider how you would arrange the four cards in

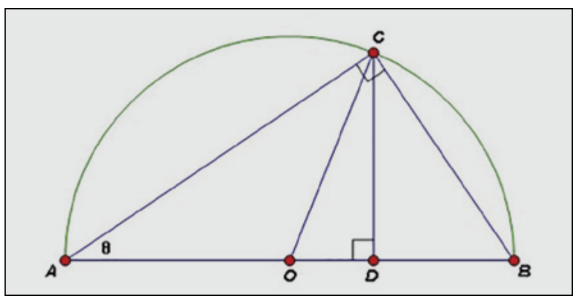


Fig. 1 This diagram can be used to prove two double-angle trig identities.

figure 2. Is there more than one way to arrange the cards? What theorems can you prove?

Four different ways to order the cards suggest meaningful reasoning. **Figure 3** shows that when two parallel lines are cut by a transversal, the alternate interior angles are congruent. The diagrams lead an explanation of why this is true: Lines *AC* and *BD* are parallel and cut by a transversal (**card *d***), so we have corresponding angles formed by parallel lines and $\angle 1 \cong \angle 2$ (**card *b***). Also, vertical angles are congruent, so $\angle 1 \cong \angle 3$ (**card *a***). Thus, by the transitive property, we can write $\angle 2 \cong \angle 3$ (**card *c***). On the other hand, **figure 4** shows that a different arrangement of the cards can illustrate the converse. That is, if two lines intersect a third line and form congruent alternate interior angles (**card *c***), then the two lines are parallel (**card *d***). In yet other arrangements, the cards can demonstrate the corresponding angles theorem and its converse. After my students have arranged the cards, the diagram guides them to write the accompanying statements and reasons in paragraph, two-column, or flowchart form.

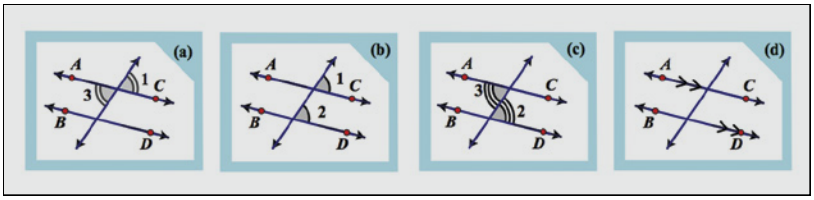


Fig. 2 A Proof without Words uses four cards.

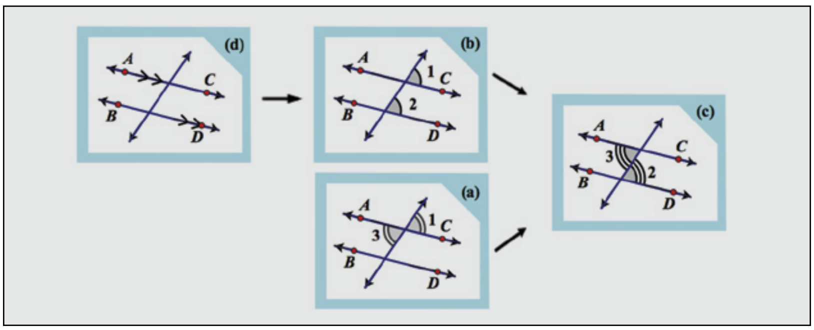


Fig. 3 This PWW demonstrates that if two lines are parallel, then the alternate interior angles are congruent.

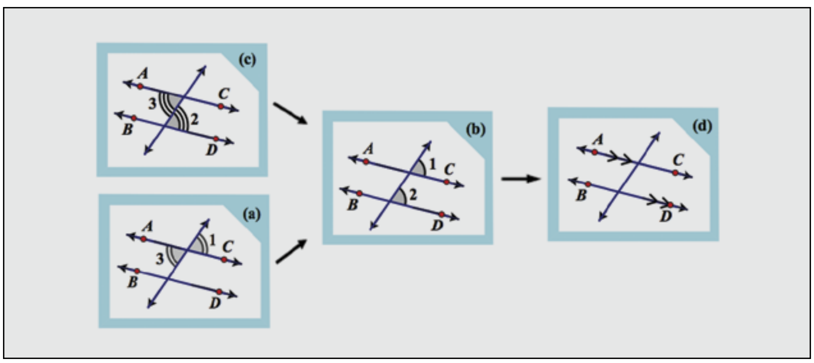
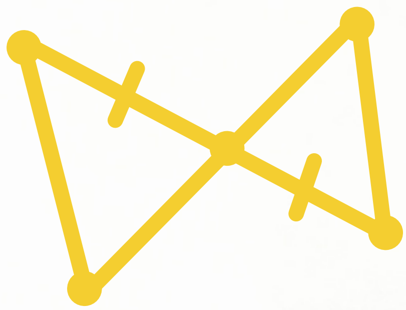


Fig. 4 This PWW demonstrates the converse of **figure 3**. That is, if two lines are cut by a third line and the alternate interior angles are congruent, then the two lines are parallel.

In my geometry class, I use PWWs as a bridge between informal reasoning and writing the complete proof. Preceding a five-part proof progression, students make many conjectures in small-group investigations both with and without dynamic geometry software. During the progression, students prove most of these conjectures (a few of them multiple times using different representations). In the first three stages of the progression, students reason informally with diagrams, put proofs in order when provided with the statements and reasons, and fill in missing details of given proofs. The fourth stage uses PWWs as scaffolding to help students write all the statements and reasons of a proof formally. In the final stage, students write complete proofs as well as critique and evaluate full proofs.

Depending upon the amount of scaffolding that students need and the complexity of the proof, I use PWWs in different ways within the fourth stage of the progression. Minimally, as with the activity associated with **figure 2**, I give students the set of cards and ask for arrangements that prove as many different results as possible. To add more support, I have sets of cards that also include a statement of what students are to prove. For example, **figure 5** shows the nine cards, plus the card with the theorem, already arranged in a flowchart proof. This PWW illustrates that if a quadrilateral has a pair of parallel and congruent sides, then it is a parallelogram. Note that **card *a*** and **card *g*** are the same. Duplicate cards emphasize that the parallel sides are used at two different points in the proof: first, as part of the given information (**card *a***), and then as a condition to apply the definition of a parallelogram (**card *g***).

For some theorems listed in the Common Core that are more complicated for students to prove, I provide a flowchart arrangement of the cards, and students need only to write the statements and reasons by analyzing the diagrams. I also use bold print and color in the diagram, to help students focus on important parts. **Figure 6** shows the PWW cards for the theorem "the medians in a triangle are concurrent." Before students begin working, I explain that **cards *a*** and ***c*** show medians $\overline{DB}$ and $\overline{AE}$ intersecting at *F*. Then, based on **card *b***, I explain that $\overline{CF}$ is extended to *H*, and that if we can conclude that intersection point *I* is the midpoint of $\overline{AB}$, then this would imply that $\overline{CI}$ is a median. Thus, all three medians in the triangle are concurrent at *F*. It would be nearly impossible for students (and most geometry teachers—including me) to come up with the proof themselves, but by using PWWs along with the short explanation I give



Working on PWWs is low-risk for students because they can continue to revise the arrangement of the pieces before they begin to write down the statements and reasons.

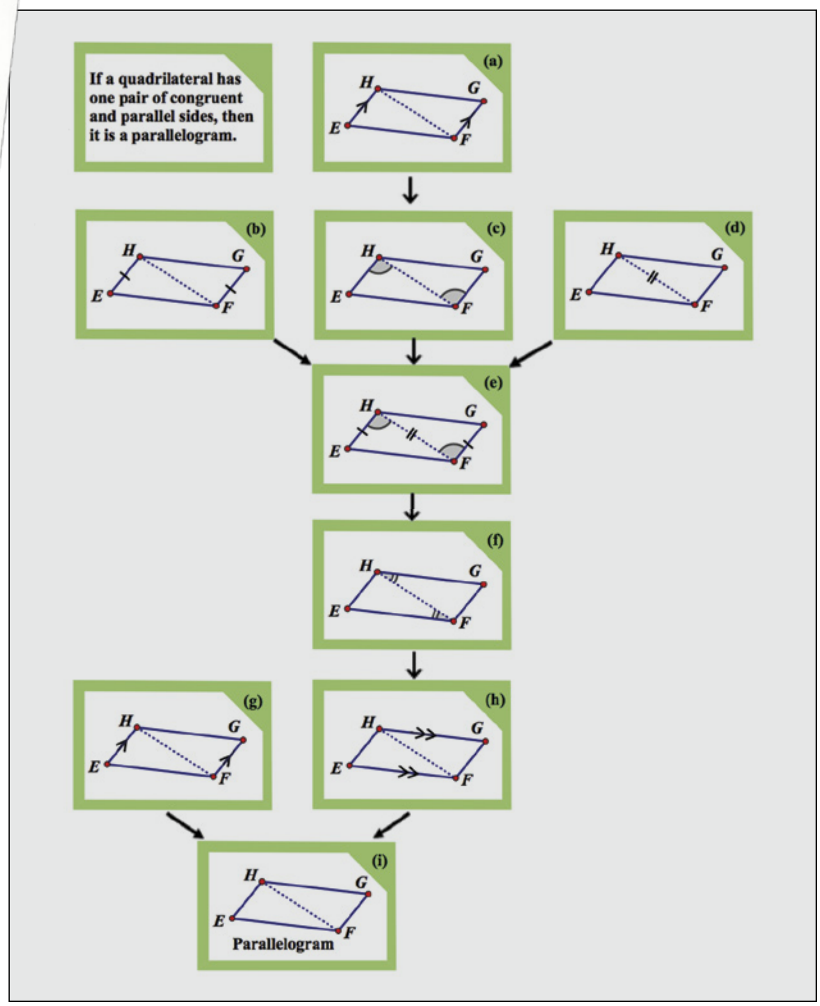


Fig. 5 Including a card with the theorem statement directs students in the proof.

before students begin, my students are able to write the statements and reasons needed to complete the proof. The proof in **figure 6** uses two previously proven theorems (both also found in the Common Core): namely, the midsegment in a triangle is parallel to the third side (**cards** ***f*** and ***g***), and the diagonals of a parallelogram bisect each other (**card** ***i***).

Finally, to increase the complexity of the task, I include more cards than necessary to form a single proof. **Figure 7** (p. 584) shows four of fifteen different cards that students can use to prove a geometric fact. (All fifteen cards are available as more4U content at **www.nctm.org/mt**.) Some cards in this set are extraneous: **Card** ***k***, for example, shows AAA similarity, which would not be needed for proofs of congruence.

I ask each student to glue cards onto paper in a flowchart that proves a result. **Figure 8** (p. 584)

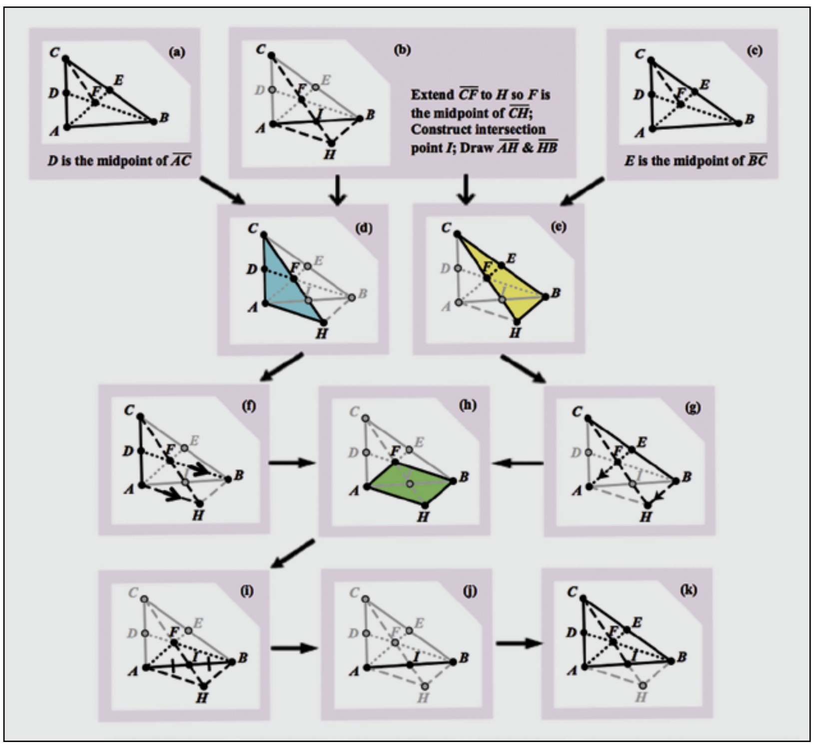
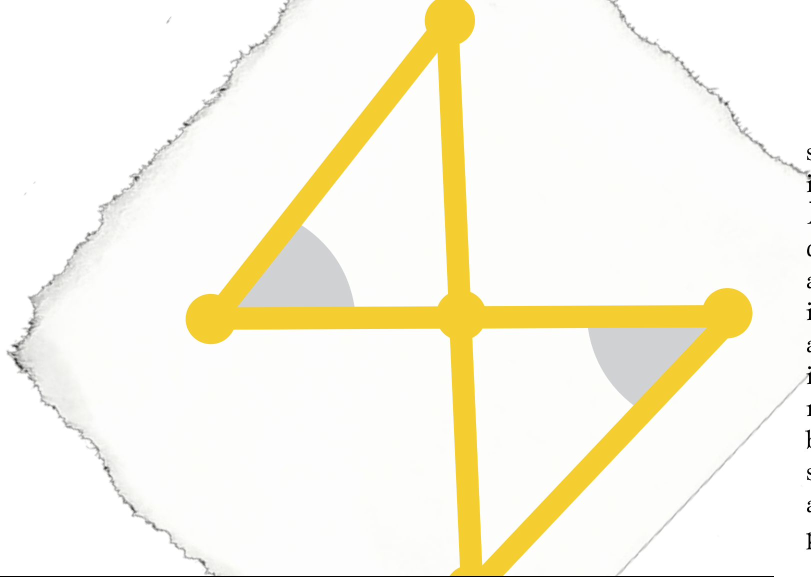


Fig. 6 This PWW demonstrates that the three medians of a triangle are concurrent.



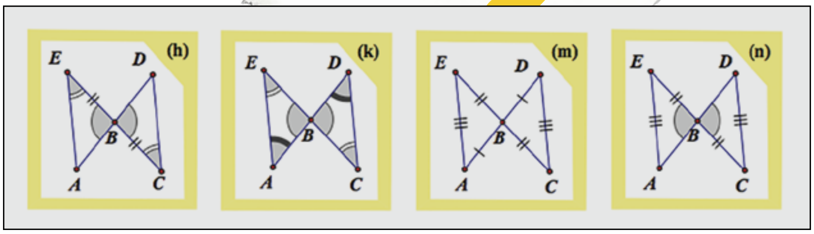


Fig. 7 A set of fifteen cards includes these four. Some, such as card *k*, are extraneous for proofs of congruence.

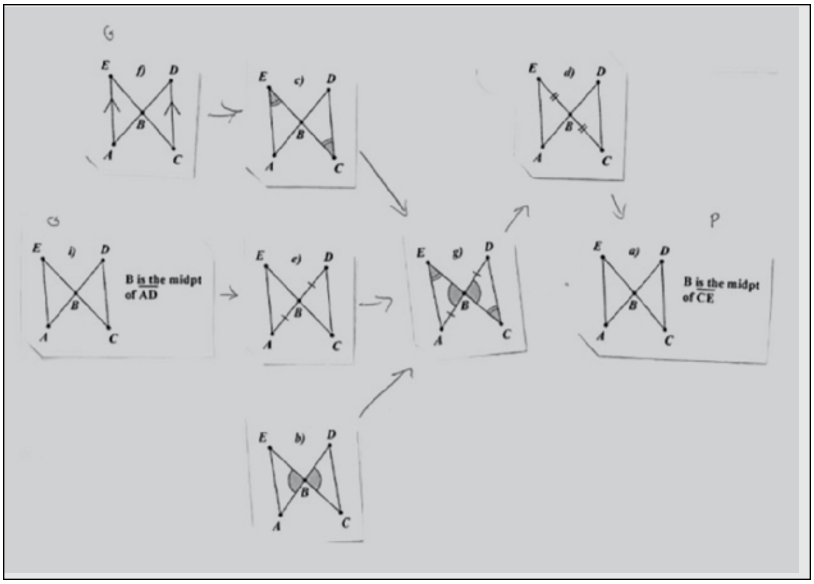


Fig. 8 A student uses cards to prove that *B* is the midpoint of segment *CE*.

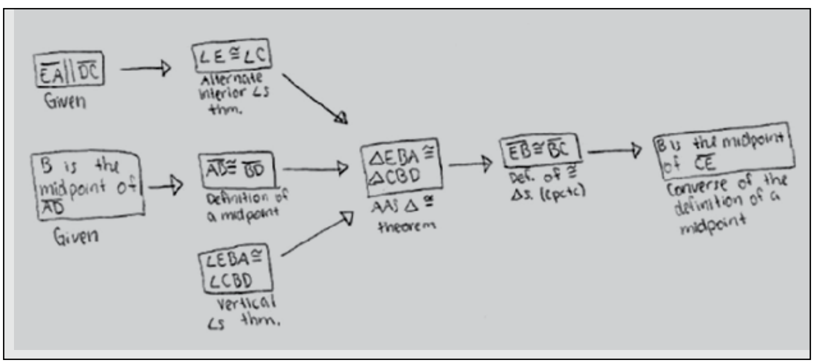


Fig. 9 A different student writes the statements and reasons for the classmate's work shown in **figure 8**.

shows how a student used eight cards to prove that if $\overline{EA}$ is parallel to $\overline{DC}$ and B is the midpoint of $\overline{AD}$, then B is also the midpoint of $\overline{CE}$. Next, students exchange papers, and they write statements and reasons corresponding to the flowchart. I also instruct them to make corrections if they think anything is wrong with the flowchart they received in the exchange. **Figure 9** shows a student's statements and reasons for the PWW (**fig. 8**) created by a classmate. As an alternative activity, I give students multiple copies of the same PWW cards and ask them to prove as many different results as possible each time using a new set of cards.

In the past, many of my students struggled to understand how the proofs for a theorem and its converse were different. However, students can use the same set of cards to create paired PWWs when I introduce a two-sided card, printed on a different sheet of colored paper to attract their attention. For example, in **figures 10** and **11**, different sides of **card *f*** are used to prove the two parts of this biconditional statement: A quadrilateral is a parallelogram if and only if the diagonals bisect each other. The forward implication (**fig. 10**) is read from the top down, whereas the converse is read from the bottom up (**fig. 11**). Note three differences between the two flowcharts: **Card *f*** has been turned over, **card *e*** is moved, and the direction of the arrows is reversed. Also, note that in the forward direction, **card *c*** uses a previously proven theorem (the opposite sides of a parallelogram are congruent).

Proving this biconditional has benefited students who are unsure whether the congruent alternate interior angles or the parallel lines comes first in the proof. Some students have incorrectly attempted to make a generalization that one must always follow the other no matter what statement they are proving.

When I am implementing a set of cards that prove a biconditional, I do not tell the students what to prove, but instead I ask them to use the cards to prove a result. When they have something ready in flowchart form, I check it, offering feedback if necessary. When it is correct, students write the proof on paper, and then I check it. Once that flowchart and proof are correct, I ask students to try to reverse the proof. That is, they start with the conclusion they just proved as given information, and they try to prove that the originally given assumptions hold true. Because they created both proofs, they are able to see the two directions more clearly. Afterward, as part of a class discussion, I use a large bulletin board to display full-page versions of their PWWs. Student volunteers come up one at a time to begin arranging the cards. Once we have one direction of the proof on the bulletin board, we discuss reversing the proof. Student ownership of this PWW, and others like it that I have created, has helped students understand the differences between the proofs of a theorem and its converse.

The PWW approach to proof supports Sinclair, Pimm, and Skelin's (2012) notion that working with diagrams is central to geometric thinking. According to Sinclair and colleagues, diagrams are artifacts that have a story to tell, but we must interpret the diagrams to learn the story. They suggest that although diagrams sit in silence, we need to improve upon hearing what they are telling us. PWWs put diagrams in the foreground instead of relegating them to the background where "marking the diagram" is only a part of the ritualized norm of "doing proof" (Herbst et al. 2009). PWWs emphasize that interpreting diagrams and reasoning are at the heart of geometric proof. The cards are a collection of individual still frames that students can organize to tell a coherent story of why a particular geometric invariance must be true.

Based upon the results from classwork, homework, and quizzes, my students have been very successful with proof when using PWWs. I believe this success is due to three reasons. First, arranging a PWW to prove a result is akin to solving a puzzle, and most people (including my students) enjoy working on puzzles and recreational mathematics with physical models. Second, focusing on diagrams initially frees students from getting bogged down with the formality of writing statements and reasons, even though they will have much of the same discussion in their groups as they complete a PWW. Third, working on PWWs is low-risk for students because, based upon discussion they have in groups and feedback they get from me, they can continue to revise the arrangement of the pieces of the PWWs before they begin the final process of writing down the statements and reasons. In summary, PWWs, such as the five examples described in this article, have made proof more accessible to more of my students while supporting them in developing their reasoning skills and in seeing the structure of how proofs are "assembled."

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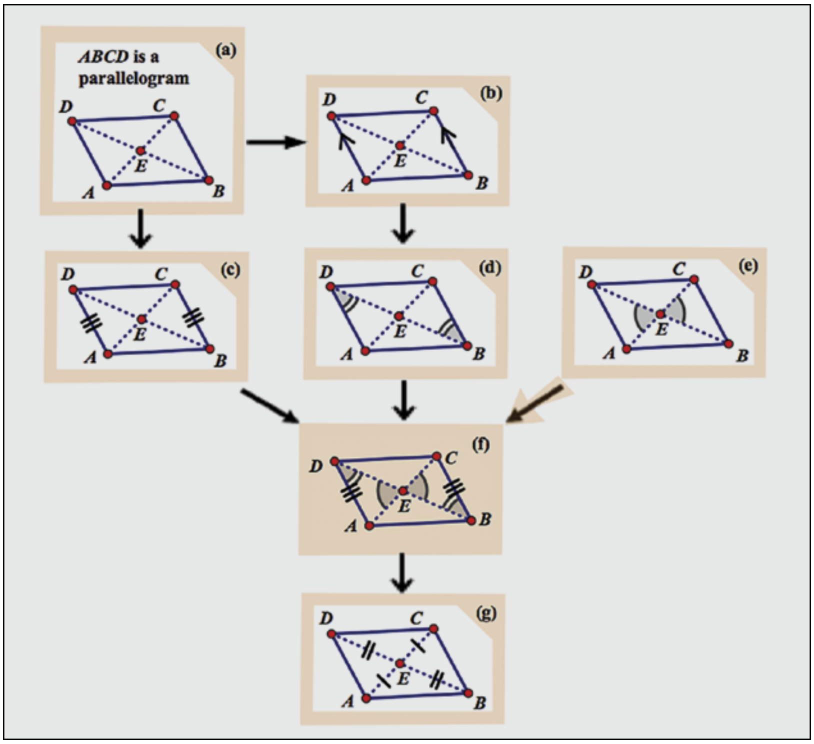

Fig. 10 This PWW shows that if a quadrilateral is a parallelogram, then the diagonals bisect each other.

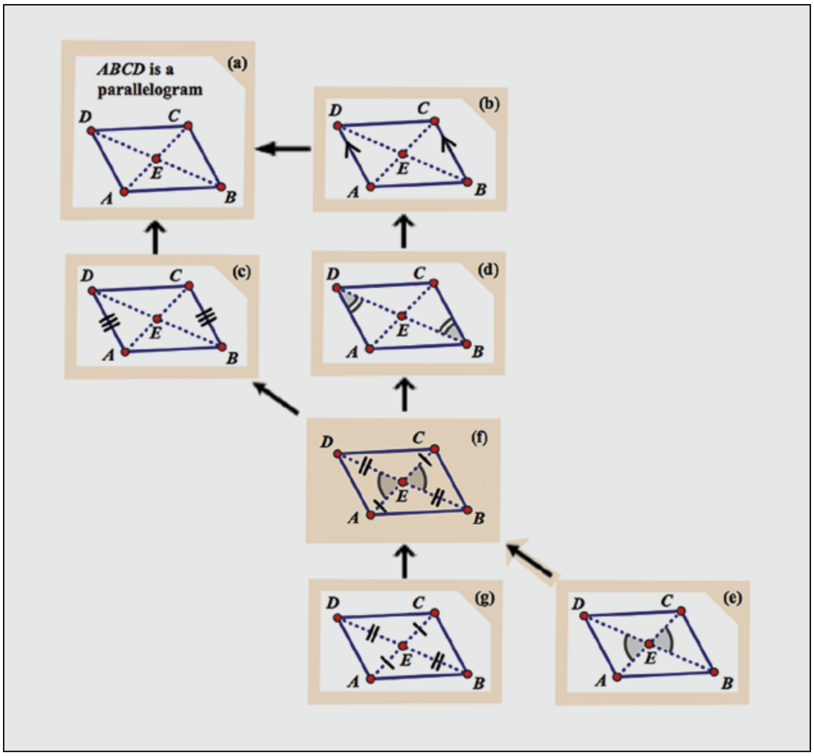

Fig. 11 This PWW uses the same cards as in **figure 10** to show that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

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more4U

A downloadable set of cards (.pdf and .gsp) to use for Proof without Words activities is available online at **http://www.nctm.org/mt**. This more4U content, an additional benefit, is for members only.