MATHEMATICAL lens

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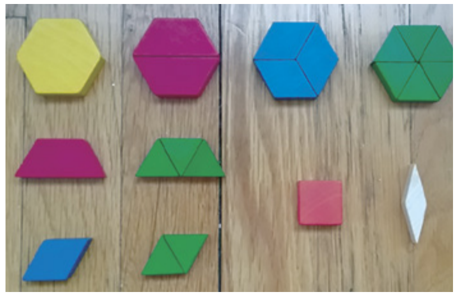
Semiregular Tessellations with Pattern Blocks

Pattern blocks are multifunctional instructional tools with a variety of applications in various strands of mathematics (number sense, geometry, measurement, algebra, probability). The six pattern blocks are an equilateral triangle (green), a blue rhombus, an isosceles trapezoid (red), a regular hexagon (yellow), a square (orange), and a white rhombus. The sides of all pattern blocks are congruent, considered to be 1 unit in length for this article. **Photograph 1** depicts a wall painting with squares and rhombuses found in Jersey City, New Jersey.

Photograph 1

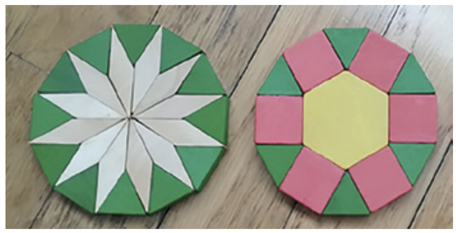


Photograph 2

1. (a) Is it possible to tessellate the plane using only squares and the blue rhombuses of the pattern blocks in a manner similar to that in the mural in **photograph 1**? Show the tessellation and identify the tessellation unit (generator).

   (b) A semiregular tessellation (also called an *Archimedean tessellation*) covers the plane without any gap by using two or more regular polygons in such a way that the same polygon order (clockwise or counterclockwise) is followed at each polygon vertex. Explain how to convert your answer to question 1(a) into a semiregular tessellation.

2. A blue rhombus is equal in area to two green triangles. A red trapezoid is equal in area to three green triangles. A yellow hexagon is equal in area to two red trapezoids, three blue rhombuses, and six green triangles. The areas of the orange square and the white rhombus are not expressible as integral multiples of the area of the green triangle (see **photo 2**). Determine the area of each pattern block in square units.



Photograph 3

3. Use the fact that the two congruent regular dodecagons (of side length 1) made of pattern blocks (see **photo 3**) have the same area to demonstrate that the white rhombus has an area of 0.5 square units. Find several other ways to demonstrate this fact.

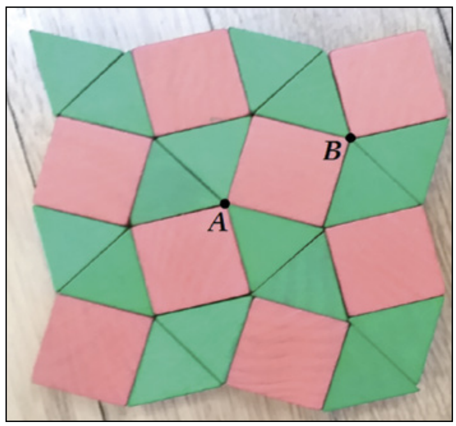


Fig. 1

4. A sequential notation based on the number of regular polygon types surrounding any randomly chosen vertex is used to name a given Archimedean tessellation. The Archimedean tessellation obtained in question 1(b), for instance, could be named 3.3.4.3.4. As depicted in **figure 1**, any randomly chosen polygon vertex (e.g., vertex A or B in the figure) demonstrates

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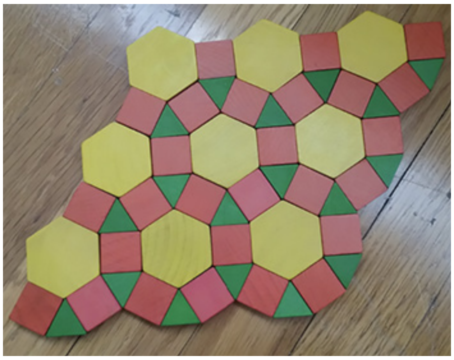


Photograph 4

Photograph 5

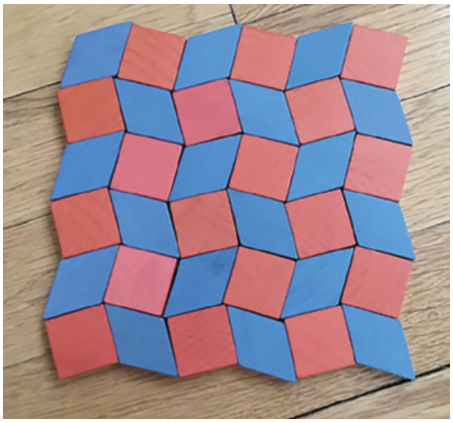
the 3.3.4.3.4 formation in that order where 3 and 4 represent, respectively, the number of sides of the equilateral triangle and the square. Determine all possible semiregular tessellations that can be generated via pattern blocks. In each case, specify the generator (tessellation unit) of the semiregular tessellation and the name of each tessellation.

5. In an Archimedean tessellation as shown in **photograph 4**, calculate the probability that a random point on the tessellated plane emerges in a (i) yellow; (ii) orange; (iii) green region.

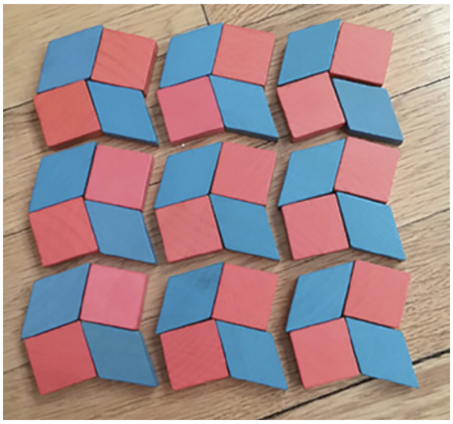
6. If the pattern shown in **photograph 5** continues, will it ever form a closed figure? If so, determine the inner polygon that forms.

7. Calculate the area of either regular dodecagon shown in **photograph 3** in two different ways: (i) by adding the areas of the pattern blocks within the dodecagon; and (ii) by using the formula for the area of a regular polygon.

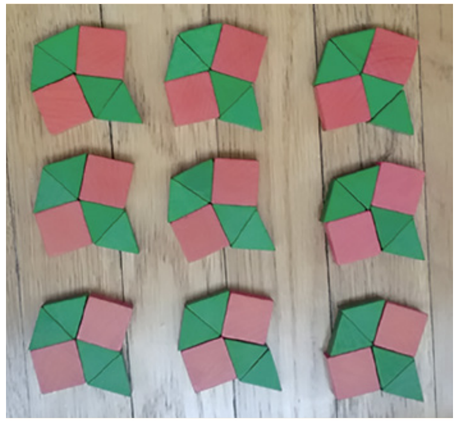
8. Using the fewest number of pattern blocks, construct a regular dodecagon with a side length of 1 unit.



Photograph 6a



Photograph 6b



Photograph 6c

1. (a) Such a tessellation is possible with pattern blocks (see **photo 6a**). The tessellation unit (generator) in this case consists of two squares and two blue rhombuses (see **photo 6b**).

   (b) Because a blue rhombus is made of two equilateral triangles of pattern blocks, a conversion to a semiregular tessellation is possible (see **photo 6c**).

2. In this article, the area of the orange square is defined to be 1 square unit. We establish the area of the green triangle (which is an equilateral triangle of side length $s = 1$) by using the area formula for an equilateral triangle:

$$Area = \frac{s^2\sqrt{3}}{4} = \frac{\sqrt{3}}{4} \text{ square units}$$

The area of the blue rhombus is therefore twice the area of one green triangle, or namely,

$$2 \bullet \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \text{ square units.}$$

The area of the red trapezoid and the yellow hexagon are, respectively, three and six times the area of one green triangle, or

$$\frac{3\sqrt{3}}{4} \text{ and } \frac{3\sqrt{3}}{2} \text{ square units.}$$

Because the white rhombus can be decomposed into two congruent isosceles triangles with leg length 1 and vertex angle 30°, as shown in **figure 2**, we can use the formula for the area of a triangle given SAS: $Area = (1/2)ab\sin C = (1/2)(1)(1)\sin 30° = 0.25$ square units and then double that result.

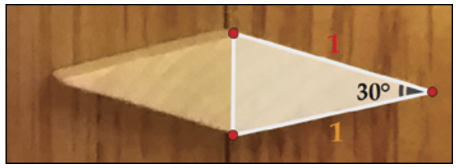


Fig. 2 The formula for the area of a triangle given two sides and the included angle can be used to find the area of the rhombus.

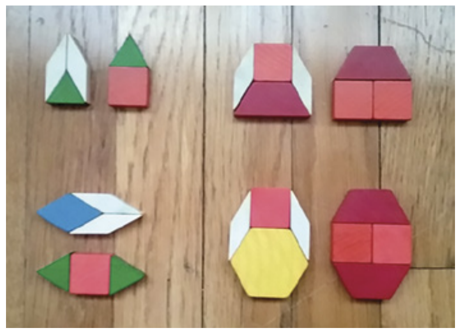
3. For the figure on the left in the photograph: Area of 12 green triangles + Area of 12 white rhombuses = Area of 12 green triangles + Area of 6 orange squares. Subtracting from both sides, we have Area of 12 white rhombuses = Area of 6 orange squares = 6 square units. Therefore, the area of one white rhombus is equal to 0.5 square units.

   In this problem, the idea is to use pairs of congruent polygons to show that the white rhombus has an area of 0.5 square units. Many other congruent polygon pairs can be used to demonstrate this fact. Let x denote the area of one such white rhombus.

   For the first pair in **photograph 7**, for instance, let g denote the area of one green triangle. Because the two congruent pentagons have the same area, we can write $2x + g = g + 1$. From this we see that $x = 0.5$.

   For the hexagons in the upper-right corner of **photograph 7**, if r denotes the area of one red trapezoid, then $2x + r + 1 = 2 + r$, and again, $x = 0.5$.

   For the hexagons on the bottom-left, the corresponding equation would be



Photograph 7

$2x + b = 2g + 1$. Since $b = 2g$, we obtain our familiar result.

   Finally, the equation for the pairs in the lower-right corner of **photograph 7** is $2x + 1 + y = 2r + 2$; since $y = 2r$, we are left with $2x + 1 = 2$ and our familiar answer.

4. Of eight possible semiregular tessellations, it is possible to make five semiregular tessellations by using the pattern blocks. **Table 1** outlines these five semiregular tessellations along with their generators (tessellation units) and the names.

Photograph 8

5. We need to focus on the generator, consisting of one yellow hexagon, two green triangles, and three orange squares (see **photo 8**). The area of this generator is numerically equal to the area of three squares plus the area of eight equilateral triangles, or

$$3+8\bullet\frac{\sqrt{3}}{4}=3+2\sqrt{3}\text{ square units.}$$

Here are the desired color probabilities:

$$\text{Yellow: }\frac{\frac{6\sqrt{3}}{4}}{3+2\sqrt{3}}\approx 0.402$$

$$\text{Green: }\frac{\frac{2\sqrt{3}}{4}}{3+2\sqrt{3}}\approx 0.134$$

$$\text{Orange: }\frac{3}{3+2\sqrt{3}}\approx 0.464$$

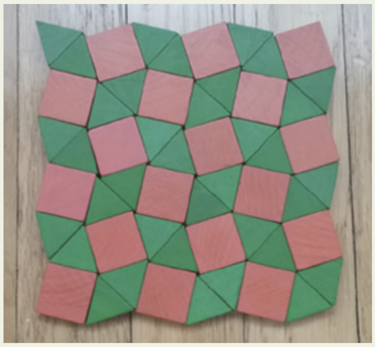
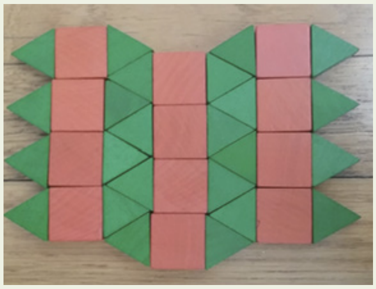
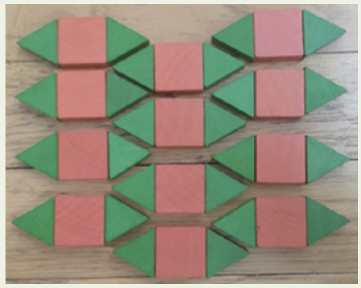
Note that if we add the numerators of the three fractions, we obtain

$$\frac{3\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+3=2\sqrt{3}+3,$$

so the sum of the three fractions is 1. **Table 2** (see the **more4U** at **http://www.nctm.org/mt**) shows the color probabilities—on the basis of generator representations—for the complete set of semiregular tessellations made of pattern blocks.

6. The inner polygon that forms is a regular dodecagon (see **photo 9**). One of the congruent interior angles is equal to 150°. For n-sided regular polygons, the sum of the interior angles is $(n-2)\bullet 180°$. We set that expression equal to 150° and solve, obtaining $n = 12$.

Table 1 Archimedean Tessellations Made with Pattern Blocks

Semiregular Tessellation	Generator Representation	Name
		3.3.4.3.4
		3.3.3.4.4
		3.6.3.6
		3.3.3.3.6
		3.4.6.4

7. (i) The regular dodecagon in **photo 3** (on the right) is made of thirteen pattern blocks (six squares, six equilateral triangles, one regular hexagon), which is equivalent in area to the sum of the areas of six squares and twelve equilateral triangles or $6+3\sqrt{3}$ square units.

Photograph 9

Photograph 10

(ii) We let $a = 1$ and $n = 12$ in the formula:

$$Area = \frac{na^2}{4\tan\left(\frac{\pi}{12}\right)}$$
$$= \frac{12}{4\tan\left(\frac{\pi}{12}\right)} = \frac{3}{\tan\left(\frac{\pi}{12}\right)}$$

To evaluate the tangent function exactly, we use one of the half-angle formulas, say,

$$\tan\left(\frac{A}{2}\right) = \frac{1-\cos A}{\sin A},$$

and letting $A = \pi/6$, we get

$$\tan\left(\frac{A}{2}\right) = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{2-\sqrt{3}}{1}.$$

Returning to the formula for the area, we have the following:

$$Area = \frac{3}{\tan\left(\frac{\pi}{12}\right)} = \frac{3}{2-\sqrt{3}} \bullet \frac{2+\sqrt{3}}{2+\sqrt{3}}$$
$$= \frac{6+3\sqrt{3}}{4-3} = 6+3\sqrt{3}$$

This is the same area as in (i).

8. Using two hexagons, three squares, and six white rhombuses seem to generate a regular dodecagon with the smallest number (11) of pattern blocks. We offer some other possibilities (see **photo 10**) and encourage students to generate other regular unit dodecagons with pattern blocks.

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