Using the 5 Practices in Mathematics Teaching

Selecting and sequencing student work with cognitively demanding tasks in a group environment can teach important mathematical ideas.

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What might effective mathematics instruction look like if we were to see it? Engaging students with "challenging tasks that involve active meaning making and support meaningful learning" (NCTM 2014, p. 9) is one possible description. Staples in our classrooms include problem solving with cognitively demanding tasks, working in teams to formulate and solve problems, communicating mathematically through written and spoken channels, and critiquing or assessing the work of others. This article highlights three of the eight Mathematics Teaching Practices (MTP) published in NCTM's *Principles to Actions: Ensuring Mathematical Success for All* (2014, p. 10): facilitating meaningful mathematical discourse (MTP 4), posing purposeful questions (MTP 5), and eliciting and using evidence of student thinking (MTP 8).

In this article, we have several objectives. We open with a brief discussion of the meaning of the term *active learning*, and we discuss the five practices (Smith and Stein 2011) as a particularly illuminating model. The five practices offer a powerful framework that we have used to activate our mathematics classrooms. Next, we share two vignettes of classroom learning from a first-year calculus class at a small university. Note that such teaching and learning experiences span all levels, K–16. The article's focus is on the implementation and management of active instructional practices, irrespective of mathematical content. Thus, we encourage the curious reader to join us in this experience (even if you do not teach calculus). The article closes with examples of student feedback from having experienced active learning in their college mathematics class.

ACTIVE LEARNING AND THE FIVE PRACTICES

Although many definitions of "active learning" exist, most describe the same core qualities, regardless of the discipline or environment in which they are used. As early as 1991, active learning was described as "involving students in doing things and thinking about what they are doing" (Bonwell and Eison 1991, p. 5). In a calculus class recently taught by the lead author,

Table 1. Descriptions of the Five Practices

Practice	Description	Question(s) a Teacher May Ask Himself or Herself	Miscellaneous
0: Identifying the Goal or Objective	Identify the specific goals of the lesson before class.	1. What do I want students to know and learn? 2. How should they know it?	Find and develop rich mathematical tasks where students may easily gain entry but from which interesting and relevant mathematics is likely to emerge.
1: Anticipating	Teacher predicts how students will solve the problem.	1. What will students do? 2. How will they do it? 3. What misconceptions are likely?	The teacher should solve the problem using a variety of strategies. Doing this allows the teacher to interpret a solution that was not anticipated more easily.
2: Monitoring	Teacher identifies the strategies used by visiting with groups, and answering and asking questions. Teacher begins documenting who is doing what.	1. What are students doing? 2. What strategies are being used?	If a group has misconstrued the problem, the teacher may wish to steer those students back on course.
3: Selecting	Teacher determines which groups should share their work.	1. Why should this group's work be showcased? 2. Why might other work not be shared?	This selection is driven by the goals and objective of the lesson (Practice 0).
4: Sequencing	Teacher determines a specific sequence that makes pedagogical sense. Those selected will present and discuss their work in this predetermined order.	1. What presentation order makes sense? a. Informal to formal? b. Simple to sophisticated? c. Common to unusual? 2. Should misconceptions be addressed immediately or later?	The sequence should allow students to see connections from one group's solution to the next and offer opportunities for evaluating and critiquing work.
5: Connecting	Teacher directly makes connections in the approaches discussed or indirectly makes connections through questioning/focusing.	1. What is the story I want to tell with student work? 2. Are there other ideas that should be discussed—ideas that did not appear in students' efforts?	Student work is used to meet the goal of the lesson. A student or group may ask about a method not publicly shared and the teacher may have further opportunities to connect.

active learning strategies were implemented in which students were problem solving, discussing, and explaining their results to their classmates. A typical fifty-minute class ran as follows:

1. Preclass Phase

Students arrived to class "prepared" through a short preclass reading. The purpose of this reading was to cover fundamental principles, notation, and other ideas so that students could immediately engage with the content and with one another.

2. Problem Solving and Group Discussion

Once in class, students undertook a cognitively demanding task. Students worked in pairs or occasionally in groups of three, sharing ideas and talking about the problem. Eventually, they provided a solution on a mini whiteboard. Whiteboards were chosen as a display tool because they symbolize a common sharing space for the group's efforts and they proved suitable to assimilate group thinking.

3. Whole-Class Discussion

Students shared their solutions with the class. Some group members explained their work at the front of the room; others explained it from their seats (a classmate from the group elevated the whiteboard for peers to see). Still others preferred the projection of the whiteboard contents on the document camera for easy whole-class viewing. These "sharing sessions" were the primary vehicle used to teach the day's content and meet the goals of the lesson.

A fifty-minute block had enough time to orchestrate two or three tasks (and their solutions), depending on the nature and content of the tasks. The details of how class was conducted are described in *5 Practices for Orchestrating Mathematics Discussions* (Smith and Stein 2011), which succinctly captures a way to bring social interaction and active learning to mathematics classrooms. Although the book focuses specifically on K–12 levels, we feel that with suitable tasks, the practices can be a successful instrument at the college and university levels. The five practices are the following: (1) Anticipating, (2) Monitoring, (3) Selecting, (4) Sequencing, and (5) Connecting.

Smith and Stein contend that Planning/Goal Setting could be called "Practice 0," as this is something teachers need to do before orchestrating a productive discussion. **Table 1** summarizes the five (or six) practices and describes salient characteristics of the implementation of each.

The table indicates various stages of teaching implementation. Practices 0–1 happen before the class meets, whereas practices 2–5 indicate active learning. In particular, practice 5 is the only part of the framework that—to an outsider—seems like "teaching." Students are the key players in the learning process (practices 2–4), and then once again when solutions are displayed and discussed (practice 5).

Before sharing classroom vignettes, some notes are in order. First, an essential piece of implementing the five practices is using high-level, cognitively demanding tasks. Groups are unlikely to "discuss" the mathematics if a task is too straightforward. The task should have a low entry point for engagement, a high bar for success, and be amenable to different approaches—all while meeting the specific goals of the lesson. This is a nontrivial cocktail of characteristics. We offer some resources at the end of this article to let readers know where to find mathematically rich tasks and how to repurpose "cookbook" tasks into more meaningful experiences (practice 0). The richer the task, the more scenarios the teacher will likely need to anticipate (practice 1).

Second, pedagogy of this sort embraces *using student work to teach mathematical content*. Practices 3–5 are manifestations of student work, so trusting what students are capable of producing is an absolute necessity. This phenomenon is not new to mathematics instruction (e.g., Rasmussen and Marrongelle 2006), but it is far from the norm in tertiary education. Third, it is important to emphasize a classroom culture that values mistakes and learning from them. This pedagogy allows one to display common errors, build knowledge from these errors, and then connect this knowledge to valid mathematics. Finally, the five practices are not the same as a "show and tell" exhibit of student work (Smith and Stein 2011). The Selection stage is carefully aligned to the goals of the lesson, and Sequencing is purposely done to make explicit the Connection phase for students. Thus, "more" is rarely synonymous with "better." The quantity and quality of the solutions should facilitate a productive discussion, which takes practice and skill on the teacher's part.

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CLASSROOM VIGNETTES USING THE FIVE PRACTICES

In this section, we share classroom vignettes from two different units of first-semester calculus (one on limits and one on applications of the derivative). Our purpose is to highlight the similarities and differences of using the five practices with different types of problems and different types of student responses. As we emphasized earlier, teacher moves are situational and vary depending on the goals designated for the day's lesson. We zoom in on the practices of Selecting, Sequencing, and Connecting because these practices examine what work the teacher chose, why he or she chose it, and how this directed a productive mathematical discussion.

Vignette 1: Limits

The task given to groups was as follows:

> TRUE or FALSE: If $f(x) < g(x)$ for all $x \neq a$, then
> $\lim\limits_{x\to a} f(x) < \lim\limits_{x\to a} g(x)$.
> Justify!!

The goal of the task was for students to internalize that "operating" on a true statement with a limit may alter its truth value. A secondary goal supports the well-known fact that $f(a)$, if it exists, has no bearing on

$$\lim_{x\to a} f(x),$$

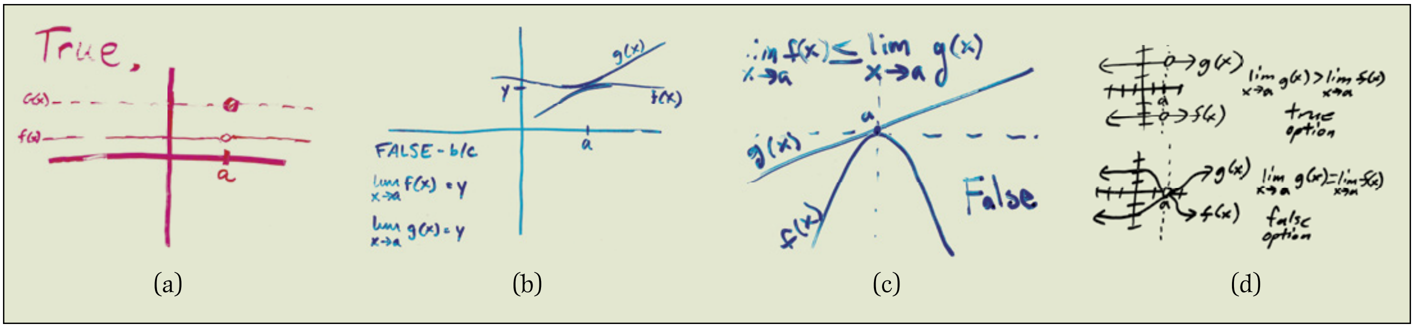

Fig. 1 Group solutions were presented to the class in the order a, b, c, and d.

should the latter exist. This task is considered fairly complex, as it wraps these ideas into one simple true/false statement, and students are asked to defend their position. The whiteboards that were selected and sequenced (practices 3 and 4) are seen in **figure 1**.

Board 1 (see **fig. 1a**) was chosen to start the discussion for two reasons. First, two groups had precisely this response and thought that if $g(x)$ was "higher" than $f(x)$, then this relationship should remain true for the limit. Second, showcasing this board allowed the teacher to ask a pointed question, such as, "Are other pictures possible?" Students were quick to suggest making f and g closer to each other, which was a perfect segue into board 2 (see **fig. 1b**). This board was shared next because this group had grasped the basic principles but were not confident about their answer. An observer can see that the group engaged in much discussion of the values of f and g near $x = a$ (note the heavier lines there). Even though they had a valid response and two limit statements as added support, the group members were uncertain what was going on at $x = a$. They graciously confirmed this uncertainty with the class.

Board 3 (see **fig. 1c**) came next, as this group claimed "false" by considering the original statement in the problem,

$$\text{“}\lim_{x \to a} f(x) < \lim_{x \to a} g(x),\text{”}$$

and modifying it to read

$$\text{“}\lim_{x \to a} f(x) \le \lim_{x \to a} g(x).\text{”}$$

This not only justifies the falsity of the statement but also is the first of the three boards to use mathematically precise language aimed specifically at the original statement. Finally, board 4 (see **fig. 1d**) was shared; it encapsulates much of the work displayed on boards 1 and 2, refines board 3, and opens the door to an important mathematical discussion (practice 5: Connecting). One sees immediately the *two* answers this group provided—one analogous to the incorrect work on board 1 and one that challenges board 3 in that neither function need be defined at $x = a$. Group 4's *incorrect* work further supports the need to address the misconception in board 1, and their *correct* work illuminates the second goal of the lesson—that a function's value need not connect to its limit (should either exist).

The discussion that followed concerned the nature of mathematical truth: What does it mean for a statement to be true? Although students in group 4 thought both options were plausible, the members of group 1 explained to the class that it would take just one example to establish falsity. Group 1 members admitted they overlooked this situation before choosing to speak about it. Thus, although group 1 opened the Connecting stage with incorrect work, these students were, in fact, the ones driving the discussion of mathematical work on board 4. An unforeseen byproduct of this dialogue was a pointed discussion of two fundamental ideas that permeate all mathematical work—(1) needing proof to establish truth, and (2) generating a counterexample to establish falsity. Although this discussion was not one the teacher had anticipated, such welcome additions proved helpful in future discussions.

Vignette 2: Derivatives and Velocity

The task here read as follows:

> A machine is causing a particle to move along the x-axis so that its position at time t is given by $x(t) = (t - 4)^2$, where t is in seconds.
>
> (a) What is the particle's velocity at $t = 2$?
>
> (b) The machine stops suddenly at $t = 3$, releasing the particle. As the particle continues, where will it be 5 seconds after the machine stops? Explain your thinking.

The goal of this task was to allow students opportunities to apply the principles of position, velocity, and acceleration to solve problems involving change. Additionally, it was hoped that students would (1) be drawn explicitly to velocity as having both magnitude and direction and (2) negotiate and

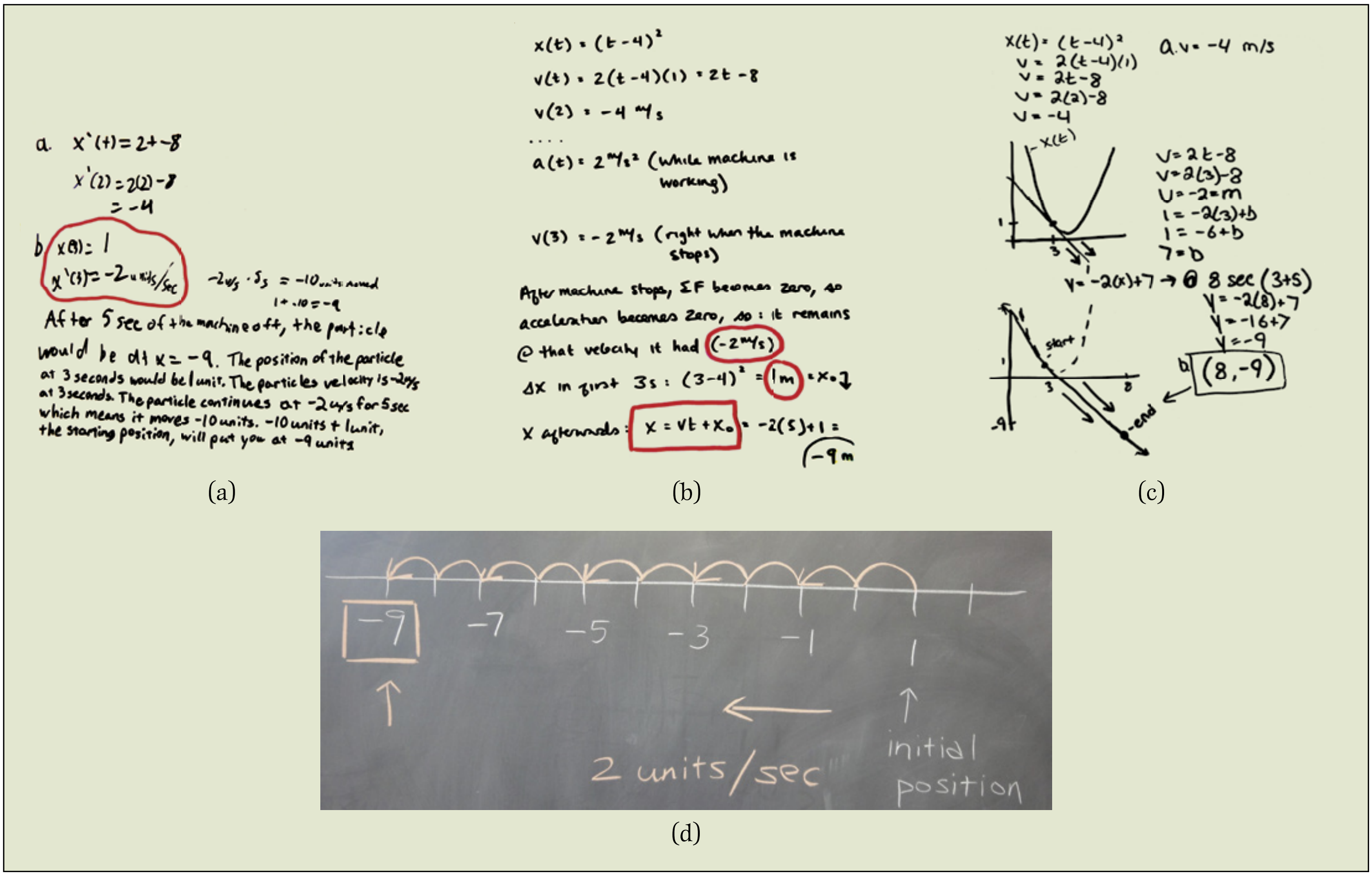

Fig. 2 The teacher provided a solution (d) on the basis of a student's suggestion.

confirm specific assumptions (e.g., a frictionless environment) prior to solving the problem. Work was shared with the class (see **fig. 2**) after adequate time was given to produce a solution.

The rationale for the selection and sequence (practices 3 and 4) was as follows. Board 1 (see **fig. 2a**) started the discussion since this was both the most common solution and the highest priority with regard to anticipation. When a group member explained the group's work—specifically that "the particle continues at –2 units/sec for 5 sec," the student was clear in articulating what this meant: The particle was moving to the left, and no external forces were acting on the particle. The teacher had anticipated an explanation because no formal prerequisite knowledge of physics (friction) was assumed. Board 2 **(fig. 2b)** followed due to its sophisticated mathematical nature. Although its contents are equivalent to board 1, it is more formal in notation

$$(\text{e.g., } \sum F \text{ and } \Delta x),$$

and it makes explicit the multiplicative relationship once the initial position is established (i.e., $x = x_0 + vt$). For those who may not have understood where this formula came from, this knowledge was built from board 1, in which the same phenomenon was explained in simple, arithmetic terms. The attention to sequencing (practice 4) was deliberate—mainly to highlight a hierarchy in student thinking and conventions in mathematics notation.

Board 3—an unusual gem—was selected next (see **fig. 2c**). Admittedly, the teacher initially questioned the validity of its contents, and not until a minidiscussion with the group members was he convinced of its correctness. The group interpreted velocity as the slope of the tangent line and used its fixed slope as an indicator of the particle being in a vacuum, similar to the first explanation. The group then let time equal 8 seconds in the tangent line equation and obtained –9. When the teacher asked what these numbers meant, one group member claimed that the line was indicative of a constant velocity and that the point (8,–9) was to be interpreted as the coordinate (time, position). Her defense was that the original graph showed time versus position, so by using a straight line, they were assuming a constant velocity and determining where something ended up at a later time—precisely the objective of part (b) of the task (practice 0). Given this explanation, practice 5 was in full swing. We had an illuminating discussion connecting the equation from board 2, $x = x_0 + vt$, to the well-known $y = mx + b$ used in board 3. One class member added, "They're both just saying that *new equals change times time plus the old*."

As we were set to wrap up the task, one member of the class asked if we could just solve the problem by counting. This prompted the teacher to illustrate some work on the chalkboard (see **fig. 2d**). Students could see that by starting at $x = 1$ and moving in spurts of 2 units to the left, one could end up with the answer. This was prompted by the structure evident in $x = x_0 + vt$, and it supported the written explanation on board 1.

DISCUSSION

The vignettes above were chosen to highlight two typical classroom discussions using the five practices in the teaching of calculus. Class discussions were a function of *students' ideas*—paving the way to meaningful understanding. As a small representative sample, below are three responses from members of the class in an anonymous, end-of-course evaluation:

- Classroom facilitated learning in a hands-on manner. Allowed students to test their knowledge as well as inspired critical thinking.
- I like how the professor put the class into groups to try and solve problems together with peers instead of constant presentation-style instruction.
- I liked the style the class was taught with. The emphasis on group work and small-group discussion helped me understand the material better than a straight lecture.

On the basis of the comments above, we see that students acknowledged the time that they were given to think, make meaning, and contribute to mathematical discussions. Meanwhile, the teacher received a steady flow of information vis-à-vis "How are my students doing?" Because the assessment of students and groups was embedded in classroom teaching, this feedback then guided the teacher for the next lesson.

Classroom teachers have asked us such questions as, "How does a student take notes in this environment?" and "What if a student or group misses the point entirely?" These are thoughtful questions, and our answers provide evidence of a paradigm shift in our teaching. For example, students are supplied with a written record of each classroom discussion through photographs of the whiteboards, often embellished with teacher comments. Knowing this ahead of time, students are less concerned with "taking notes" and more likely to make meaning of the mathematics being shared. The written record also addresses the second question of missing the objective of the lesson. Should a student or group fail to understand the material or even miss a day of class, the written document informs the student what he or she missed and provides pictures (literally!) of classmates' work and examples of student thinking. Generally, it is a win-win scenario for all. Moreover, much of the discussion above easily transfers to both high school and junior high school audiences.

CONCLUSION

The classroom examples shared here demonstrate what the five practices might look like in any mathematics classroom. Students are at the center of the learning, and the teacher navigates the terrain to ensure equitable, meaningful, and deep discussions about important mathematics. We have reshaped and repurposed many of the courses we teach to reflect an atmosphere in which students ask, explain, and connect. Without a doubt, our students are the greatest beneficiaries of this change.

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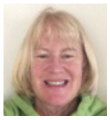

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Let's Chat about the Five Practices

On Wednesday, March 28, at 9:00 p.m. ET,

we will discuss "Using the 5 Practices in Mathematics Teaching" (pp. 366–73).

Join the discussion at #MTchat.

We will also Storify the conversation for those who cannot join us live.

Mark your calendars for #MTchat on the fourth Wednesday of each month.