LOOKING FOR AND USING STRUCTURAL REASONING

Examples of solving equations and inequalities, analyzing quadratic expressions, and reasoning with functions show three ways to engage students in this mathematical practice.

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Both the Common Core State Standards (CCSSI 2010) and the NCTM Process and Content Standards distinguish between Standards for Mathematical Practice (SMP) and standards for mathematical content. We believe this distinction is important and note that students often acquire knowledge of mathematical content without necessarily developing the related mathematical practices. In fact, we would argue that students grappling with mathematical content without mathematical practices are developing a different understanding. For example, consider the following task:

Find a value for x that satisfies $x/(x + 3) = 1$.

Jake multiplies both sides by $x + 3$ to get $x = x + 3$, and then eliminates the x on each side. He then writes, "no solution," applying the rule he has been taught that such nonsense statements should be answered in this way.

Madi reasons that because $x + 3$ is always 3 more than x, this means that the ratio of $x + 3$ to x can never equal 1. Although both approaches arrive at a correct solution, Madi's approach invokes the mathematical practice to "look for and make use of structure" (CCSSM 2010, SMP 7, p. 8). But what does it mean to "look for and make use of structure," and how can we as teachers support students in developing this practice?

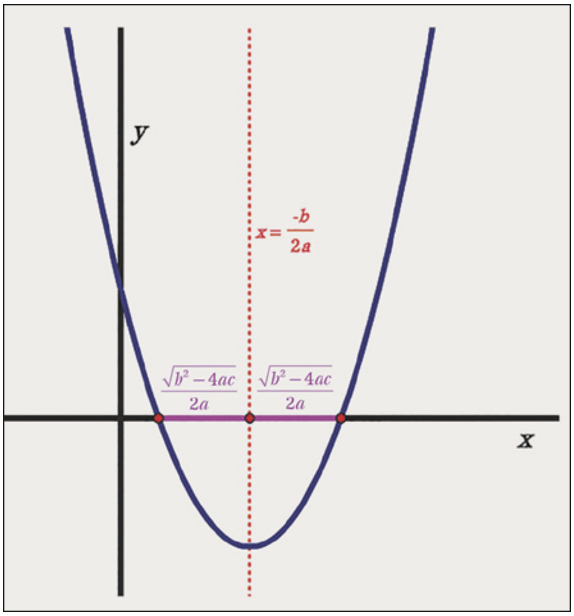

Fig. 1 This representation connects a quadratic graph to the quadratic formula.

Before unpacking these questions, we offer a problem to illustrate structural reasoning. We invite you to find two ways to reason about the values of x that make the following inequality true: $|x - 3| > -4$.

This problem can be approached using algebraic techniques that leverage rules associated with symbolic notation. To do so, students might write two separate inequalities, $x - 3 > -4$ and $x - 3 < 4$, solve each separately with the correct conjunction, $x > -1$ or $x < 7$, graph their solution sets, notice that their union covers the entire number line, and conclude that the solution is all real numbers. Although this approach reflects one kind of mathematical understanding that we aspire to instill in students, this problem could also be approached by examining the structure of the inequality. Some students might observe that this absolute value will be positive for all values of x and thus will always be greater than –4. Yet other students might leverage the interpretation of absolute value as representing distance or magnitude, noting that because distance is never negative, then *all numbers* are a distance greater than –4 from 3. Although all these understandings are critically important for students, too often we focus on the former. These latter approaches are not only efficient but also exciting to recognize.

Structural reasoning involves first taking a step back and looking for properties that are embedded in mathematical representations before selecting a procedure to use to solve a problem. Inviting students to search for and examine relationships and properties can foster not only a greater understanding of mathematics but also a sense of self-efficacy surrounding mathematical problem solving.

STRUCTURE AND REPRESENTATIONS

In general, structure denotes characteristics of how objects are built. In mathematical terms, structure refers to the embodiment of properties and their relationships in mathematical objects. To build and access mathematics, we use representations of mathematical ideas, such as symbolic, graphical, verbal, contextual, and tabular representations. Depending on the representation used, the resulting structure takes on different forms. For instance, the symmetry of a quadratic equation can be seen more easily when graphed than in the symbolic equation $y = ax^2 + bx + c$. Because different representations offer different insights into properties (Cuoco 2001; Lesh et al. 1987), an object's structure becomes clearer when we analyze a property across multiple representations.

One example of how engaging with a mathematical property across multiple representations may highlight structure can be seen when finding the x-value of the vertex of a quadratic $y = ax^2 + bx + c$. Analyzing the problem *graphically*, we can see a symmetric parabola (see **fig. 1**). In *words*, we recognize this property by saying that the vertex lies on the axis of symmetry. Focusing on specific points, the axis of symmetry occurs halfway between the two x-intercepts. Having noticed this property graphically and described it verbally, we now look for its instantiation in *symbolic* form.

DIFFERENT REPRESENTATIONS OFFER DIFFERENT INSIGHTS INTO PROPERTIES.

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Using a lens of structure, the quadratic formula can be transformed from a rule for calculating zeros of a function to reasoning about an embodiment of symmetry. This property is highlighted by analyzing the quadratic through multiple representations. To first capture the symmetry of the parabola, we split up the quadratic formula into two fractional expressions:

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Connecting the various parts of the expression to their graphical representations,

$$\frac{\sqrt{b^2 - 4ac}}{2a}$$

is the distance between the axis of symmetry and each of the two roots. This means that $x = -b/(2a)$ must be the center, consequently the x-value of the vertex. In summary, we can highlight structure resulting from the embodiment of symmetry within the symbolic form by describing and analyzing symmetry of parabolas through a graphical representation.

STRUCTURE AND GOALS

In addition to acknowledging the role that representation plays in structure, it is important to recognize that any structure that one sees in a mathematical representation will also depend on the mathematical goal. For instance, a student who does not consider the goal of a task may just see the symbolic representation of the expression $4x^2 - 9$ as a collection of disconnected symbols: 4, x, 2, and 9. It is not until the representation intersects with the goals that a student may begin to see structure in the notation.

• Goal: to factor the expression $4x^2 - 9$.
We can view $4x^2 - 9$ as the difference of two squares. This may be conveyed more clearly as $(2x)^2 - 3^2$, which can be factored as $(2x + 3)(2x - 3)$.

• Goal: to solve an equation using the quadratic formula.
We could overlay the general symbolic form of $ax^2 + bx + c = 0$ onto the given algebraic expression. Students might see that the absence of a middle term bx means a coefficient of zero and the operation of subtracting 9 as a constant term of -9. Such a view might be highlighted by rewriting $4x^2 - 9$ as $4x^2 + 0x + -9$. We could then evaluate the quadratic formula where $a = 4$, $b = 0$, and $c = -9$, resulting in

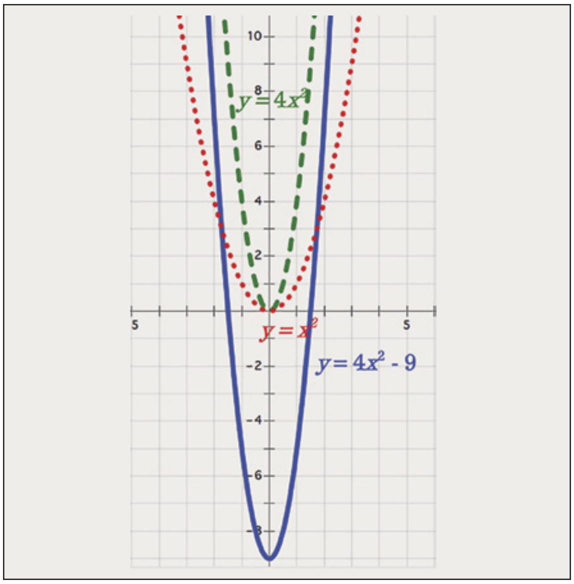

Fig. 2 This graphical representation shows transformations associated with $y = x^2$.

$$x = \frac{-0 \pm \sqrt{0^2 - 4(4)(-9)}}{2(4)}.$$

• Goal: to solve a quadratic by applying inverse operations.
Focusing on the three operations involved in the expression (squaring, multiplying by 4 and subtracting 9) a new structure emerges, one that can be emphasized by rewriting the expression as $4(x)^2 - 9$. With these operations identified as separate chunks, we can set this expression equal to zero and apply inverse operations in reverse order. This results in $4(x^2) = 9$, $x^2 = 9/4$, and finally

$$x = \pm\sqrt{\frac{9}{4}}.$$

From this point of view, $4x^2 - 9$ and $4x - 9$ have comparable structures. Although a common perception is that quadratic and linear equations fundamentally possess different structure, the two equations can be viewed as having similar structures when thinking about steps in isolating an unknown variable.

• Goal: to graph the quadratic.
We can see the symbolic representation $y = x^2$ as a base graph, with a vertical stretch of 4 and horizontal shift of 9 units down. Such transformations of the original $y = x^2$ might be symbolized as $y = 4(x^2) - 9$ and graphed as in **figure 2**.

With each of these four goals, the expression's structure was not an inherent feature of the symbolic notation. Instead, we were able to perceive the structure only when we recognized mathematical

1. Recognizing equivalent or similar mathematical properties in different forms and multiple representations.
2. (a) Seeing a mathematical expression (or parts of a mathematical expression) as an object as well as a process.

   (b) Decomposing (or chunking) algebraic expressions into a variety of sub-structures based on the context and goals at hand.
3. Making sense of appropriate manipulations that productively uses the structure instead of automatically applying a set procedure.

Fig. 3 Three components of structural reasoning involve recognizing, decomposing and making sense of properties, expressions and manipulations.

properties associated with the goals in the representation. It is through this connecting process that structure emerges—it exists for a student only when the student sees it. This stands in contrast to structure being an intrinsic attribute that exists in mathematical representations.

To summarize, we might help students by thinking of structure as existing when we connect mathematical understanding to the mathematical representations, with a particular goal in mind. We now provide three ways that have helped our students look for and make use of structural reasoning.

Helping Students Develop a Structural Lens

Along with the role that representations and goals play as we look for and make use of structure, we find it useful to think of structure as a lens through which mathematics can be viewed. One must develop a structural lens just as one develops any productive habit and learn when and how to use it to be effective. This lens is similar to what has been referred to as *structural thinking* (Mason, Stephens, and Watson 2009), *structural sense* (Hoch 2003), or *structural reasoning* (Bishop et al. 2016). These terms suggest a disposition where one looks for, uses, and connects underlying mathematical properties in representations. In contrast to the way we learn a technique or a procedure (Mason et al. 2009), we must develop structural sense over time. This is an understanding that teachers must think about developing during the course of the entire year by repeatedly drawing attention to this practice. We build on the work of Hoch and Dreyfus (2005) by providing three components involved in structural reasoning (see **fig. 3**). We also provide examples to illustrate each of their meanings.

1. Recognizing equivalent or similar mathematical properties in different forms and representations

The first component of structural reasoning is the ability to *recognize equivalent or similar mathematical properties in different forms and across multiple representations*. Rather than rely on contextual characteristics, this skill involves connecting similar ideas that may be represented in multiple ways. For example, we teach the slope-intercept and point-slope as two distinctive forms for linear equations. Our teaching experiences suggest that students often do not see these forms as connected. By taking a structural lens, one can see both forms as instantiations of two properties that describe a line: *a fixed point* (e.g., an intercept in one form and a general point in the other) and a *direction* (e.g., the slope).

Seeing both the slope-intercept and point-slope forms as representations involving a fixed point and a direction can be highlighted through graphing. Although notationally $y = mx + b$ and $y = m(x - x_1) + y_1$ look very different, their equivalent structure becomes more apparent by connecting these forms to their graphical representation and asking the question, "For what value of x can we evaluate each expression so that a point on the line can easily be identified?" For a linear function in slope-intercept form $y = mx + b$, one answer is $x = 0$, producing $y = b$. This means that the line passes through the y-intercept $(0, b)$. In the point-slope form $y = m(x - x_1) + y_1$, the choice of $x = x_1$ makes the first part of the expression equal to zero, leaving $y = y_1$. This means that the line passes through (x_1, y_1). By finding the x-value that readily yields a y-value, we can identify the coordinates of a point in each form. Consequently, students can understand a graphed line knowing a point and slope, whether that equation is written in slope-intercept form or point-slope form.

Additionally, the Common Core further emphasized the relationship between these two forms through the elevation of transformations. With this lens, the slope-intercept form $y = mx + b$ can be interpreted as a vertical shift of the line $y = mx$, and the point-slope equation $y = m(x - x_1) + y_1$ can be interpreted as a horizontal shift of x_1 and a vertical shift of y_1, meaning that $y = mx$ now passes through (x_1, y_1).

2. (a) Seeing expressions as objects as well as processes

The second two components of structural reasoning are interrelated, one being an understanding and the other an associated skill. The first is an understanding that enables students *to see a mathematical expression (or pieces of a mathematical expression) as a single object that can be operated on.* This interpretation contrasts seeing an expression as individual symbols combined by operations. Algebraic expressions can simultaneously represent the process of a computation and the object of that process (Sfard 1995). For example, the expression $x + 3$ can be interpreted as the *process* of adding three to an unknown quantity. It can also be interpreted as an object in and of itself, which is the result of three more than the quantity x. This difference can be emphasized contextually. For example, if the cost of a dinner is x dollars, and the tip is \$3, from a process perspective, $x + 3$ is the process of adding three dollars to the cost of the dinner. From the object perspective, $x + 3$ would represent the total cost of dinner.

With numerical operations, the distinction between process and object is easier to see because we typically use a different symbol for the object that results from the process (i.e., the process 12 + 3 can be represented by the single object 15). With algebraic expressions, no alternative exists to highlight the resulting object, as the result of $x + 3$ is $x + 3$. It may be challenging for students to see expressions both as individual symbols combined by operations and as a single object (the result of these operations). Consequently, students often struggle with the property of closure, not seeing $x + 3$ as a viable answer (Tabach and Friedlander 2008). Not accustomed to seeing arithmetic expressions as objects, students may feel compelled to simplify algebraic expressions incorrectly, adding unlike terms, such as writing $3x$ in place of $x + 3$.

(b) Chunking algebraic expressions into substructures

Once students are able to interpret an algebraic expression as an object, they have access to a different way of thinking. Students are able *to decompose algebraic expressions into a variety of substructures according to the context and goals at hand*. Take the equation $2(x - 4) = 10$. Students who see the left side of the equation as a set of operations are able to solve this equation by undoing the processes being applied to x in the reverse order (divide by 2 and add 4). In contrast, for those who are able to take a step back and see $x - 4$ as an object, the question then becomes what number, multiplied by 2, gives 10? This leads to an understanding that the object $x - 4$ must be equivalent to 5, and x is 9. Such an approach of viewing the expression as embedded chunks has been referred to as the "cover-up" method (Herscovics and Kieran 1980).

Cuoco, Goldenberg, and Mark (2010) refer to this component of structural reasoning as chunking. Chunking is often associated with factoring, but such an ability supports students in a wide variety of mathematical contexts. The understanding associated with chunking is critical when solving quadratics where factoring is necessary or with more challenging rational expressions (e.g., $11 - 50/(x - 2) = 6$). We can also use chunking when finding the domain and range of the function

$$f(x) = 8\sqrt{1 - \frac{25}{x^2}}.$$

We can interpret $f(x)$ as a series of function decompositions. This requires reflecting on the structure and identifying various symbolic pieces as separate, individual chunks. First, looking for the domain, imagine the radicand $g(x) = 1 - 25/x^2$ as a function. For the square root of $g(x)$ to be real, the output of $g(x)$ must be nonnegative. Likewise, by interpreting and analyzing $25/x^2$ as a new chunk or expression that is subtracted from 1, we can see this chunk must be less than or equal to 1 for the radicand to be nonnegative. Further decomposing $25/x^2$, we realize that the denominator x^2 must be larger than or equal to the numerator 25 for the fraction $25/x^2$ to be less than or equal to 1 and the output of $g(x)$ to be nonnegative. Therefore, x must be greater than or equal to 5 or less than or equal to –5. Similarly, with the range, we leverage the fact that x^2 must always be positive to reason that $25/x^2$ will also be positive. We can conclude that $1 - 25/x^2$ will be strictly less than 1. Therefore, the range will be less than $8\sqrt{1}$, which is equal to 8. The smallest value will be 0 because the radical must be nonnegative, which occurs when $25/x^2$ is equal to 1.

IT IS THROUGH THIS CONNECTING PROCESS THAT STRUCTURE EMERGES—IT EXISTS FOR A STUDENT ONLY WHEN THE STUDENT SEES IT.

Although chunking may be a specific case or even the result of an object understanding of notation, we see these two components of structural reasoning as mutually supportive. As students develop the ability to take a step back and see algebraic expressions as objects, they are able to see decompositions of expressions as multiple pieces according to the goals and context. Likewise, encouraging students to identify different chunks within algebraic expressions leads to an understanding of algebraic expressions as an object. Although details of this relationship are beyond the scope of this article, we encourage others to explore what activities might support students to develop connections between chunking and process/object reasoning.

3. Using the lens of structure to make sense of appropriate manipulation

Finally, as students begin to possess the skills to decompose representations and chunk expressions into objects, they may engage in the last component of structural reasoning. This component requires students *to pause to examine the structure and decide whether one manipulation may simplify a problem more than another.* This contrasts with automatically applying a set procedure to solve a problem. Making sense of the next steps that take advantage of structure is difficult to develop, as demonstrated by Hoch and Dreyfus (2004). When college-bound juniors were asked to solve $1/4 - x/(x - 1) - x = 6 + 1/4 - x/(x - 1)$, close to 90 percent of them multiplied both sides of the equation by a common denominator to convert it into a linear equation, rather than observing that the expression $1/4 - x/(x - 1)$ occurs on both sides of the equation. Noting this similarity in structure can help students see the original equation as equivalent to $-x = 6$. Although all students in the study were exposed to the "substitution method" in solving quadratics (i.e., substituting u for $(x - 4)$ in the expression $2(x - 4)^2 - 5(x - 4) + 3$), very few applied this technique. This may be because students viewed such a method as just that, a specific technique, not an overall orientation that permeates their thinking.

IN THE EYE OF THE BEHOLDER

Whether an elementary school student is asked to complete the equation $123 + 98 = 122 + \_\_\_$; a middle school student, to solve for x in the inequality $|\, x - 3 \,| > -4$; or a high school student, to use the quadratic formula to interpret the relationship of the roots to the vertex, structural reasoning is an important process and practice that shapes students' understanding of mathematics. With fear of stating the obvious, we note that we can only see what we look for. By taking a step back and looking for properties embedded in multiple mathematical representations, students can develop abilities to see expressions as both processes and objects, to chunk expressions into substructures, and to evaluate their next steps before automatically applying procedures. We find that thinking about structural reasoning as a lens for interpreting mathematics to be powerful for supporting students in learning mathematics.

The three qualities described here involved in looking for and making use of structural reasoning (see **fig. 3**) can be developed by teachers and used by students while problem solving. Although such an approach to mathematics is undoubtedly challenging to develop, focusing on structural thinking provides a powerful new way to reason mathematically. Because structure is in the eye of the beholder, teachers can play an important role in developing structural reasoning in students. As students practice looking for and using structural thinking across mathematical representations, they will begin to draw connections between previously compartmentalized topics. In this process of connecting mathematics, students will find greater enjoyment in developing their mathematical practices. We encourage others to further consider ways to develop structural reasoning in students.

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Let's Chat about Structural Reasoning

On Wednesday, January 23, at 9:00 p.m. ET,

we will discuss "Looking For and Using Structural Reasoning," by Casey Hawthorne and Bridget K. Druken (pp. 294–301).

Join the discussion at #MTchat.