(continued from pp. 116–23)

Fun with Triangular Numbers

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1. a.

Number of people	1	2	3	4	5	6	7	8	9	10
Number of handshakes	0	1	3	6	10	15	21	28	36	45

b. The number of handshakes is not increasing by a constant value. To go from 2 to 3 people, the number of handshakes increases by 2. To go from 3 to 4 people, the number of handshakes increases by 3. To go from 4 to 5 people, the number of handshakes increases by 4 and so forth.

c. The first person shakes hands with 49 people, the second person shakes hands with 48 people, the third person shakes hands with 47 people, and so forth. The second to last person shakes hands with only 1 person, and the last person would have shaken everyone's hand. By adding all these, we obtain the following sum (S):

$$S = 49 + 48 + 47 + 46 + \cdots + 4 + 3 + 2 + 1$$

If we reverse the order of the addends, we get the same sum S.

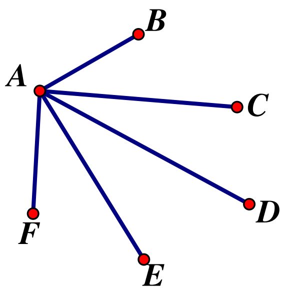
$$S = 1 + 2 + 3 + 4 + \cdots + 46 + 47 + 48 + 49$$

Adding these two sums vertically, we get

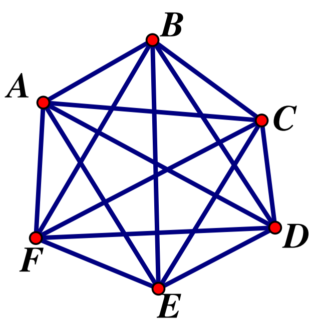
$$2S = 50 + 50 + 50 + 50 + \cdots + 50 + 50 + 50 + 50$$

Because we are adding 50 to itself 49 times, we can write this as $2S = 50(49)$. We want the value of only one S, so we divide by 2 to get $S = 50(49)/2$, resulting in 1225 handshakes when there are 50 people.

2. If we label the 6 points A through F, then we can see that there will be five lines connecting point A to all other points. Moving to point B, there will be four more lines created by connecting point B to points C, D, E, and F. We do not count the line from point B to A because that line was counted already. Going around the rest of the points, we find the maximum number of lines to be $5 + 4 + 3 + 2 + 1 = 15$.



Another way of thinking is to recognize that the final picture will have 6 points, with five lines from each point connecting to the other points, totaling 6(5) = 30 lines. Each line is counted twice because it connects 2 points; therefore, we must divide by 2 to eliminate the double counting, to get 15 lines.



3. The third triangular number is 6, and the six dots can be organized as a triangle with three rows of 1, 2, and 3 dots, respectively. The base of the triangle also has three dots. The 3rd triangular number is 1 + 2 + 3 = 6.

3rd

*

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The 4th triangular number is 10, and its figure is obtained by adding a row of four dots at the bottom. Hence, the fourth triangular number is 1 + 2 + 3 + 4 = 10. One can imagine how the figure for the 100th triangular number will be designed, with 100 dots on the bottom row, 99 dots on the next row, and so on, until we get to the last two rows containing 2 and then 1 dot. Then the 100th triangular number, $T(100)$, is $100 + 99 + 98 + \cdots + 3 + 2 + 1$. Using the same technique used in problem 1, we obtain

$$T(100) = 101(100)/2 = 5050.$$

The nth triangular number, $T(n)$, will form a triangle with n dots on the bottom row, $n - 1$ dots on the next row, and $n - 2$ dots on the following row until we reach the top row with 1 dot, resulting in $T(n) = (n + 1)(n)/2$.

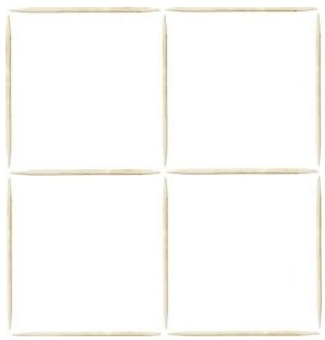
4.

n = number of lines	1	2	3	4	5	6	7	8	9	10
$R(n)$ = number of regions	2	4	7	11	16	22	29	37	46	56

Note that the number of regions is one more than a triangular number. Then:

$$R(n) = \frac{n(n+1)}{2} + 1$$

5.



n = side length	1	2	3	4	5	6	7	8	9	10
$H(n)$ = number of toothpicks	4	12	24	40	60	84	112	144	180	220

Notice that the number of toothpicks on a square of side length n is four times the nth triangular number. Therefore:

$$\begin{aligned} H(n) &= \frac{4n(n+1)}{2} \\ &= 2n(n+1) \end{aligned}$$

One can also obtain this result by counting the number of toothpicks horizontally and vertically. For example, the square with side length of 4 toothpicks has 4 rows and each row has 5 toothpicks. There are 4 columns, and each has 5 toothpicks. The total number of toothpicks for this case is

$$H(4) = 4(4 + 1) + 4(4 + 1) = 40.$$

In general,

$$H(n) = n(n + 1) + n(n + 1) = 2n(n + 1).$$

6. a.

n	1	2	3	4	5	6	7	8	9	10
$R(n)$ = reciprocal of nth triangular number	1	1/3	1/6	1/10	1/15	1/21	1/28	1/36	1/45	1/55

b. Start adding the consecutive reciprocals:

$$\frac{1}{1}=1$$
$$1+\frac{1}{3}=\frac{4}{3}$$
$$1+\frac{1}{3}+\frac{1}{6}=1+\frac{3}{6}=\frac{9}{6}$$
$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}=1+\frac{6}{10}=\frac{16}{10}$$
$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}=1+\frac{10}{15}=\frac{25}{15}$$
$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}=1+\frac{15}{21}=\frac{36}{21}$$

The numerations are the square numbers, or
1 added to ratios of triangular numbers.
This gives

$$R(n)=1+\frac{T_{n-1}}{T_n}=\frac{2n}{n+1}.$$

c. As n gets bigger and bigger, this sum increases up to 2 but does not become bigger than 2. We say that the limit is 2.