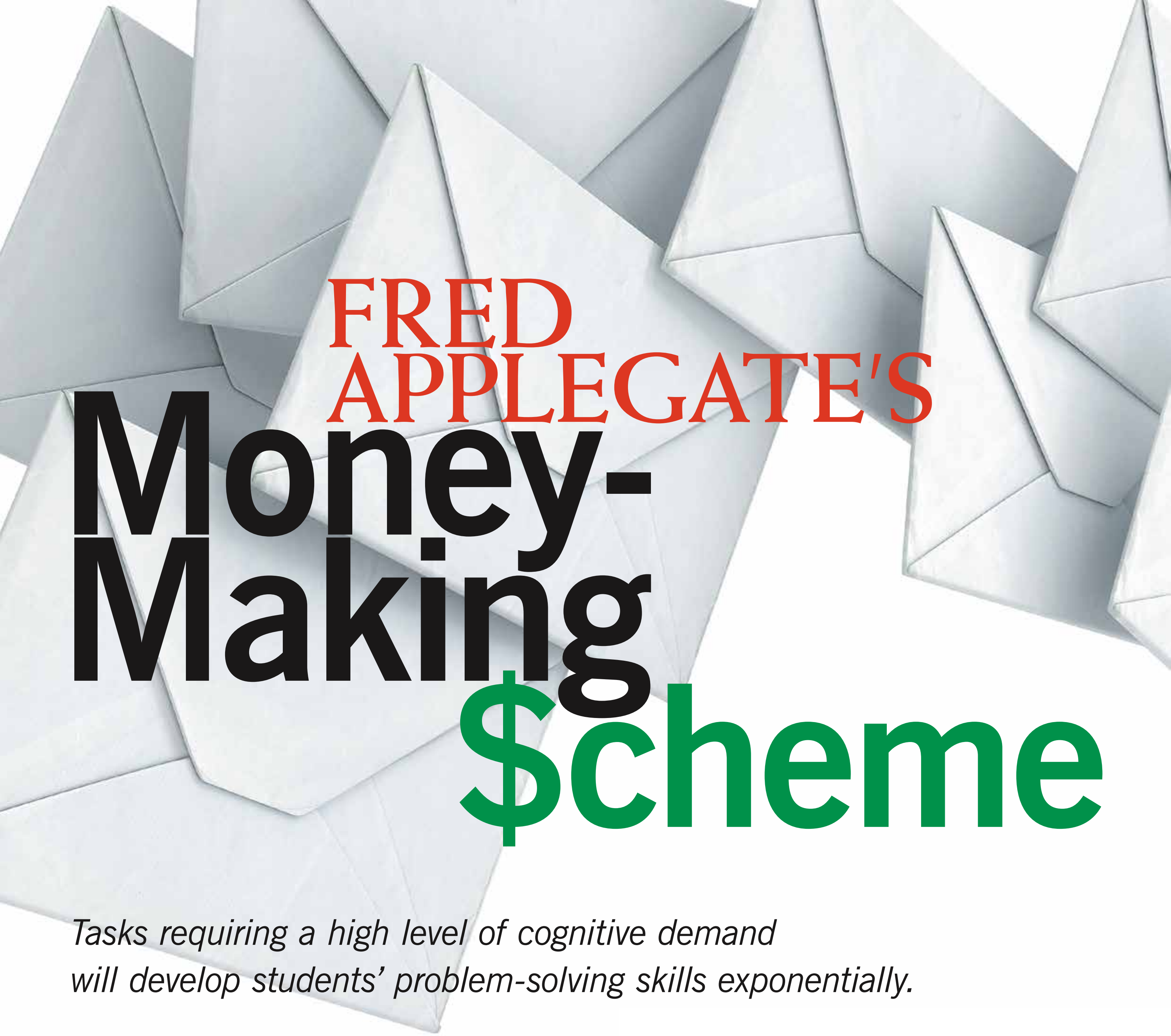

FRED APPLEGATE'S Money-Making $cheme

Tasks requiring a high level of cognitive demand will develop students' problem-solving skills exponentially.

Annette Ricks Leitze and Kristen L. Soots

Teachers across all grade levels agree that problem solving and reasoning are areas of weakness in students. Assessments among U.S. students indicate that these weaknesses persist (NCTM 2014) in spite of repeated calls that date back more than thirty years for increased problem solving, reasoning, and sense making in our schools. The NCTM is leading the charge for an increased emphasis in these three areas.

Furthermore, NCTM (2014) describes eight teaching practices to help facilitate problem solving, reasoning, and sense making. In this article, we focus on three of those practices: implement tasks that promote reasoning and problem solving, support productive struggle in learning mathematics, and facilitate meaningful mathematical discourse (NCTM 2014).

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THE TASK

Our goal was to develop students' problem-solving skills by creating a

high-level cognitive demand task that would be accessible and interesting to the students while maintaining a focus on important mathematics (Kahan and Wyberg 2003). The task was inspired by an actual chain letter received by one of the authors. The letter contained a list of names and addresses and instructions for the recipient to send $1 to the first name on the list. The recipient was then instructed to remove the first name and address from the list, add his or her own name and address to the bottom, and duplicate the new list. The instructions said that seven more people of the recipient's choosing were then to be added. The letter promised great wealth to anyone choosing to follow through with the instructions. Because letters of this type are real-life examples of exponential structures familiar to middle school students, it was adapted into a high-level cognitive demand task with which students could develop problem-solving skills. The story written for the task follows.

Fred Applegate got a neat letter in the mail today. It says he'll be rich if he follows the directions. Fred likes that idea. He'd love to buy a new RV next summer. The letter contains a list of 17 names

> and addresses. It says to send $1 to the name at the top of the list, and then cross it off and add his name to the bottom. Then, he has to photocopy the letter and list of names and send it to 10 people. Gradually, all the names will get crossed off the list, and Fred will be the one getting all the money! Fred's wife, Mary Alice, is fuming. She says Fred is a con artist, and she won't bail him out of jail when he gets locked up for stealing from their friends. If there are 16 names before Fred's, and every recipient sends out 10 letters, how much money will Fred make? Write your answer in three different ways.

We consider this to be a high-level cognitive demand, "doing math," task as categorized by Smith and Stein (1998). According to this categorization, it requires students to engage in nonalgorithmic thinking, explore and understand the process of how many letters will be mailed and received, access relevant knowledge, examine the task and constraints, and display considerable cognitive effort.

Mathematically, students are expected to discover that the layers of money sent are exponential and to correctly express Fred's earnings in standard notation, exponential notation, and scientific notation. Fred is to send $1 to the name at the top of the list and 10 letters. The 10 recipients are each supposed to send $1 to the second name on the list, along with 10 more letters. The next round of recipients is to follow the same routine. Students should discover that the pattern results in $\$10^0$ for the first name on the list, $\$10^1$ for the second name, $\$10^2$ for the third name, and so on, until Fred's name is reached. Since Fred is the seventeenth name on the list, he is theoretically supposed to receive $\$10^{16}$.

PRODUCTIVE STRUGGLE

Good teaching practices include supporting the productive struggle of students as they learn mathematics (NCTM 2014). It is important for teachers to make it clear to students that struggle will be expected in the normal course of learning. Because problem-solving tasks were a relatively new concept for our students, they were prepped at the beginning of the lesson with a review of this expectation as well as other behavior expectations while working in cooperative groups.

During the initial instructions, the teacher explained to students that she would act as a facilitator; it would be their responsibility to devise appropriate methods and solutions for the problem. By including this preparation time before starting the task, students were better prepared for the challenge ahead because they began the task with an understanding that they were responsible for developing their own solution methods. Students were then assigned a cooperative group composed of three or four students with differing abilities. The teacher circulated around the classroom as the groups began working on the task.

“We should have kept going” was a comment made by many students.

Most cooperative groups began the task by reading the problem together. It was interesting to note that almost every group contained a student who made a quick judgment after the initial reading. That quick judgment incorrectly assumed that Fred would receive $10 because the problem stated that each of the 10 recipients sends out $1. Discussion in each group varied in response to this answer.

Some groups began debating the validity of the solution simply because it was the first assumption shared. When supporting productive struggle, it is important to give time for each group to work through this initial discussion while wrestling with the task. Watching students struggle can sometimes be painful for teachers. The temptation is for teachers to intervene too soon, depriving students of the opportunity to carefully think through the task and reach some understanding.

As difficult as it may be, teachers must initially withhold input in an effort to encourage students to continue thinking. Without teacher input, a few groups decided to read through the problem again to test the validity of their answer against the description of the chain letter. Additional readings of the problem helped these groups notice that the number 10 represents letters sent out by Fred rather than letters received by Fred. Because of this understanding, these groups continued to think about different ways to solve the problem.

Allowing productive struggle means that teachers will allow students to work through strategies that prove ineffective for the given task. The hope is that as students think through an inappropriate strategy, new ideas for better strategies will come to mind. Because students are often working through new scenarios within a problem-solving task, they will need time for trial and error. In this case, students in one group remembered that some problems can be solved by drawing a picture. At this point, the group began to draw pictures of the letter while the teacher observed without comment. Although drawing a picture was ineffective as a strategy, it helped build ideas for other possible solution paths after students realized its ineffectiveness.

Next, students tried multiplying 10 by 16, thinking that Fred will receive $10 from each person in the list. This method also proved ineffective once the students read through the problem and understood that it did not match the description given. Without teacher input, the group finally decided to add more detail to the original picture strategy, which resulted in the group re-enacting the process of the chain letter. This strategy finally led to an appropriate answer for the group. Because the teacher allowed them time, students could attempt inappropriate strategies and use trial and error to develop an effective strategy through their own discussions.

ROADBLOCKS AND DISCUSSION

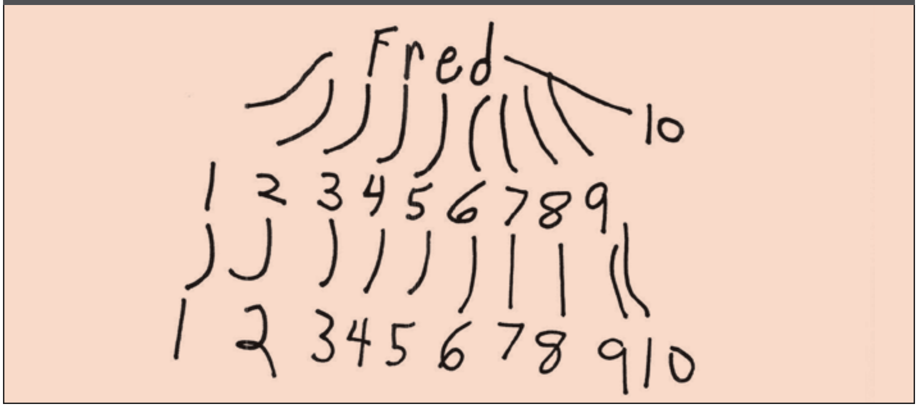
Some groups spent quite a bit of time in more heated discussion, resulting in a roadblock. Roadblocks, which are different from productive struggle, are unproductive because no progress is being made. When they occur, it is important for teachers to intervene by asking such questions as, “If you disagree, how could you prove your point? Is there something in the problem that might help?” This type of questioning was helpful for the groups because it promoted student discussion without the teacher giving away critical information that would change the cognitive level of the task. As a result of the questioning, students were able to reach an agreement and the high-level cognitive demand of the task was maintained.

As groups continued to discuss the problem, it became clear that many were treating the problem as a “treasure hunt,” thinking that the solution would be hidden within the story itself. This situation presented the teacher with another opportunity to ask questions and elicit discussion. As students worked, the teacher began asking groups which methods they had tried. Many groups commented they had searched the story for key words, but did not know how to proceed. Students were getting lost within the description of the chain letter itself and needed guidance from the teacher in a way that would promote ideas for new solution paths while still allowing the math to be problematic for them.

The teacher prompted the group to think about other problem-solving methods that were available to them. Referring to a problem-solving poster in the classroom, the teacher reminded students of other strategies they might attempt. The group decided it might help to make a diagram or list. This strategy seemed to be the prompting that groups needed to help them focus on the overall structure of the problem rather than the small details within the story. This prompt allowed them space to continue discussing the problem.

One group began drawing tree

Fig. 1 One attempted solution path involved drawing a tree diagram; it captured the first 10 letters in the process, but it did not allow for the stage after that.



diagrams to represent outgoing letters, beginning with letters sent by Fred (see **fig. 1**). The initial stage of the diagram was correct, representing the 10 letters that Fred is to send to new recipients, but the next stage of the diagram became problematic. The group struggled to represent how many letters were to be sent out by the next group of recipients; they incorrectly demonstrated that each recipient was to send out 1 letter. The teacher elicited discussion by asking the students to explain the meaning of their diagram. When the students began to think through their work, they realized the mistake they had made in the second row of the diagram, and began to discuss ways to fix it.

To elicit such productive discussion among students, teachers may sometimes need to reveal information. The challenge is knowing when and how much to reveal to maintain the cognitive demand of the task. One example of this occurred after a group decided to re-enact the directions given in the story. This group's strategy was interesting because it demonstrated their understanding of the exponential nature of the problem.

The students understood that each name on the list was to receive letters in powers of 10, but struggles with reading comprehension prevented the group from finding the correct answer. One student was asked to read the story out loud to the group. The teacher confirmed that students had made a correct choice of strategy but pointed out that they had made an error while carrying it through. Students had demonstrated that they had developed some level of understanding about the task. Because the students knew that their chosen strategy was a good choice, they were able to focus on details within the story to correct their mistake, rather than starting over entirely.

Another group also understood the exponential nature of the number of chain letter mailings. This group began the problem by multiplying 16 by 10, incorrectly determining that each name on the list before Fred will send him \$10. The group felt that this answer was not sufficient, because Fred concluded that he would be rich. The group continued to multiply by 10 eight more times, resulting in $\$1.6 \times 10^{10}$. Next, the group made a second mistake by representing their incorrect multiplications as $\$16^{10}$. When the students were asked to explain their answer to the teacher, they struggled to justify why they had multiplied 16 by 10 nine times, but had represented that process as $\$16^{10}$. The group's method involved compounded errors but also demonstrated an understanding that the chain letter mailings were exponential. Therefore,

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Students discussed the likelihood that a chain letter could produce $\$10^{16}$ for a recipient.

Careful and specific teacher intervention allowed students to connect exponential growth to a real-world situation.

the teacher added a bit of productive information, as done with other groups. As the students demonstrated understanding, the teacher stepped away and allowed them to finish the process without additional help.

SIMILARITIES AMONG SUCCESSFUL SOLUTIONS

As groups reached the correct solution, several similarities began to emerge. Each correct answer was the result of a group using the list of names and re-enacting the directions in the story. No other strategies resulted in correct responses. As groups began to solve the problem with this type of method, they began to understand the chain letter situation and make sense of it mathematically, which is a trait we seek in mathematically proficient students (CCSSI 2010). In addition, the groups that reached a correct answer were better able to describe the chain letter to a third party afterward. In this particular problem, understanding the chain letter itself always resulted in a correct solution. Once this understanding was reached, students easily represented the answer in exponential form. In this problem-solving task, understanding also prompted excitement in students. Groups began to discuss the likelihood of this type of chain letter in real life and what an amount of $\$10^{16}$ actually meant.

AN ACCESSIBLE, INTERESTING EXPERIENCE

Students' understanding became evident when groups were given the opportunity to share their solutions in a class discussion. Each group had opportunities to share their methods by either describing a strategy that helped them understand or describing a strategy that proved ineffective. Many groups shared the knowledge that lists or simulations were particularly beneficial for this problem, and many commented that they were surprised by how long it took to find the correct answer. The class discussion also proved to be encouraging for those groups who were unable to solve the problem. One group noticed that the students who were able to reach a correct answer tried many inappropriate strategies first. This group recognized that many of these early attempts were the same as their own. "We should have kept going" was a comment made by many students.

The goal of this task was to provide an accessible and interesting mathematical problem-solving experience for these eighth graders. In the process, these students learned that not all mathematics problems can be solved quickly. Although learning to allow students to experience productive struggle may be a challenge for teachers, it is well worth the effort in the long run. In this problem-solving task, careful and specific teacher intervention allowed students not only to develop their understanding of the chain letter scenario but also to connect exponential growth to a real-world situation. As more problem-solving tasks were presented throughout the remainder of the year, students demonstrated a greater willingness to try their own strategies when a solution path was unclear. Students also showed growing persistence in their willingness to spend greater periods of time developing solutions.

REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Kahan, Jeremy A., and Terrence R. Wyberg. 2003. "Mathematics as Sense Making." In *Teaching Mathematics through Problem Solving: Grades 6–12*, edited by Harold L. Schoen, pp. 15–25. Reston, VA: NCTM.

National Council of Teachers of Mathematics (NCTM). 2014. *Principles to Actions: Ensuring Mathematical Success for All.* Reston, VA: NCTM.

Smith, Margaret S., and Mary K. Stein. 1998. "Selecting and Creating Mathematical Tasks: From Research to Practice." *Mathematics Teaching in the Middle School* 3 (February): 344–50.

Any thoughts on this article? Send an email to **mtms@nctm.org**.—*Ed.*

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