Exploring Algebraic Misconceptions WITH TECHNOLOGY

Long-standing common student misinterpretations concerning expressions and equations can be addressed by using an app such as iSolveIt: MathScaled.

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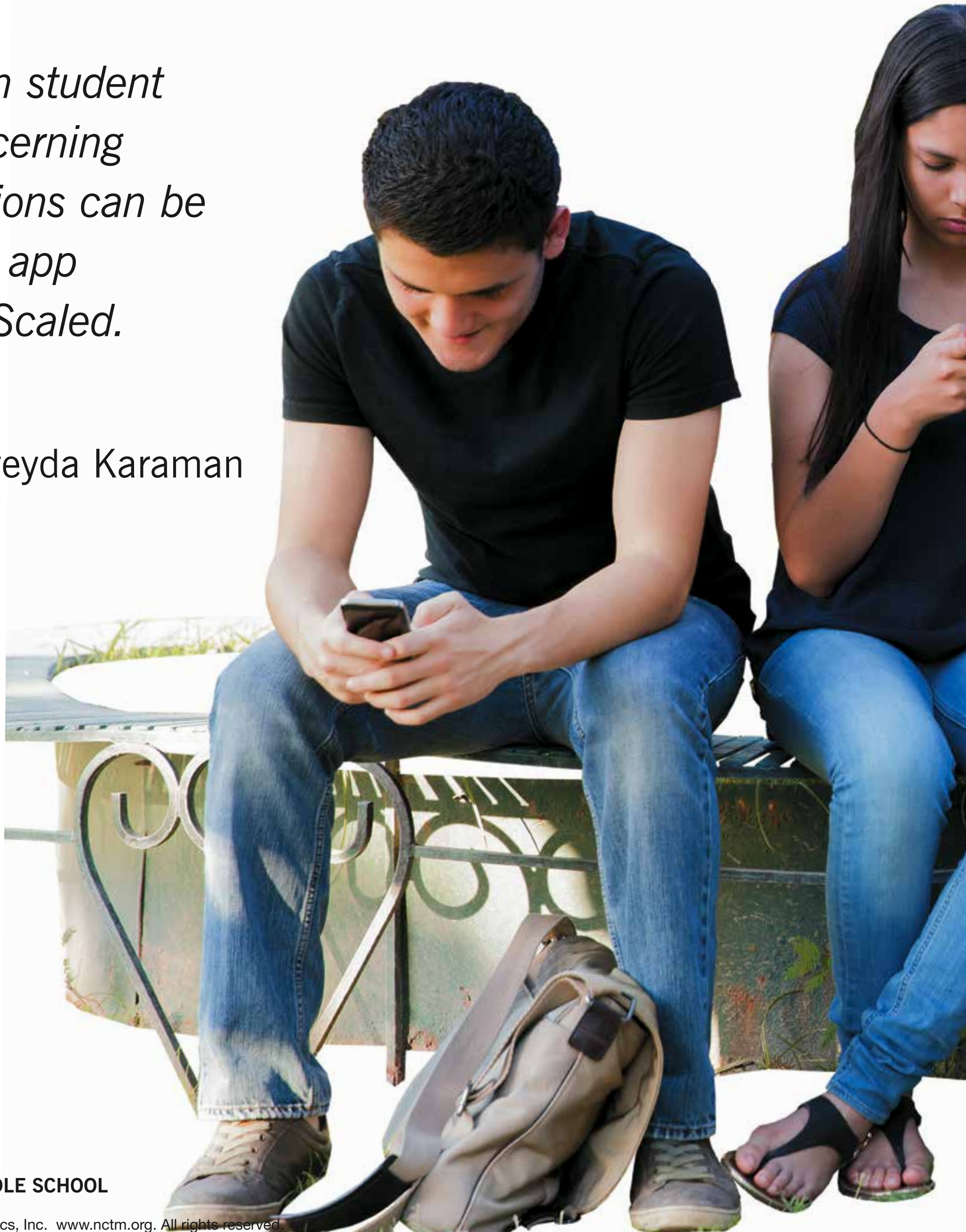

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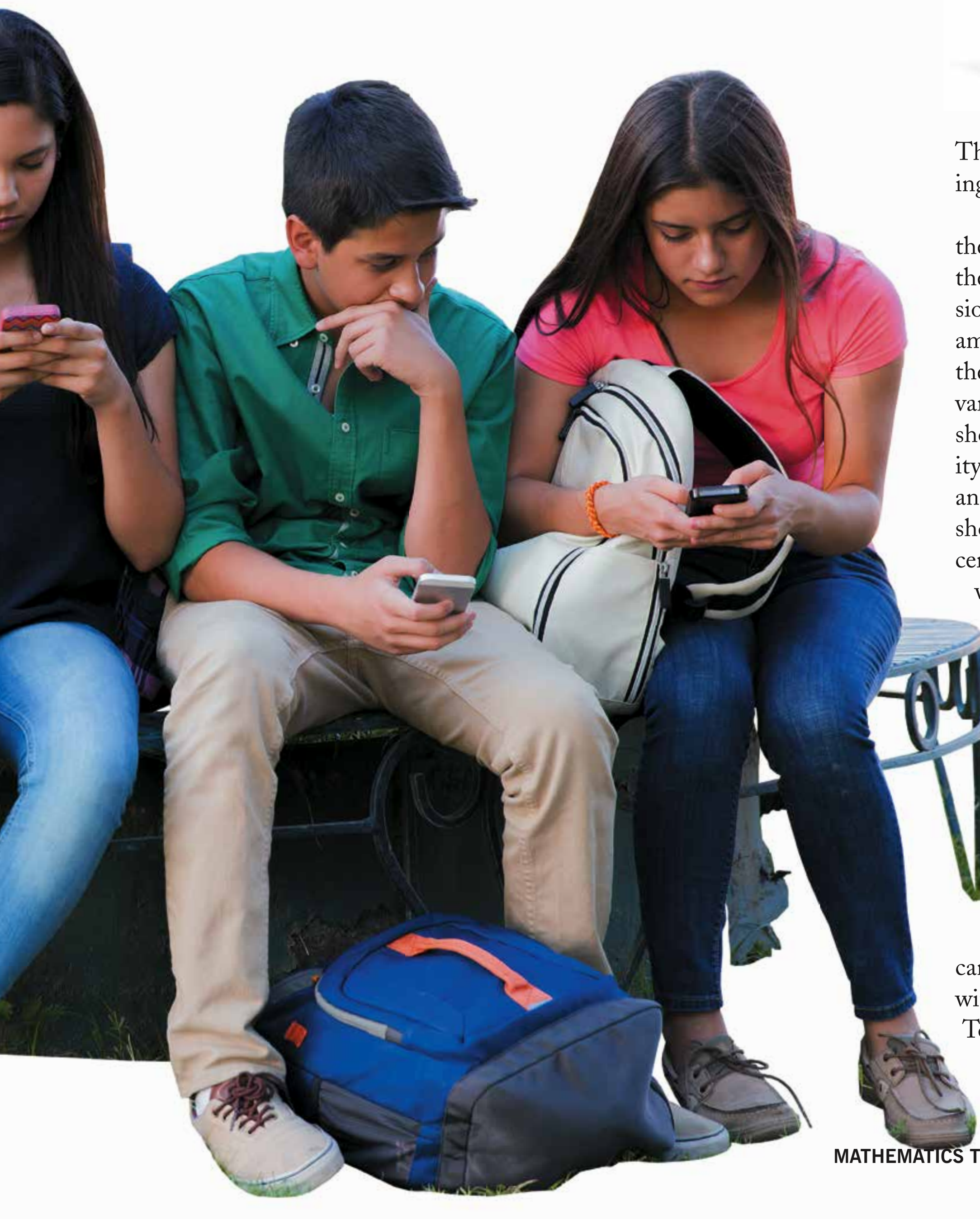
Many students struggle with algebra, from simplifying expressions to solving systems of equations. Students also have misconceptions about the meaning of variables. In response to the question “Can $x + y + z$ ever equal $x + p + z$?” during a student interview, the student claimed, “Never . . . because p has to have a different value from y. . . . [If] it didn’t have a different value, then you wouldn’t put p, you’d put y” (Booth 1988). Twenty-three of the thirty-five students interviewed similarly believed that a variable represents a specific value rather than a potential range of values (Demana 2000).

Nearly thirty years after Booth’s (1988) report on students’ difficulties in algebra, data from the Algebra Screening and Progress Monitoring (ASPM) project (Foegen and Dougherty 2011) showed that misconceptions like the example above have persisted despite our best efforts. The ASPM surveyed 1502 ninth-grade students to examine their understanding of typical algebraic problems through progress monitoring tools throughout the year.



The results highlighted some interesting misconceptions.

One test administered through the ASPM focused on three topics: the behavior of variables in expressions and equations, relationships among equations or expressions, and the effect of changing the value of the variable. Two problems from this test, shown in **table 1**, resulted in a majority of incorrect responses (Foegen and Dougherty 2011). This evidence showed that students tended to share certain misconceptions, some of which are also listed in the table.

Recent developments in technology might offer solutions for correcting students’ misconceptions. Technology is becoming an incredibly prevalent part of the American youth experience, with many teenagers owning an iPad® or an iPhone® (Piper Jaffray 2014). Teenagers are increasingly comfortable using these tools—tools that can be incorporated into a classroom with minimal instruction on their use. To take advantage of this familiarity, we have chosen to examine how teachers can effectively and

Table 1 Two problems from a test resulted in incorrect responses.

Problems	Representations of Students' Common Misconceptions		
Problem 1: "Bart said, '$m + 3$ is less than $m + 5$' ": Always true, sometimes true, never true.	Sometimes true. You do not know what m represents. (There is no way to tell.)	Sometimes true depending on the value of m. (Is m positive or negative? Greater than or less than the integers?)	Sometimes true depending on where m is located. (Is m represented before or after the integer?)
Problem 2: If $h + m = 7$, what does $h + m + 7$ equal?	$hm7$ or $7hm$	$7(h + m)$	hm^2

organically adapt an iPad app for use in the classroom.

The app we chose to help with student misconceptions of variables was selected with regard to several factors. First, the app had to be free to make it accessible for all students. Next, the app had to be related to learning goals addressed in the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) for working with expressions and equations. Finally, the app had to be exploratory or one that permitted individuals to engage in flexible learning experiences. In other words, an exploratory app would allow a student to take charge of his or her own learning, instead of following a fixed, procedural and perhaps ineffective track to achieve learning outcomes.

To find an app that would meet these criteria, we evaluated forty-five apps that we found after searching the Apple store for the word "Algebra." We agreed that iSolveIt: MathScaled (CAST) met all three criteria. This free app functions as a puzzle in which players are given a hanging scale and asked to use the variously shaped weights to achieve balance (see **fig. 1**). As the player progresses, he or she unlocks increasingly challenging systems of scales with more weights. The puzzles that follow are almost impossible to solve without the user developing a strategy.

MathScaled developers are aware of this challenge, providing built-in supports that help the player develop strategies. Two such support features are within each puzzle: Compar-o-Tron and Scratch Pad (see **fig. 2**). The Compar-o-Tron opens to the screen shown in **figure 3**. This feature allows players to balance the various weights in a specific puzzle against each other on a single scale, recording images of successful balances for future reference in the original puzzle. For instance, **figure 3** shows the comparison of weights within the Compar-o-Tron function for the puzzle shown in **figure 2**.

Once the player recognizes the ratios between the weights of various figures, he or she can use the Scratch Pad mode on the bottom-left side of the screen to enter in values (both numbers and letters) for each type of weight on the balance's arms (see **fig. 4**). However, it is ultimately left to

Fig. 1 The MathScaled app met these criteria: It was free, it related to CCSSM, and it allowed for student-guided exploration.

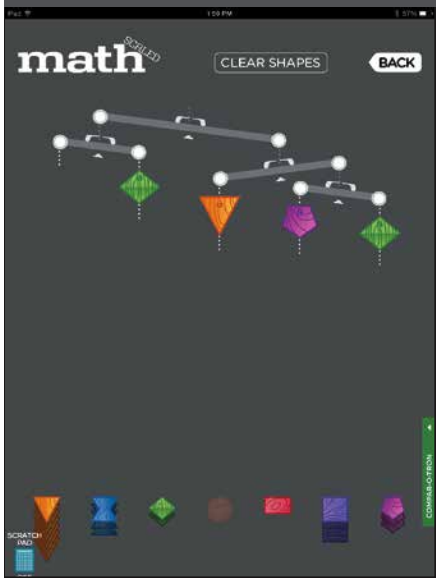


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Fig. 2 Compar-o-Tron and Scratch Pad help users maneuver through increasingly difficult puzzles.

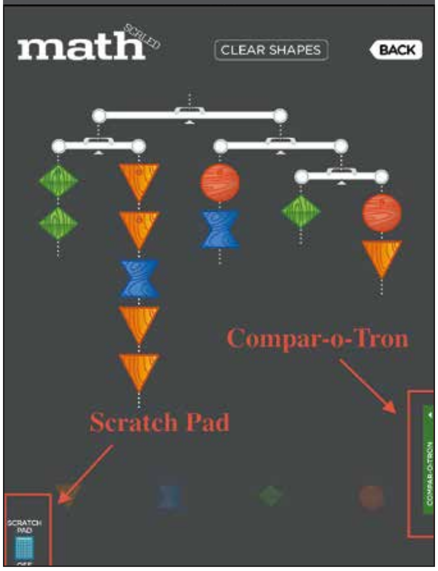


Fig. 3 A comparison of weights within the Compar-o-Tron function helps answer the **figure 2** puzzle question.

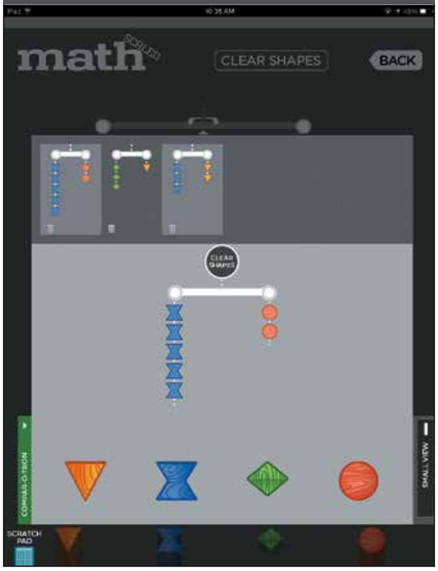


Fig. 4 The app allows players to decide on their own how to balance the shapes.

players to decide their own procedure for balancing the system.

By exploring the app and solving the various levels with the Compar-o-Tron, we realized that by not knowing the values of the provided weights, we were actually comparing, relating, and equating expressions with unknowns or variables. This led us to consider the following two hypothetical questions:

1. How could we use MathScaled to address and explore the misconceptions underlying the two problems from the ASPM?
2. How could using the app help students explore these concepts more effectively?

EXAMPLE 1: COMPARING EXPRESSIONS

Recall the following prompt discussed earlier: Bart said, "m + 3 is less than 5 + m." This prompt was given to ninth-grade students, who were to indicate if this statement is always true, sometimes true, or never true. However, to solve this problem, students must understand the meaning of the term *variable*, the effect of properties of operations applied on equations, and how to compare two expressions without being given specific values.

According to CCSSM, students should begin to develop these skills in sixth grade. However, only 21 percent of the ninth-grade students participating in the ASPM answered this question correctly, with most failing to provide an explanation for their answers. Forty-eight percent of participants failed to even attempt an answer. These results seem to indicate that erroneous concepts of variables and properties of operations in algebraic expressions can persist for years after they are first encountered.

A similar prompt was posed to a sixth-grade class: "Bart said, '$t + 3 < 5 + t$'." Most students stated that Bart's problem was always correct and provided strong evidence of understanding the concept of a variable through their explanations.

Some individuals, however, selected the correct answer but still showed some confusion about variables, as shown in **figure 5**. We see that the student claimed the statement is always true "as long as t equals the same value for each expression." This modifying phrase implies that the student reserves doubts that t always represents a consistent value within a mathematical phrase, despite the student's initial answer.

How can a teacher effectively support someone like this student when the written answer and conceptual understanding do not match? We propose that the iSolveIt: MathScaled app could be used to allow students to individually explore these ideas, with some guidance. For example, a teacher could divide the class into pairs or small groups of students, asking them to discuss their initial ideas about the question from **table 1**. As shown in the table, a further whole-class discussion may reveal three or more forms of common, erroneous responses. To address any misconceptions, the teacher can have students open the Compar-o-Tron feature within any puzzle in MathScaled to use its individual scale to represent balancing Bart's inequality.

When students then engage with Bart's problem, they can apply their experiences from previous puzzle games to decide how they could compare the two expressions m + 3 and 5 + m using the scale. First, a

teacher can explain that each shape in the puzzle has a corresponding weight that could be anything; thus, the values of the weights are variable. He or she can then ask how a student knows which expression has a greater value when using the balance.

Students can model these expressions within the Compar-o-Tron while forming a new understanding of variables and expressions; therefore, the teacher should provide as few interventions as possible during this process. A necessary intervention, however, should be to assign a value of 1 to a shape. (Here, we decided to use triangles.) Then, allow students to decide how to model the expressions $m + 3$ and $5 + m$ on the scale. This will require the student pairs to determine a weight to represent m as well as an order in which to hang the various weights. After students finish working, they can send a screenshot of the balance to the teacher for assessment (see **fig. 6**).

It is important to note a potential problem within this exploration. Students must learn that a variable represents a value, not an object itself. For example, the m in the problem can refer to the weight of a shape but not the actual shape. We suggest that teachers be aware of this possible misconception and be prepared to use appropriate language to clarify the distinction.

After the students' results have been submitted, the teacher should pose two questions to the class. First, would the order in which the shapes appear change which side is heavier? Allow students to show their different examples, or generate new ones, to examine this idea with the entire class. Second, would using different shapes to represent m change which side is heavier? Again, allow students to explore these ideas with the class. They should see that these aspects of a variable in an expression do not change the truth of Bart's statement.

When the Compar-o-Tron is used to explore Bart's problem, students can visualize that the order of variables and constants do not matter because every conceivable combination of $m + 3$ and $5 + m$ on either side of the scale would not change the fact that $5 + m$ is always heavier, or greater, regardless of the exploratory strategy chosen. Engagement with the app allows students to participate in self-driven discussion concerning the misconceptions previously shown in **table 1**.

A student's lack of confidence could be resolved through the discussion and exploration noted above. The work with the app shows that m (or t) is always equal to itself within an expression, regardless of what value each actually stands for. Thus, after using the MathScaled app, we would hope for students to achieve the level of understanding evident in the student response in **figure 7**. Here, the student explicitly demonstrated his understanding of t's equality through his justification.

EXAMPLE 2: EVALUATING EXPRESSIONS

Table 1 showed a second ASPM problem: "If $h + m = 7$, what does $h + m + 7$ equal?" Evaluating expressions, when they include letters that represent numbers, and understanding properties of operations within expressions are two of the sixth-grade expectations articulated by CCSSM

Fig. 5 Some confusion about variables continued to exist, as shown by a student explanation.

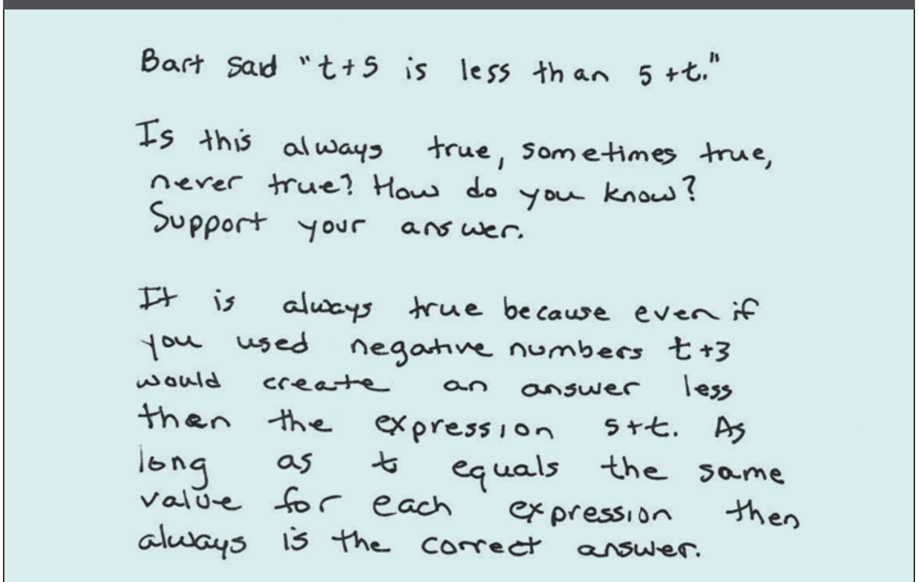
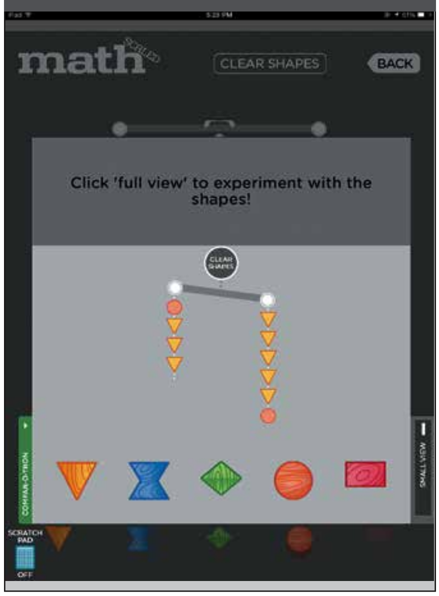


Fig. 6 A screenshot can be sent to a teacher for assessment purposes.



(CCSSI 2010). However, Demana (2000) refers to Booth's study and the finding that, to students, expressions like $h + m$ "were not legitimate expressions—rather they were directions to add" (p. 5).

Accordingly, results from the ASPM showed that only 23 percent of students gave the correct answer of 14. Through the misconceptions from **table 1**, we see that students were clearly confused about the properties of addition with expressions and that they were not viewing the variables as distinct quantities. Instead, students were tempted to transform the expression $h + m + 7$ into expressions such as $hm7$ or hm^2, akin to how one might add letters to form words.

To represent this question from the ASPM, one can use the fifth puzzle in the very first Level 1 puzzle group (see **fig. 8**). The teacher can ask students to determine how the shapes can be balanced in the Compar-o-Tron, so that they accurately represent $h + m = 7$, without giving explicit directions on how to accomplish it and instead asking students to discuss their reasoning about their assigned values with other groups. Those discussions may conclude that the weights of the two shapes, the hexagon and diamond on one side of the scale, represent h and m. Thus, the balanced scale will indicate that the weight of the triangle must be 7.

Next, to find the sum $h + m + 7$, the students can be left to explore how to adjust and rebalance their scales to reflect this new expression. For students to see the relationship between the equation and its model, it may be beneficial to have the pairs discuss their goals for this problem. For example, students can be asked to discuss what they expect their final answer of $h + m + 7$ to look like. Will it be a group of variables? Will it be a number? Have the students determine what answer makes sense to them. When students are finished with their balances, each pair can send a screenshot to the teacher, who can assess their current understanding. These screenshots may also help the teacher elicit the discussion to follow.

Some students may approach balancing the scale by adding another "7 weight" to the opposite side of the scale, as seen in **figure 9**, revealing the answer of 7 + 7, or 14. However, other students may add the two variable weights (hexagon and diamond, or h and m) again to the opposite side of the scale, representing $h + m + 7 = 2h + 2m$. This provides an opportunity for rich discussion because student pairs can compare any differing approaches and determine how to make sense of their results, connecting back to their previously established goals

Fig. 7 A student exhibits a hoped-for level of understanding.

t+3< 5+t

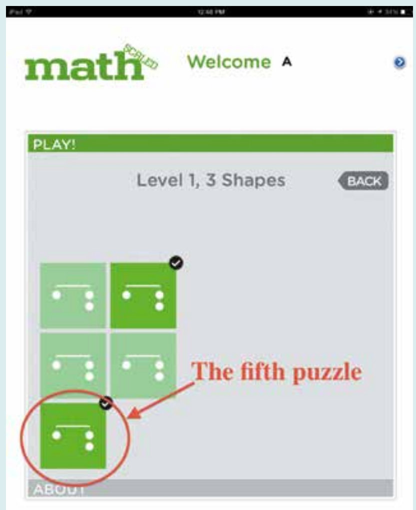
This is true.

3 is always going to be less than 5. and when you're adding the same amount to both numbers it would still work,

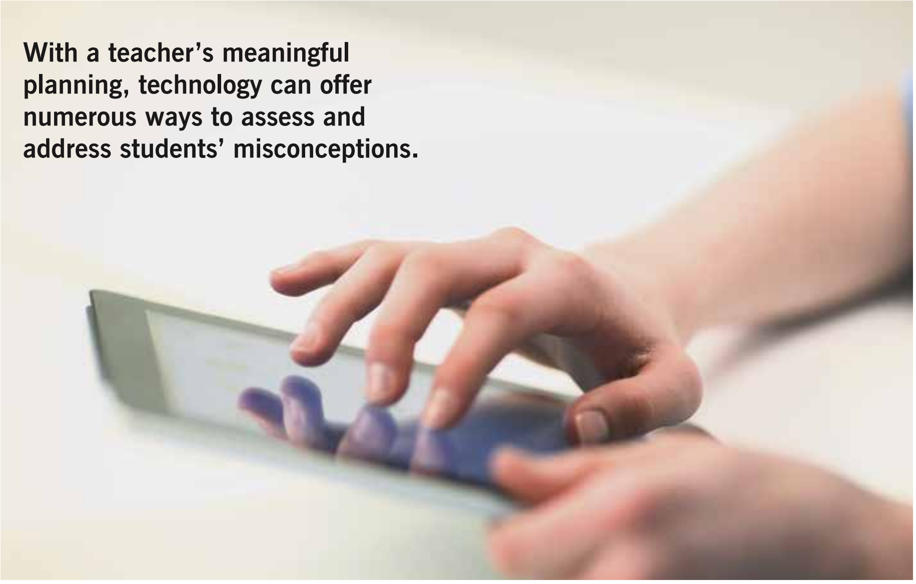
it must always represent the same amount

Fig. 8 Students were to work independently, asking questions of other groups, to accurately represent $h + m = 7$.

(a)



(b)



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for their answers. (Does $2h + 2m$ mean anything more than $h + m + 7$?) Such discussion can help students understand as well as learn to make a habit of justifying their work in mathematics.

Through MathScaled, students are afforded the opportunity to address different levels of structure, engaging both individual shapes and whole groups of shapes as they are added on one arm of the balance. For example, thinking about a hexagon and a diamond (instead of a hexagon-diamond, as the incorrect responses from **table 1** suggest) as separate quantities that

Fig. 9 Adding another "7 weight" to the opposite side of the scale revealed the answer.

can be added is crucial to the resolution of this problem. The modeling performed through MathScaled may help students then refrain from the erroneous thinking highlighted in the ASPM.

CLEARING UP ALGEBRAIC CONFUSION

An app like MathScaled affords a teacher the opportunity to engage students in exploring common algebraic misconceptions concerning variables in ways that are impractical without technology. For example, a teacher may be able to use physical balances in the classroom to demonstrate the algebraic ideas within this article; however, these demonstrations are limited by the available weights and the difficulty of representing variable weights. MathScaled's weights change in value from problem to problem, erasing student notions of specific values for variables. Furthermore, the app allows students to save screenshots of their work to assist the teacher in efficiently assessing understanding and providing individualized support. The use of technology then, illustrated through the use of the app MathScaled, can provide a teacher with ways to help students truly understand a variable's uncertainty and fluidity.

Although MathScaled was not explicitly designed to explore variables and algebraic expressions, the app affords students a chance to develop their conceptual understanding through exploration, visualization, and modeling. Likely, a teacher can also effectively adapt the app to other algebraic topics. For example, MathScaled's use could help students develop equations based on the balances they achieve in the various puzzles, promoting further understanding of the ideas of equality and symbolic representation. Transposition might also be related to the idea of keeping a scale in balance, providing an opportunity to extend the students' discoveries from MathScaled into other areas of formal symbolic algebra. One can also relate it to ratios and finding patterns. However, the most effective use of MathScaled, or any app, should be determined by the teacher as based on the content and contextual needs of their students.

Using apps in the classroom requires creativity and courage on the part of the teacher to fully develop into the exploratory, engaging, concept-building tool that our students need. Many educational apps or technologies intended for classroom use and advertised as hassle-free may remain ineffective without a teacher's meaningful engagement and planning. Only the instructor knows the specific context, challenges, and needs of his or her classroom; only the instructor can fully realize an app's potential. Thus, apps and technologies specifically presented for educational purposes may not be the only ones

CCSSM Practices in Action

6.EE.2: Write, read and evaluate expressions where letters stand for numbers.

6.EE.3: Apply properties of operations to generate equivalent expressions.

6.EE.4: Identify when two expressions are equivalent.

able to enrich student academic performance. Many other free apps that are available might also be adapted beyond their original purpose to great effect, providing efficient teaching opportunities for fostering high-level thinking. Modern tools like tablet apps may help middle school teachers end the thirty-year stagnation and put these algebraic misconceptions to rest at last.

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Any thoughts on this article? Send an email to **mtms@nctm.org**.—*Ed.*

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