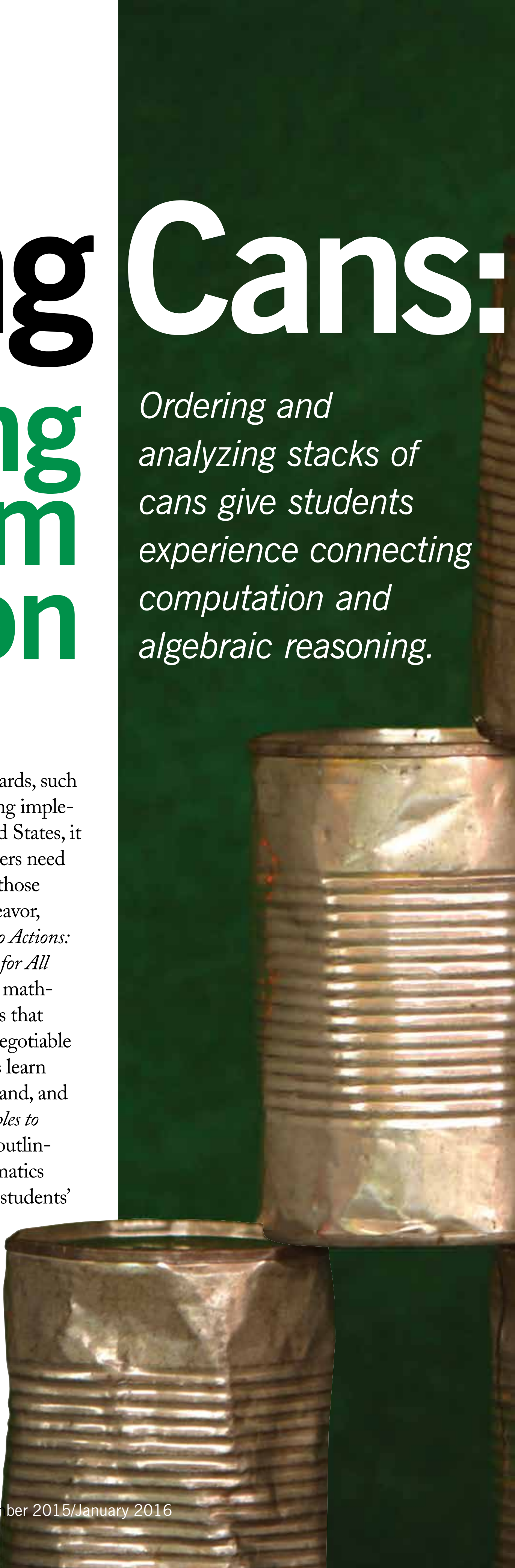
Stacking Cans: Abstracting from Computation

Ordering and analyzing stacks of cans give students experience connecting computation and algebraic reasoning.



As current mathematics standards, such as the Common Core, are being implemented throughout the United States, it has become evident that teachers need support to enact the tenets of those standards. To help in this endeavor, NCTM published *Principles to Actions: Ensuring Mathematical Success for All* as a guideline to emphasize to mathematics education stakeholders that "effective teaching is the nonnegotiable core to ensure that all students learn mathematics" (Brahier, Leinwand, and Huinker 2014, p. 656). *Principles to Actions* provides a framework outlining research-informed mathematics teaching practices to promote students' mathematical reasoning and problem solving (NCTM 2014). Consistent with the aforementioned points, the goal of this article is to describe an algebraic task that illustrates the connections between *Principles to Actions* and current mathematics standards.

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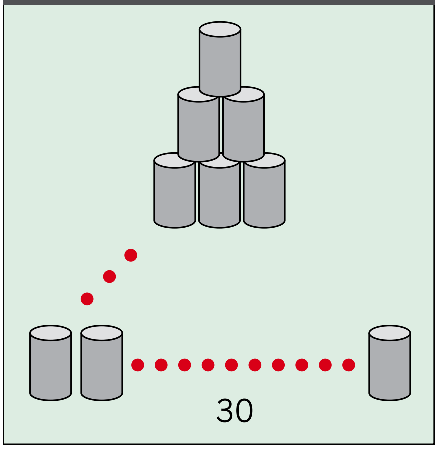
HEMERA TECHNOLOGIES/THINKSTOCK

Fig. 1 How many total cans are in the whole stack if 30 cans are in the bottom row? Explain your reasoning.



A MIDDLE SCHOOL TASK: STACKING CANS

We created an introductory, high-cognitive-demand task (Stein and Smith 1998) that would allow a class of seventh-grade students in algebra to begin exploring figurate numbers (see **fig. 1**). The mathematical goal of the lesson was to use functions to model relationships between quantities by engaging students in these standards for mathematical practice: (a) look for and express regularity in repeated reasoning; and (b) reason abstractly and quantitatively (CCSSI 2010).

When designing the task, we initially considered using 10 cans or 100 cans in the bottom row, but purposefully chose 30 cans for the bottom row. We reasoned that 30 cans would provide an appropriate foundation for students to eventually reason abstractly when answering a follow-up question. A bottom row with 10 cans would be too easy for many students to solve and would not create the need to consider abstracting. Although selecting 100 cans for that row would link to a historical connection, we felt that although students could add the first 100 numbers, many would mistake the complexity with the effort needed to add 1 to 10 and disengage with the task, saying it was too hard.

Furthermore, we also decided that we wanted students to solve the task without the use of calculators because using the calculator could constrain the students' engagement in the algebraic habits of mind as abstracting from computation, thus making certain aspects of the task more trivial (Driscoll 1999). In the end, we anticipated that students would employ a variety of approaches to solve the task, including verbal, pictorial, numerical, and tabular representations. These approaches would provide the foundation for future algebraic explorations by emphasizing connections across representations (Lesh, Post, and Behr 1987).

STUDENTS' STRATEGIES TO SOLVE THE TASK

By encouraging varied solution pathways to engaging tasks, teachers give students the opportunity to foster mathematical thinking, communication, and reasoning (Cai and Kenney 2000). To aid this process, the stack of cans in **figure 1** was shown to seventh-grade students, and students were asked to make observations. Students noticed the growth pattern of the stack of cans and correctly hypothesized that as the stack increased in height, another row of cans was added to the bottom of the

Fig. 2 This student's pictorial and written strategies are both incomplete.

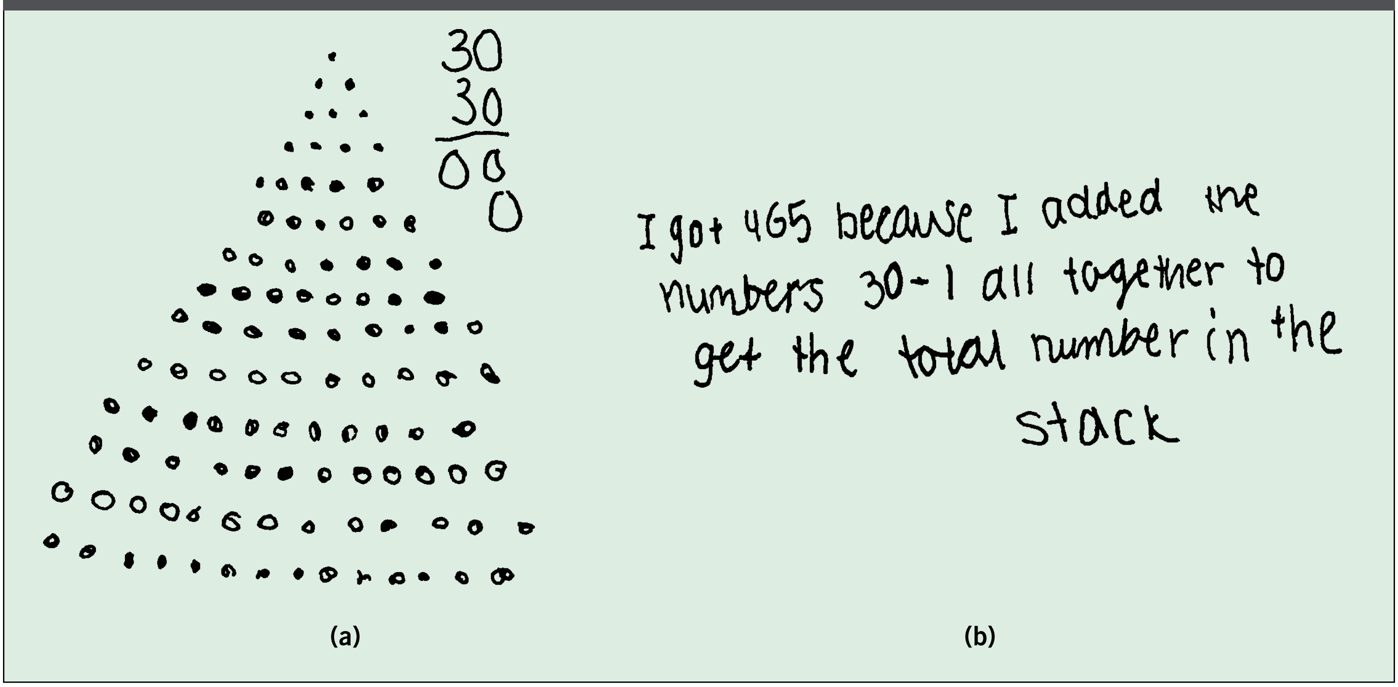
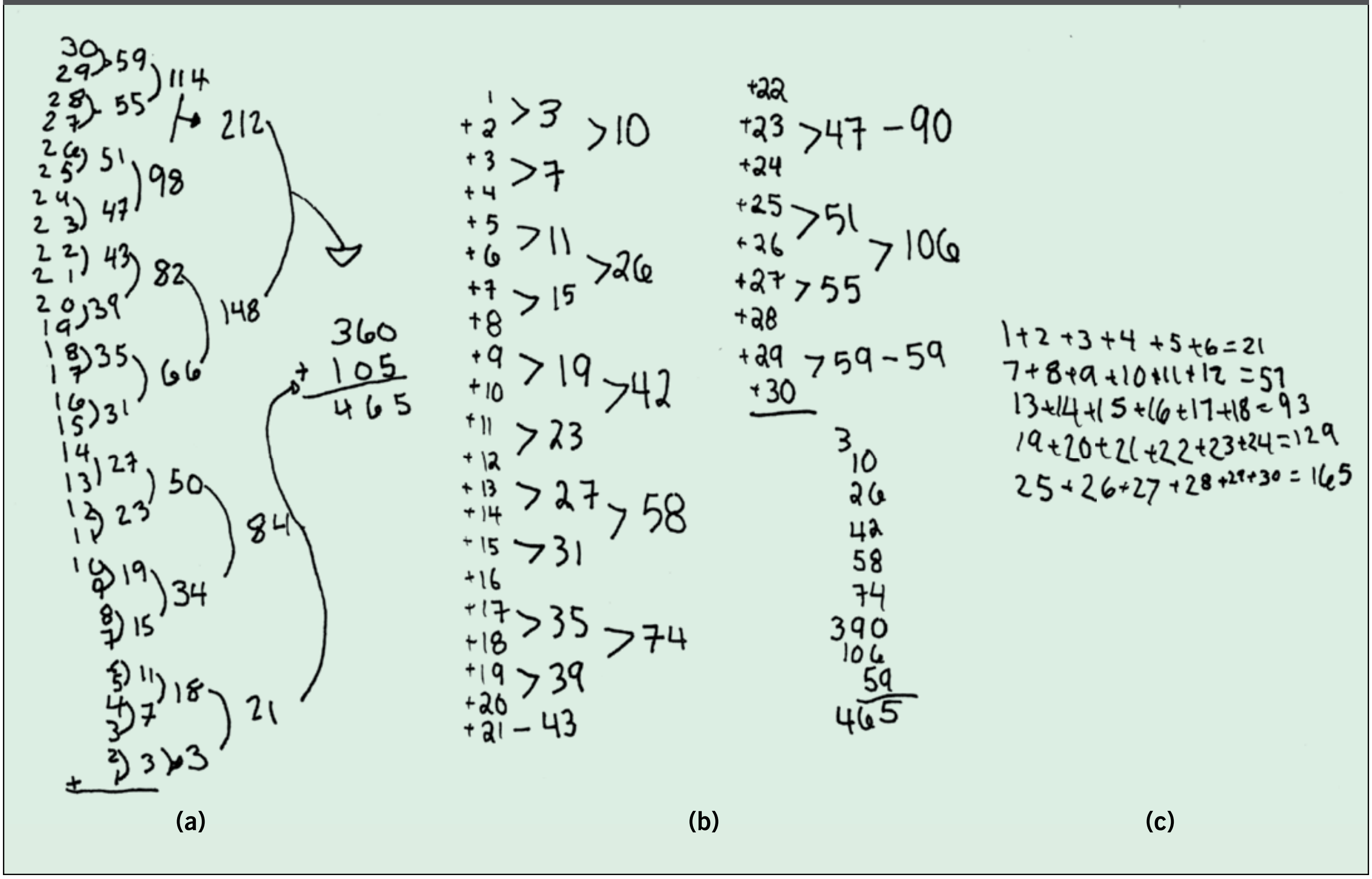


Fig. 3 Three flexible solution strategies emphasize addition.



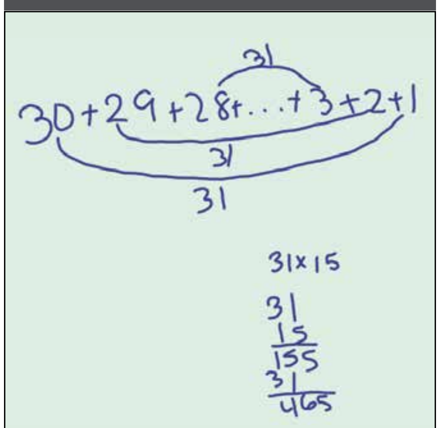
stack. Furthermore, this row had one more can than the bottom row of the previous stack.

Students initially engaged with this task by first using pictorial approaches and then applying strategies using words. For instance, when reasoning pictorially, one student drew dots to represent the cans, beginning with one can on top, then adding subsequent rows of cans underneath, until reaching the fourteenth row of cans (see **fig. 2a**). The student then computed 30 × 30 but did not finish the work. An explanation merely stated that the student used addition to solve the problem without documenting the computation (see **fig. 2b**). A third line of reasoning, used by a majority of the seventh-grade students, involved addition strategies (**fig. 3** illustrates three such approaches).

On first glance, one may initially think these strategies are rather simple and not worthy of class discussion. However, the teacher felt it was vital to discuss them with students. First, the strategies documented how students meaningfully accessed the task. Second, these approaches served as valuable transitions to more efficient and sophisticated additive strategies that students used when looking for and expressing regularity in repeated reasoning (see **figs. 4** and **5**). Furthermore, we believe that whole-class discussions about mathematical relationships are critical to student understanding.

Consequently, several students were asked how they used repetition to flexibly group the cans. When describing her reasoning to her classmates, Kathy said that she realized that she could make groups of 31 (see **fig. 4**). As the students listened, she described and documented her thinking using the interactive whiteboard by adding 30 + 1, 29 + 2, 28 + 3, and so on. In the whole-class dialogue that ensued, the teacher prompted the rest of the students to engage in meaningful mathematical discussion by verbalizing

Fig. 4 A grouping of 31 cans is reminiscent of a Gaussian explanation.



their understanding of Kathy's explanation.

Mariella: Kathy ordered them from 30 to 1 just in a shorter way, so she wouldn't have to write them all out. So then she added from the highest number and the lowest number, adding them so it will make the 31 each time. Then multiply them by how many groups she had.

Teacher: [Called on Bentley to respond.]

Bentley: I think she is going to multiply 31 times 15 because she would be adding two numbers, so you would have to divide 30 by 2.

Teacher: Why did Bentley say "31 times 15" not "31 times 30"?

Izzy: Well, when you are doing two numbers to get one number, like the 31 stands for two numbers, so if you multiply by 30 that would be like the 31 standing for one number instead of both.

Teacher: What does she [Izzy] mean by that?

Neeka: When you are done with all of the 31s, there would be 15 left, 'cause you are taking two numbers and making one. So you are slicing everything in half. Fifteen 31s.

By having the rest of the class make sense of Kathy's method, the teacher gave students the opportunity to listen to various lines of reasoning when interpreting Kathy's thinking as well as establish a shared understanding of her particular solution strategy. In the end, Kathy provided the total number of cans by computing 15 groups of 31, or 465 cans.

After the class made sense of Kathy's reasoning, Neeka presented a second strategy by making groups of 30 (see **fig. 5**). When verbalizing her reasoning, Neeka realized that she could make groups of 30 by adding 30, 29 + 1, 28 + 2, 27 + 3, 26 + 4 and so on. The teacher then stopped her so that the class could begin to make sense of her reasoning.

Teacher: She has a different answer than Kathy. Why does she have 450 [cans] and Kathy had 465 [cans]? Think in your head. Andrew?

Andrew: Because, since it's an even number (30 is an even number), you have an odd number in the middle, so she did not add 15 yet.

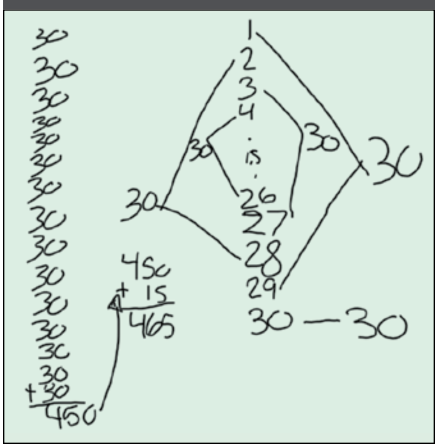
Mariella: You know how Kathy did the 31s, she added by 31s, she [Neeka] is forgetting the one part; it's kind of what Andrew said, it's kind of like an even number and it doesn't really split evenly.

Tiffany: Well kind of like Kathy. Hers were all 31s, and she had fifteen 31s. So if the ones, if you don't put them in, she would have 15 left, so Neeka has the 30 but she is forgetting 15.

Teacher: She did not forget it on her paper; she just stopped describing her strategy because I asked her to; I asked her to stop so that I could have you think about what she was doing. Neeka.

Neeka: I added all of the numbers together that would get me 30, then I added all of the 30s together and got 450. Since 15 was the only number that didn't have another number together with it to get 30, I added the 15 to the 450 and got 465.

Fig. 5 Neeka found groups of 30 in her list of numbers.



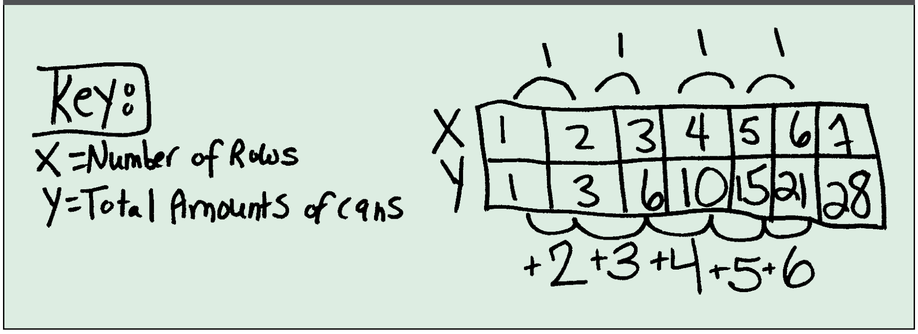
At the end of the conversation, Neeka finished her line of reasoning by adding 15 cans to the 15 groups of 30, or 450 cans, equaling 465 cans. Together, both lines of reasoning and the corresponding classroom conversation developed a quantitative foundation for more purposeful ways of pairing values to arrive at a sum.

ABSTRACTING

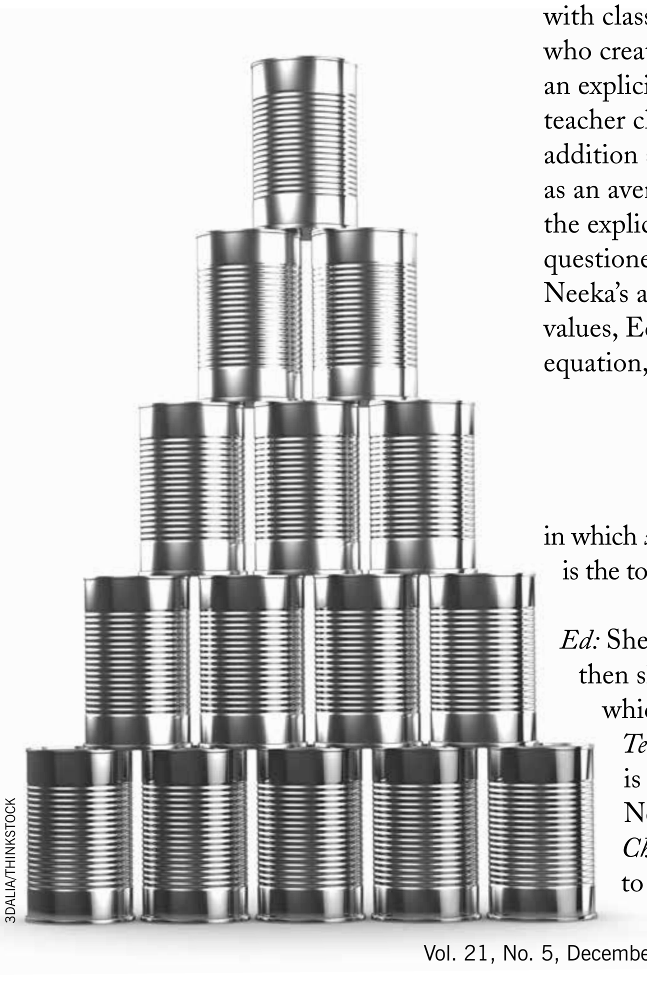
Although the students took multiple pathways to get the total number in the stack, at this point we felt the task could no longer be described as a high-cognitive-demand task. As stated by Smith and Stein (1998), tasks that require students to think conceptually steer them to a very different set of thinking processes than tasks that require students to perform a memorized procedure. To increase the mathematical rigor as well as engage students in the mathematical practice of reasoning abstractly and quantitatively, we required them to grapple and reason beyond the computations demanded in the task and to decontextualize to any stack of cans by representing it algebraically using symbols. At the end of the conversation, we challenged the seventh-grade students to write an equation to determine how many total cans would be in a stack of cans of any size.

One way that the students abstracted from the computations involved using a table. We encouraged students to pursue the table as a line of reasoning as one way to bridge their computations beyond the context of the problem and as a way to find an equation. Generally, when exploring a set of values in a table, students can demonstrate their reasoning in two ways. A rule for the relationship can be thought of

Fig. 6 A function table is another example of a student-driven representation.



recursively or explicitly, depending on the portion of the functional relationship to which they attend (Rubenstein 2002). When students think recursively, they use previous output values to determine the next output value. Alternatively, when students think explicitly, they generate an output by using its input value. The table in **figure 6** provides an example of how the students reasoned recursively about the stack of cans.



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Although these students would eventually find a solution for any size stack of cans, their perspective created a major challenge in that they would need to derive each prior input-output pair in the table to determine the desired output. The seventh-grade students realized this barrier and determined that finding an explicit equation would allow them to determine any output from its unique input. However, even after exploring and having discussions with classmates, none of the students who created a table was able to derive an explicit equation. As a result, the teacher chose to connect the two addition strategies described earlier as an avenue for students to arrive at the explicit equation. As the teacher questioned the students to detail Neeka's addition strategy used to pair values, Ed and Charles posted the equation,

$$y = \frac{x}{2} + x\left(\frac{x}{2}\right),$$

in which x is the number of rows and y is the total cans.

Ed: She had 30 times 15 to get 450 then she added 30 divided by 2, which is 15, to get 465.

Teacher: OK. Charles, what is your reasoning that this is Neeka's way of thinking?

Charles: I know that Neeka had to add the 15 at the end of it and that one has a plus sign with the other 15 on it.

To explore the reasonableness of the algebraic equation, the teacher and the students then referred to the introductory stack of 30 cans on the bottom. Ultimately, they arrived at the numerical expression 15 + 30(15) as a way to check their reasoning.

As the conversation progressed, Michael and Jordan, another pair of students, came to the interactive white board to discuss their understanding of Kathy's strategy. They posted the following equation,

$$y = (x+1)\left(\frac{1}{2}x\right).$$

Michael: Well, what I was thinking about? Make an equation determining the total number of cans, y, where $(x + 1)$ is the number of rows plus 1 times 1/2 the number of rows.

Jordan: All right. And we got $(1/2)x$; you take 1/2 the number of rows. We were thinking about the way Kathy did the problem. She was pairing the two numbers together and then halving it. That's how we got $(1/2)x$.

Again, the teacher and the students referred to the stack with 30 cans on the bottom. By doing so, they found the numerical expression

$$y = 31\left(\frac{1}{2} \times 30\right)$$

as a way to confirm their reasoning.

In the end, the students were able to acknowledge different aspects of the numerical relationship by looking for and expressing repeated reasoning. Consequently, the students were able to generalize by using functional thinking about the computations freed from the numbers in the problem while

exploring the equivalent algebraic expressions, referring to the context of the problem (Driscoll 1999).

HISTORICAL CONNECTION

Natural connections exist between this Stacking Cans task and a problem posed to renowned mathematician Carl Friedrich Gauss. When Gauss was a young boy, he and his classmates were given the task of adding all the numbers from 1 to 100 (Eves 1990). Gauss was able to solve the problem quickly by grouping the numbers ascending from 1 to 50, under which he lined up the numbers descending 100 down to 51 (see **fig. 7**). He then added the grouped pairs equaling 101 and realized that he had 50 groups of 101 and as a result would compute 50 × 101 to get the answer of 5050. If one were to abstract from computation, one would derive the expression

$$\frac{x}{2}(x+1) \text{ or } \frac{x^2+x}{2} \text{ or } \frac{1}{2}x^2+\frac{1}{2}x.$$

FINAL THOUGHTS

By integrating tasks that promote problem solving and mathematical sense making, middle school teachers have the potential to captivate their students and engage them in fruitful mathematical experiences. These experiences not only impact the time that students spend on the task as well as the cognitive demand placed on them but also influence students' perceptions about what it means to do mathematics and ultimately what they will learn. As Lambdin (2003, p. 11) articulated, "When challenged with appropriately chosen tasks, students become confident about tackling difficult problems, eager to figure things out on their own, flexible in exploring mathematical ideas and trying alternative solution paths, and are willing to persevere when tasks are challenging."

Fig. 7 How to make 50 groups of 101, per Gauss.

1	2	3	...	50
+100	+99	+98	...	+51
101	101	101	...	101

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Any thoughts on this article? Send an email to **mtms@nctm.org**.—*Ed.*

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