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How Many Jelly B

Students' mathematical intuition about estimation can serve as an entry point for tasks exploring measures of center.

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eans Are in the Jar?

Who will make a better estimate concerning the number of jelly beans in a jar, a single person or a group of people? On one side of the debate is the notion that a person would make a better decision because he or she uses unique knowledge that the group may not possess. On the opposite side of the argument is the claim that because of their breadth of responses, the collective wisdom of a group will arrive at a better conclusion.

This is the very dilemma that finance professor Jack Treynor posed to his students regarding *The Wisdom of the Crowds* (Surowiecki 2005). After reading this book, we wondered how middle school students would respond to the question. Moreover, we thought the question could be used as an entry point for us to leverage middle school students' mathematical intuitions regarding estimation and then link those estimates to data representation and analysis. In particular, we focused our effort on students' understanding of measures of center, as our experiences have suggested that although middle school students often use the word *average*, they do not necessarily specify which measure of central tendency they are referencing: mean, median, or mode or some completely different mathematics concept.

According to NCTM's *Principles and Standards for School Mathematics* (2000), making reasonable estimates is foundational to a student's understanding of number and operations. We found that using middle school students' estimates can also bridge an investigation of complex data sets. Having students focus on the reasonableness of a computation or on more abstract estimations, such as guessing the number of people in a stadium by using a picture, provides an avenue for middle school students to engage in the critical components of statistical problem solving. Analyzing data and interpreting and communicating results are just a few of these components (Kader and Perry 1994).

COLLECTING DATA

To leverage middle school students' intuitions regarding estimation and connect them to both graphical representations and numerical summaries of measures of center, we asked a class of students to determine how many jelly beans were in a container (see **fig. 1**).

After their initial estimate, we asked the twenty middle school students to reevaluate their estimates two more times. The first revision came after they were shown 10 jelly beans in a small cup. This additional information presented students with an opportunity to visualize the cup of 10 jelly beans and to consider how many cups of 10 would fill the jar. Another opportunity to reevaluate their estimates came after hearing a classmate's estimate and the rationale. This allowed the rest of the students in the class to reflect on their estimation strategy, as well as

Fig. 1 How many jelly beans are in the container?

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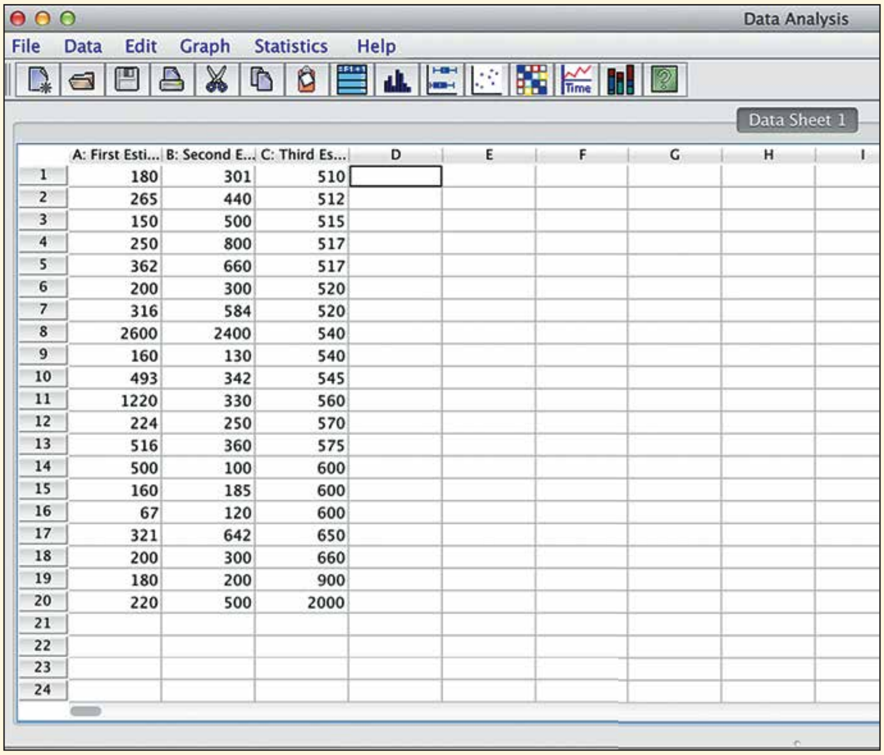
Fig. 2 One student explained how an average could play a role in estimating.

I think that the group will answer the question better than just one person because you can find the average of what other people said and you would have a more accurate estimate.

Fig. 3 Students entered their estimates into a spreadsheet.



	A: First Esti...	B: Second E...	C: Third Es...	D	E	F	G	H	I
1	180	301	510						
2	265	440	512						
3	150	500	515						
4	250	800	517						
5	362	660	517						
6	200	300	520						
7	316	584	520						
8	2600	2400	540						
9	160	130	540						
10	493	342	545						
11	1220	330	560						
12	224	250	570						
13	516	360	575						
14	500	100	600						
15	160	185	600						
16	67	120	600						
17	321	642	650						
18	200	300	660						
19	180	200	900						
20	220	500	2000						
21									
22									
23									
24									

the reasonableness of their prior two estimates.

Following these refined estimates, we asked the students to conjecture if an individual's estimate or the group's estimate would be closer to the actual number of jelly beans in the jar.

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Initially, some students thought that an individual would make a better estimate. One student declared, "I think an individual would do better, because they won't have as many different answers to pick from." Another student declared that the group would make an estimate that was more sound by explaining, "I think the group because then everyone gets to hear other opinions." In contrast to these general responses, one student's response was more mathematical in nature and alluded to his or her intuitions regarding "average" (**fig. 2**).

The responses gave us the opportunity to connect the students' estimates to explore which "average" could be used as a "typical" measure of center. Together the estimates and mathematical intuitions provided a convenient approach to engaging the students in exploring the data using technology to see what various data representations would tell them.

Technology is an essential tool in helping students' reason about mathematics (e.g., Dick and Hollebrands 2011; Lee, Hollebrands, and Holt 2010; NCTM 2000). By shifting the mathematical focus to move beyond solely the computational aspects of measures of center, technology provided a vehicle to emphasize its essential aspects by linking key graphical and numerical representations. Consequently, the NCTM's Core Math Tools® (http://www.nctm.org/coremathtools), a suite of software tools designed to support implementing the Common Core State Standards for Mathematics, was used to compile in a table the students' three estimates collected in the exploration (see **fig. 3**).

REPRESENTING AND ANALYZING THE DATA

Graphical Representations

For the given task, students' estimates were represented graphically, so that they could reason visually about measures of central tendency and measures of dispersion. This focus allowed us to assess their understandings of the distribution of data using a series of histograms and box plots.

The first graphical representation presented was a histogram. Since a student in the class suggested it, we displayed the third estimates the students made on the histogram with a bin width equal to 50 (see **fig. 4**). We capitalized on the suggestion because we knew in advance that the bin width would yield important insights into the shape of the distribution of the data. We then asked questions that emphasized the following three components of graph comprehension: (1) reading the data; (2) reading between the data; and (3) reading beyond the data, as suggested by Friel, Bright, and Curcio (1997).

The students addressed "reading the data" by determining the bin width of intervals in the histogram, the number of estimates that the students in the class made, and the range in which all the data fell. They also addressed "reading between the data" by making comparisons of different-size bins within the histogram. The discussion led to "reading beyond the data." To extrapolate beyond the data presented in the histogram, we asked students to answer and justify the following question: If a new student entered the class, and was shown the histogram as a point of reference, what do you think his or her guess would be?

Out of the twenty students, seven provided informal reasoning regarding the mode as being the measure of central tendency that would best answer the question. For example, one student stated that the new student would guess 500 jelly beans "because that has the most people who guessed that." Additionally, twelve of the twenty students provided a justification that either formally or informally related to the median. First, some students described "middle" in relationship to the top-two bins [500–600]. A second line of reasoning was similar to the first line of reasoning; however, the students saw "middle" falling between the "most commonly guessed numbers" [500–700]. Finally, some students established that the "middle of 500–550" could be used to determine that the new student would guess 525. In each case, the students' intuitions regarding the histogram led them to use only some of the estimates to think about the "middle" or median as the "typical" measure of central tendency. These intuitions were important for us to leverage and also helped guide the class discussion

Fig. 4 The students' third estimates appeared in a histogram.

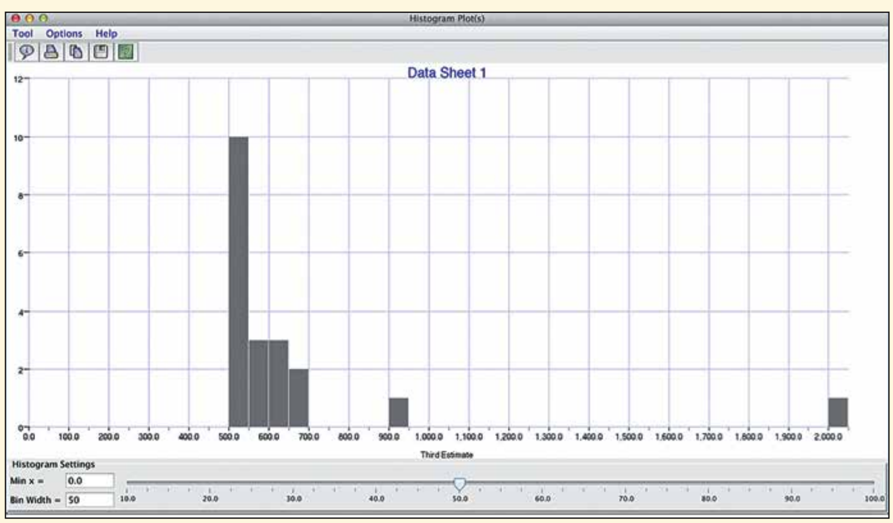
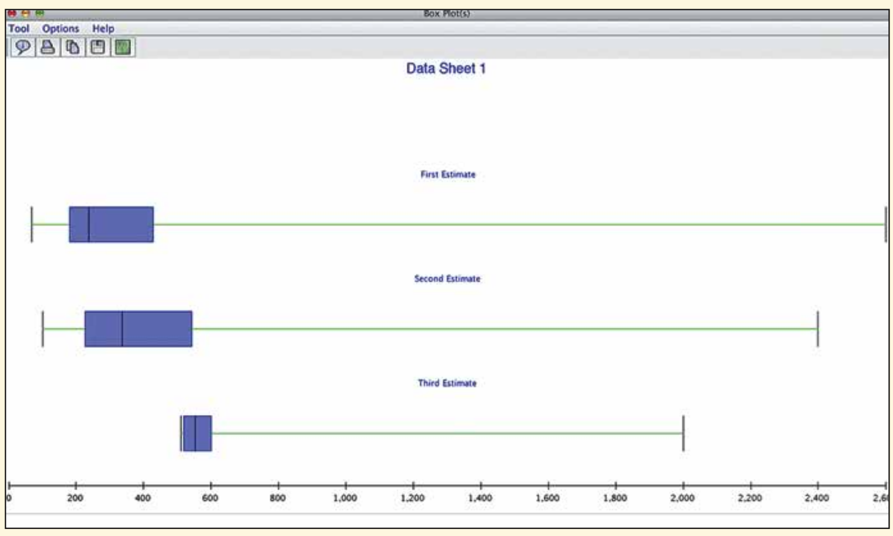


Fig. 5 First, second, and third estimates could be examined immediately in this box-plot form.



Figs. 6 Three students explained their reasoning about the box plot.

The ranges of the 3 box plots start to shrink as the 50% part of the boxplot gets thinner.

(a)

In the beginning 50% of the estimates were farther apart, but by the end the 50% was close together. The guesses over all got closer. The spread got closer as we went.

(b)

The median increases from each boxplot, and the maximum and minimum for the first two plots is about the same. However, the last range is smaller. You can tell this because the blue box (half of the data) moves to the right (increases) each time. The line that spreads out throughout the maximum and minimum also gets smaller (range gets smaller).

(c)

toward exploring box plots using all three estimates.

Using Core Math Tools, three box plots were constructed and placed on the same set of axes to allow students to reason about their estimates (Friel and O'Connor 1999) (see **fig. 5**).

Again, questions we asked where students were reading the data and reading between the data helped us access students' thinking about reading beyond the data. We asked what they noticed was happening to the box plots representing their three estimates. For instance, "What does the box represent? What does the line in the box represent? What do you observe about the spread of each of the estimates?" The responses shown in **figure 6** varied in mathematical detail and depth. For example, two students reasoned about the dispersion of the estimates using the range by emphasizing the "shrinking" of the box plot as the "minimum increases" and "maximum decreases."

Other students chose to identify the interquartile range (IQR), where 50 percent of the data were represented, in addition to commenting on the dispersion or spread of the data. For example, two students (see **figs. 6a** and **b**), identified how the IQR was beginning to "shrink" and get close together "by the end [third estimate]." Finally, one student (see **fig. 6c**) reasoned about aspects of both the measures of center and measures of dispersion. This student correctly read the graph and noticed that the median increased in the IQR of each box plot. Further, the student identified that the range was decreasing in each estimate.

The students' reasoning regarding box plots provided us with definitive ways that their thinking about median and range were still evolving. When this reasoning was juxtaposed to the students' thinking about the data represented as a histogram, it compelled us to connect their intuitions regarding both these visual representations to the equivalent numerical summaries of the data.

Numerical Summaries

After exploring the data visually, we wanted the students to connect their reasoning to the descriptive statistics of the data, so we posted all their final estimates horizontally on the interactive white board.

510, 512, 515, 517, 517, 520, 520, 540, 540, 545, 560, 570, 575, 600, 600, 600, 650, 660, 900, 2000

We then asked, "Which estimate would represent a typical guess?" to determine how the students viewed the data. Although one student responded using the range 500–600, the majority of the students thought that the mode, 600, would be indicative of a typical estimate. It was important to recognize this response: Earlier in the lesson, a majority of the students indicated that the median would be the measure of center that they would use to determine an estimate if a new student entered the class. This response was unique in our experience and inconsistent with prior iterations of the task, in which the mean and median were more appropriate measures of center and the mode was not. It was also noteworthy since, mathematically speaking, the mean and median were typically better estimations than randomly selecting any one of the estimates. Ultimately, these divergent lines of reasoning were based on the representations we presented to the students. As a result, it was imperative that we further investigate the mean, median, and mode, as well as the range of their estimates, to have students draw a conclusion about their final estimates.

Drawing a Conclusion

Since the students in the class knew how to calculate the measures of central tendency, we decided to continue to engage them in reasoning about measure of centers without computation. We told them that there were

Fig. 7 This explanation provided a window into a student's developing thinking about the average.

Because all of the different guesses come together to form an average answer. Some of the individual guesses were not close to 601, but when those guesses are combined with other guesses, the average answer is closer to 601.

Another characteristic of the mean is its sensitivity to extreme values or outliers.

601 jelly beans in the container and also gave them the measures of center and dispersion of their guesses:

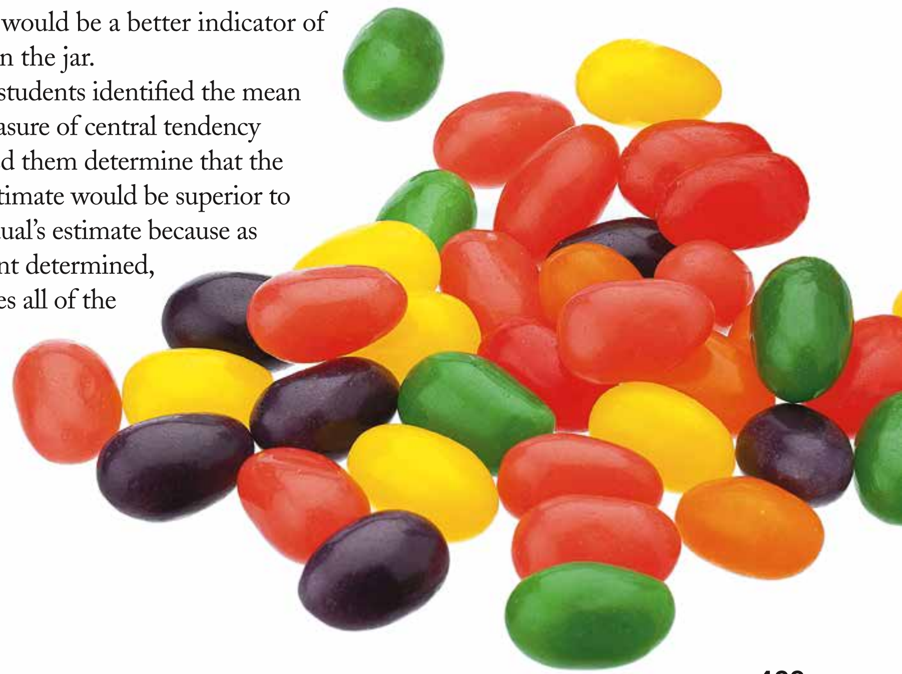
- Mean = 647.55
- Median = 552.5
- Mode = 600
- Range = 1490

We then returned to the question that drove the investigation, "Who will make a better estimate of the total number of jelly beans, an individual or a group?" The students concluded that only six out of twenty students made estimates that were closer to the total than the mean or median of the class, and that only three out of the twenty students guessed the mode. As a result, the students indicated that the group, not an individual, would make a better decision about the number of jelly beans in the jar. We were able to return to the notion of "average" to ascertain the students' evolving informal intuitions regarding measures of central tendency, as shown in **figure 7**.

The reasoning above allowed us to segue and again push the students past vague notions of average and the ambiguity of combined estimates to further synthesize numerical reasoning connected to graphical representations. For example, some students identified the mode as being the measure of central tendency that helped them determine that the group's estimate would be superior to an individual's estimate because, as one student stated, "It shows what appears and reappears multiple times because it is what a lot of the guessers guess." This reasoning allowed for discussion that connected the most popular estimate, 600, to the histogram that the students explored earlier in the investigation. The mode also allowed us to ask students to reason about instances when there were multiple modes or no mode or when data appeared and reappeared. These situations did not always guarantee that they represented all the data. The estimates in this investigation helped exemplify this point. We had three estimates of 600 and two each of 517, 520, and 540. If one of these estimates, 517 for example, was made by another student, the mode would not be an appropriate indicator of the group's wisdom since all but five of the individual estimates would be a better indicator of the total in the jar.

Some students identified the mean as the measure of central tendency that helped them determine that the group's estimate would be superior to an individual's estimate because as one student determined, "It includes all of the data. The mean was 647.55, and this included all of the data." By correctly identifying that the mean "includes all of the data," we were able to discuss another characteristic of the mean, namely, its sensitivity to extreme values or outliers (i.e., values that are 1.5 times greater than or less than the interquartile range). The measure also provided an avenue to explore box plots to help determine the skewness of the data.

This reasoning was enhanced when it was juxtaposed to the reasoning about the median; as one student stated, "It gets rid of all outliers." Since the context helped the students' intuitions surface regarding measures of central tendency, it is important for teachers to explore each of these lines of reasoning with their students. The context also provides an opportunity for students to determine the suitability of each measure of central tendency by investigating the affordances and limitations of each.



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USING MULTIPLE REPRESENTATIONS

We chose to explore data both graphically and numerically to leverage middle school students' intuitions of measures of central tendency. More specifically, we used multiple representations of data (i.e., tables, histograms, and box plots) that drew students' attention to various measures of center. Furthermore, students were able to arrive at different conclusions on the basis of the various representations, ultimately making connections across representations and measures of center. In so doing, we were able to use the key elements of statistical problem solving to link a simple estimation task to one that uses the measures of central tendency and dispersion to answer questions based on data (Watson and Wright 2008). By having students make estimates we make the case that middle school teachers have the potential to engage their students in meaningful mathematical ideas including critical content foci found in *Principles and Standards,* such as *selecting and using appropriate statistical methods to analyze data* (NCTM 2000), and in the Common Core State Standards for Mathematics, such as *summarizing and describing distributions* (CCSSI 2010).

Middle school mathematics teachers often search for tasks that introduce and connect multiple concepts to their students. Rather than covering concepts separately, these instructional tasks allow a teacher to actively engage students in the learning process while emphasizing various mathematics topics coherently as done in real life (Hodge 2009). Furthermore, NCTM's *Principles to Actions: Ensuring Mathematical Success for All* (2014) highlights the importance of making curricular decisions that connect and extend mathematical ideas across various topics. This introductory estimation problem can be used to bridge students' intuitions toward investigating data that they have generated both graphically and numerically. The Jelly Bean problem and others like it are often engaging to students, offering contexts to understand numbers and see their usefulness in the real world by having students explore a problem in a variety of ways and demonstrate persistence in problem solving.

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