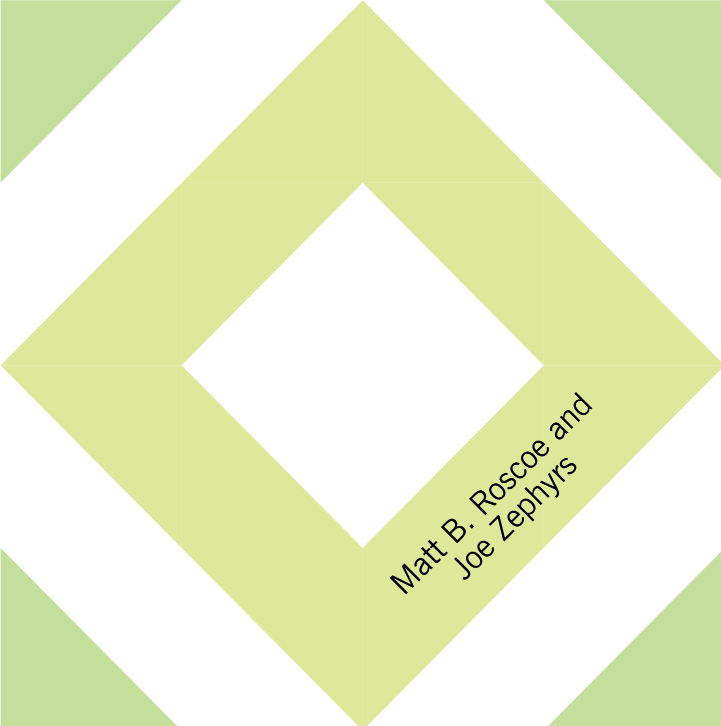


Matt B. Roscoe and Joe Zephyrs

QUILT BLOCK

Pull on the threads of congruence and similarity in a series of lessons that explores transformational geometry.

Geometric transformations have long been topics of middle school mathematics. Generations of middle school students have learned to reflect, rotate, and translate geometric objects. Historically, though, the mathematics of "movement" might have been considered a departure from other more central middle-grades geometric content areas, such as measurement, congruence, and similarity. But in the era of the Common Core State Standards for Mathematics (CCSSI 2010), the study of reflection, rotation, and translation have been given special importance. Consider the introduction to the high school geometry domain in the Common Core State Standards for School Mathematics (CCSSM):

> The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes. (CCSSI 2010, p. 74)

The document makes it even more explicit in the next paragraph, describing the approach to congruence, in which "two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other" (p. 74). Like congruence, the study of similarity in high school also rests on students understanding transformation by defining similarity as a sequence of rigid

SYMMETRIES

motions followed by a dilation, which “lead[s] to the criterion for [angle-angle] triangle similarity” (p. 74). These passages make it clear that one crucial element of students’ success in high school geometry is a firm understanding of transformation, which must be acquired in the middle grades.

One difficulty associated with teaching transformations in the middle grades is the facilitation of inquiry-based learning environments in which students explore and investigate properties of transformations in objects of their own creation. The study of transformation is a fertile setting in which students can make sense of problems, reason abstractly, construct viable arguments, and look for structure, all practices that are encouraged in the Common Core’s Standards for Mathematical Practice (2010). In addition, the study of transformations has long been associated with tools of inquiry (witness the use of patty paper, the Mira™, and interactive geometry software). But few are the opportunities where student-generated examples are the focus of mathematical investigation. To this end, we created a series of mathematical tasks that provided students with the opportunity to construct understandings of transformation through an investigation of quilt block symmetries. The following is a description of these tasks and a demonstration of their use in an eighth-grade classroom.

THE SETUP

Imagine that you are a quilt maker. Each quilt you construct is composed of individual quilt blocks, which are squares that are filled and colored. Suppose that

Fig. 1 A quilt block grid and shading restrictions were given to students.

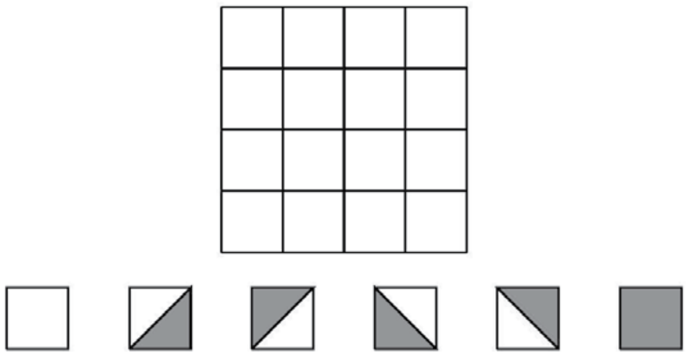
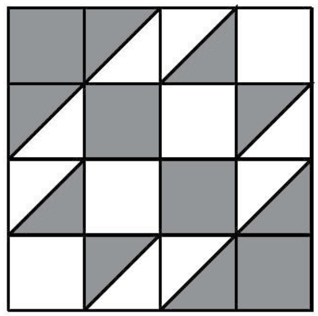


Fig. 2 This example illustrates a completed quilt block.



your quilt-making process restricts each quilt block to a 4 × 4 grid of squares that are filled in one of six ways, as shown in **figure 1**. One possible quilt block constructed according to these "rules" is shown in **figure 2**.

Since each quilt block is constructed in the shape of a square it can have, at most, the symmetries of a square:

- 90 degree rotational symmetry
- 180 degree rotational symmetry
- 270 degree rotational symmetry
- Reflective symmetry in a line defined by the midpoints of opposite sides of the boundary square
- Reflective symmetry in a line defined by the opposite vertices of the boundary square

Fig. 3 A red X helped students keep track of rotation, to then identify rotational symmetry.

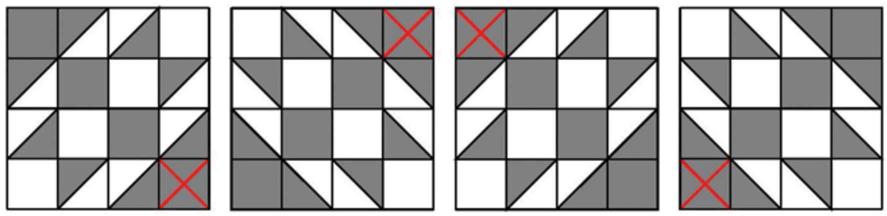
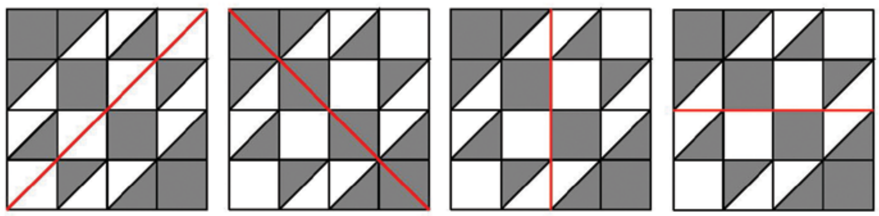


Fig. 4 Reflective symmetry was also examined and identified.



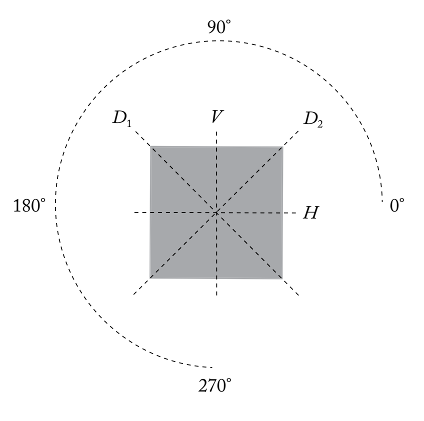
With these possibilities in mind, we can "classify" the example found in **figure 2**. The quilt block only reproduces itself when rotated 180 degrees, not at 90 degrees or 270 degrees. Thus, the quilt block does not possess 90 degree or 270 degree rotational symmetry, but does possess 180 degree rotational symmetry. This result is shown in **figure 3**. Note that a red "X" has been added to one cell of the quilt block to clarify how the block has been rotated.

The example quilt block can also be analyzed for reflective symmetry. We see that the block can be reflected across either of its two diagonals and reproduce itself. However, when the quilt block is reflected across a line formed by either of the pairs of opposite midpoints of the boundary square, the block fails to reproduce itself. We conclude that the example block has two reflective symmetries in the lines defined by the diagonals of the boundary square. This result is summarized in **figure 4**.

After considering every possible line and rotational symmetry, we can conclude that the example quilt block can be classified by its symmetries as "D1, D2, 180" because it has two reflective symmetries in both diagonals of the boundary square and 180 degree rotational symmetry. The exercise opens the door to a world of new mathematical questions:

- How many D1, D2, 180 quilt blocks exist?
- What other types of quilt blocks exist?
- If a quilt block has two different

Fig. 5 The available symmetries of the square are highlighted by this diagram.



reflective symmetries, does it always have a rotational symmetry?
- If a quilt block has a rotational symmetry, does it always have at least one line symmetry?

These questions and others were explored by eighth-grade students at a Montana middle school.

DAY 1: INVESTIGATING AND CLASSIFYING

On the first day of the activity, students were given a blank quilt block like that shown in **figure 1**. After a brief discussion of the restrictions associated with filling each of the 16 cells, each student was asked to create his or her own design. Immediately, students began designing a personalized quilt block. After about ten minutes devoted to creation, the mathematical discussion began. Students were first challenged to compare their quilt block with a partner's block. Partners were asked to discuss how the two examples were similar and different in terms of the symmetries present in each. Many interesting conversations arose, which naturally primed a classification scheme that was motivated by the question, "What sorts of symmetries can a quilt block possess?"

Reflective symmetries were the first to be recognized. Students decided that each block could have one of four line symmetries: horizontal, vertical, and two diagonals. Some students objected to the use of horizontal and vertical as "labels" for lines of symmetry because of the problem of orientation. That is, what one might call a horizontal line of symmetry another might call a vertical line of symmetry under a different orientation. It was decided that the ambiguity could be resolved by recognizing that the two symmetry labels were interchangeable.

Rotational symmetries were noticed next. This transformation seemed less intuitive to students. Students identified rotational symmetries of 90, 180, 270, and 360 degrees. We pursued a discussion of 360 degree rotational symmetry with the question, "Which quilt blocks that you have constructed possess 360 degree turn symmetry?" When all answered affirmatively for all blocks, we pressed students further, "Would every possible quilt have 360 degree rotational symmetry?" Students reasoned that since a 360 degree turn of any quilt block is a "full turn," and since every full turn returns a quilt block to itself, all blocks would possess this symmetry. Further, since the full 360 degree turn returns the quilt to its original orientation, it was the same as a 0 degree rotation, that is, no rotation at all. For these reasons, students decided not to "count" 0 degree and/or 360 degrees in their classification system. It should be noted that disallowing 0 degree or 360 degree turn symmetry agrees with most elementary school definitions of rotational symmetry as being strictly less than 360 (i.e., Billstein, Libeskind, and Lott 2010, p. 980). However, in the modern mathematics of group theory, the 0 degree rotation is included as the identity transformation—behaving in a similar manner to 0 in addition or 1 in multiplication.

Once a list of these available symmetries had been generated, the class decided that these symmetries would be symbolically represented according to the following:

- 90: 90 degree rotational symmetry
- 180: 180 degree rotational symmetry
- 270: 270 degree rotational symmetry
- H: Reflective symmetry in a horizontal line (interchangeable with V)
- V: Reflective symmetry in a vertical line (interchangeable with H)
- D: Reflective symmetry in a diagonal line (two possible)

Again, here we note that the student-generated classification system is at odds with modern group theory in its classification of diagonal line symmetry. Whereas the student-generated system labels both diagonal symmetries with the same algebraic representative, D, modern group theory would prefer each diagonal to be distinguishable similar to horizontal and vertical labels (i.e., H and V). Teachers who are re-enacting this lesson might opt to lead students in this direction. To this end, a summary of the available symmetries of the square, consistent with modern group theory, is displayed in **figure 5**. Our pedagogical decision not to promote this level of specificity was an attempt to allow students to build and test a system sufficient for the classification of the quilt blocks they had constructed; the system they invented was suitable for this task.

With the classification system decided, students were asked to classify their quilt blocks. Among the blocks constructed by the fourteen students, the following types were found:

- D, D, 180
- H, V, 180

Fig. 6 These 28 quilt block designs were given to students to sort by symmetry.

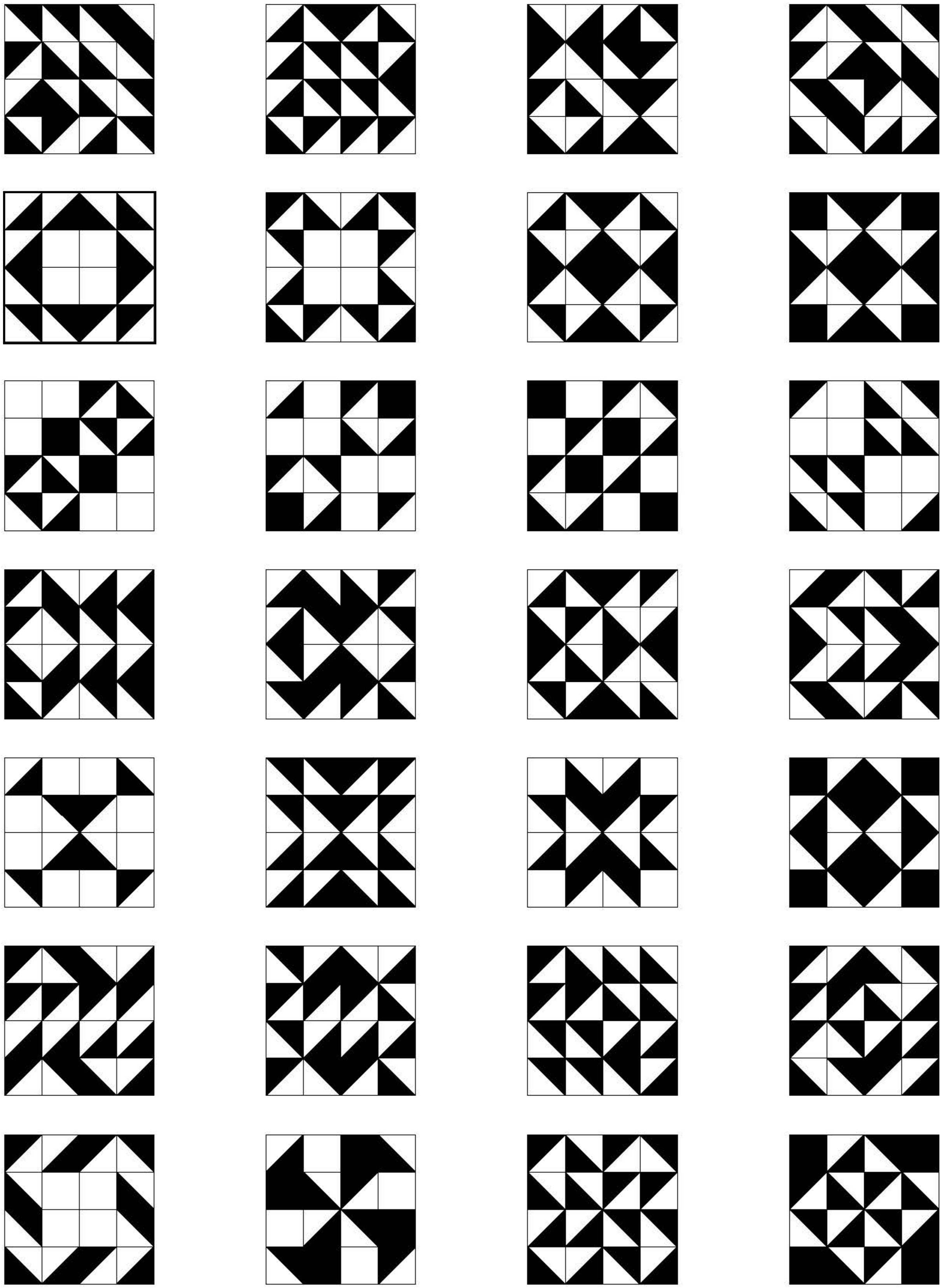
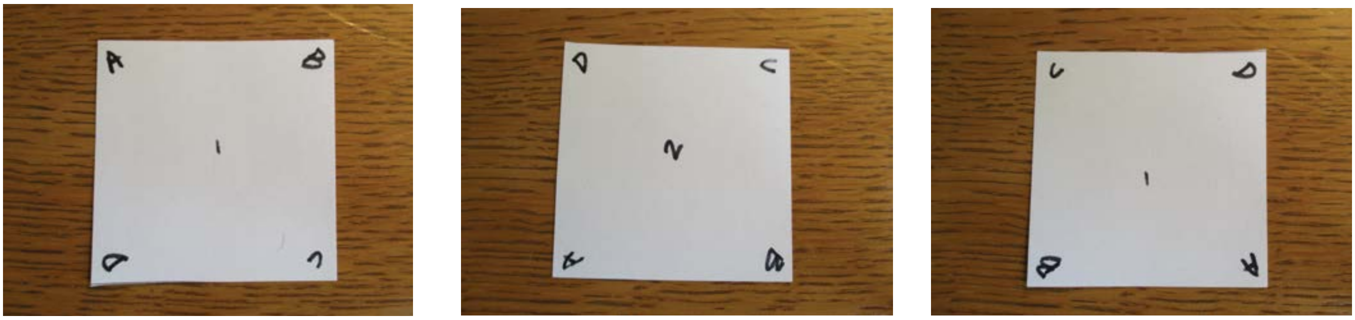


Fig. 7 A composition of horizontal and vertical reflection yielded a 180 degree rotation.



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(a)
The original orientation

(b)
Reflected in a horizontal line

(c)
A reflection of (b) in a vertical line yielded the same result as a 180 degree rotation of the original orientation

- H
- H, V, D, D, 90, 180, 270 (also called "All")
- 90, 180, 270
- None

Day 1 concluded with a brief discussion of questions for future inquiry, the most pressing of which was the open question of the existence of other classes of quilt blocks.

DAY 2: SORTING QUILT BLOCKS AND CREATING SETS

Students' newly acquired knowledge of quilt block classification was reinforced on day 2 in a quilt-block-sorting task. A Mira, a transparent mirror-like device that aids in the analysis of reflective transformation, was given to each group as a tool for inquiry for the activity. Pairs of students were then given a complete set of 28 quilt blocks, each block having a unique number from 1 to 28 written on the back. They were instructed to use the classification system that they had developed on day 1 to classify each of the quilt blocks into sets with exactly the same symmetries. These 28 quilt blocks are displayed in **figure 6**. Readers are encouraged to attempt this classification task before reading on.

Although students were quick to begin the task, the time to completion was highly variable. The fastest groups finished in about fifteen minutes, whereas other groups took nearly twice that time. Groups that finished early were challenged to investigate further and given the following prompts:

- Each quilt block set should have the same number of members. Do all your sets have the same number of members?
- Did you discover any new sets to add to those we discovered yesterday?
- Do you think that any other sets exist? Why or why not?

Disagreements gave students an opportunity to either critique or defend their mathematical understandings.

Once all groups had finished the quilt block classification task, results were shared. Although most pairs of students found similar results, some disagreed. These disagreements gave students an opportunity to either critique or defend their mathematical understandings. Eventually these conflicts were resolved, and the class came to a group consensus. For the reader who has attempted the task, each row of quilt blocks found in **figure 6** has the exact same symmetries. Using the students' previously developed symbols and ordering by row number in **figure 6**, seven different symmetry classes of quilt blocks were found, each with four members:

- D
- H, V, D, D, 90, 180, 270 (also called "all")
- D, D, 180
- H
- H, V, 180
- 180
- 90, 180, 270

Students noted that two new classes had been "discovered" in the activity: D and 180.

Once the sorting activity was completed, students were assigned a homework task. Each student was given a sheet of empty quilt blocks and instructed to create a set of 7 quilt blocks, 1 of each type that was discovered in the sorting activity. Students were told that their 7 examples would become part of a classroom set of seven quilts, each

Fig. 8 This time sequence shows the students at work in the final quilt sorting activity.

(a) (b) (c) (d)

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consisting of quilt blocks of the same class that would all be sorted once they were completed. Anticipating this future use, it was decided that each student would complete his or her set using a single color so that each quilt block's maker could be identified by color and so that each final quilt would be aesthetically multicolored.

DAY 3: LOOKING FOR OTHER SYMMETRY CLASSES

On the first day of the activity, we discovered 5 symmetry classes that were represented by student-generated examples. During the sorting activity, 2 more classes were discovered. A few students started to ask, "Why do we only have 7 of them?" and "Are there any others?" Day 3 was devoted to answering these questions.

At the beginning of class, every student was given a square note card and instructed to label the four corners of the front of the card A, B, C, and D, moving counterclockwise, starting from the top left. They were then asked to label the four corners of the back of the card as A, B, C, and D as well, so that each physical corner of the card had the same label. To distinguish easily between the sides, a student recommended that we place a 1 in the center of the first side and a 2 in the center of the second side. An example was created in front of the class to avoid mistakes in labeling (see **fig. 7**).

Students were then asked to demonstrate their understanding of each of these symmetries using the note card manipulative. Students demonstrated 90 degree rotational transformation by making a one-fourth turn counterclockwise. Similarly, they demonstrated horizontal reflection by flipping the card over across its horizontal axis. Once each movement corresponding to a transformation of the square was familiar, students were encouraged to explore the effect of repeating a transformation or the effect of combining two dissimilar transformations. They were asked, "Would new transformations arise?"

Students were encouraged to explore the effect of repeating a transformation or the effect of combining two dissimilar transformations.

Starting with their card in its original orientation (A, B, C, and D, clockwise from top left), students first explored the properties of a quilt block with 180 degree rotational symmetry by applying a 180 degree rotational transformation (always counterclockwise) to their card. Applying the transformation a second time returned the card to its original orientation. Students discussed this result. They decided that since repeated applications of 180 degree rotations only produced 180 degree rotations, or resulted in the card being in its original orientation, a "quilt block class" (some preferred the term "family") possessing only 180 degree rotational symmetry was "allowed," since combining 180 with itself failed to produce any "new" transformations. Similar results were found for a horizontal reflective transformation (interchangeable with V) and a diagonal reflective symmetry. Quilt block families possessing only horizontal reflective symmetry or only diagonal reflective symmetry were also "allowed."

We posed this question to the class, "Why did we not list a family with solely 90 degree rotational symmetry?" Using their cards as a manipulative, students began by rotating their card 90 degrees. Students quickly noticed that they would need a 270 degree rotation to return to the

Fig. 9 Some of the 7 finished quilts made by the students, each labeled with the symmetry of the blocks.

(a) H, V, 180°

(b) 90°, 180°, 270°

(c) D, D, 180°

(d) H, V, D, D, 90°, 180°, 270°

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original orientation. An important connection was made that 90 degrees and 270 degrees were tied together and that 90 degrees alone did not belong on the list. But as one student quickly pointed out, the list did not include a group consisting exclusively of 90 and 270. This was resolved by agreeing that if a quilt block had 90 degree rotational symmetry, then it must also have 180 degree and 270 degree rotational symmetries because 180 can be produced by putting two 90 degree rotations "together" and 270 can be produced using three 90 degree rotations "together." This action of "putting together" available symmetries to produce new ones was termed "composition" or "composing." We concluded that any quilt block with 90 degree rotational symmetry must also have 180 degree and 270 degree rotational symmetry. This confirmed the quilt block class "90, 180, 270" and disallowed many other types (for example, the class "90, 270" cannot exist).

Other logical necessities were explored in a similar fashion. By composing a horizontal and vertical reflection, as in **figure 7**, it was discovered that a 180 degree rotation was produced, which confirmed the "H, V, 180" quilt block class and disallowed many other types (for example, the class "H, V" cannot exist). By composing the two available diagonal reflections, it was discovered that 180 degree rotation was also produced, confirming the class "D, D, 180" and disallowing many other types (i.e., D, D). Continuing in this fashion and using the card as a manipulative, the class validated the presence of the 7 types of quilt blocks discovered on day 1 and day 2 while systematically excluding any others. This activity was particularly rich in exploration and discovery, although some students struggled with the level of abstraction. For this reason, having students work in mixed-ability groups on this activity is highly advised.

Students will never look at a basic quilt block in the same way again after working through this activity.

DAY 4: COMPLETING THE CLASSROOM QUILT CATEGORIZATION

On day 4, students arrived with their complete set of 7 quilt blocks, 1 of each type. Since there were fourteen students in the class, and it was agreed that either a 1 × 14 or a 2 × 7 quilt would not be very pleasing to the eye, we had constructed two extra sets each, with 7 quilt blocks, which were added to the students' quilt blocks so that the 16 total sets, once sorted, could be organized into 7 quilts each arranged in a 4 × 4 array.

The 16 × 7 = 112 quilt blocks were gathered together and shuffled at a large table. Students were then given the task of sorting the 112 examples into 7 groups of 16 quilt blocks whose members all shared the exact same reflective and rotational symmetries. **Figure 8** displays a time-sequence view of this process as it evolved in the classroom. Notice that students used columns ordered by color as a strategy to analyze the "completeness of categorization." Using this structural approach to the group effort, it was determined that two quilt block sets had "duplicates," thus making it necessary for the owner to "rebuild" a quilt block of the missing type. Once these issues had been resolved, students collected each multicolored set and glued them together to produce a classroom display of 7 quilts composed of quilt blocks possessing the exact same reflective and rotational symmetries. Finished quilts are shown in **figure 9**.

THE IMPORTANCE OF GEOMETRY

Quilt blocks provide an interesting and aesthetic environment in which math teachers can support inquiry-based learning activities targeting transformational geometry. This knowledge is vital in the era of Common Core (2010), because students' understanding of the rigid motions is identified as prerequisite knowledge to their deepening studies in congruence and similarity in high school. But perhaps most attractive is the opportunity that quilt blocks offer for students to practice mathematics as mathematicians do: to make sense of problems and persevere in solving them, to reason abstractly, to construct viable arguments, to critique the reasoning of others, and to look for structure. The importance of the Standards for Mathematical Practice (CCSSI, pp. 6–8) cannot be understated in terms of their potential to renovate the mathematical inheritance of the next generation of learners. We are confident that the activity described herein has contributed to this renovation by offering students the opportunity to make sense of quilt block symmetries through tasks that encourage mathematical investigation, creation, and sense making.

Students' understanding of the rigid motions is identified as prerequisite knowledge to their deepening studies in congruence and similarity.

REFERENCES

Billstein, Rick, Schlomo Libeskind, and Johnny Lott. 2010. *A Problem-Solving Approach to Mathematics for Elementary School Teachers*. Boston, MA: Addison-Wesley.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

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Let's Chat about Symmetry

On Wednesday, August 17, at 9:00 p.m. EDT,

we will expand on the article "Quilt Block Symmetries" (pp. 18–27), by Matt B. Roscoe and Joe Zephyrs. Join us at **#MTMSchat**.

We will also Storify the conversation for those who cannot join us live. Our monthly chats will always fall on the third Wednesday of the month.