All Talk and More Action

Students' understanding of proportional relationships become apparent during an algebra project that focuses on constructing mathematical arguments.

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How better to begin the study of linear equations in an algebra class than to determine what students already know about the subject? A seventh-grade algebra class in a suburban school undertook a project early in the school year that was completed before they began studying linear relations and functions. Most of the students had experienced the Connected Mathematics 3 curriculum during their sixth-grade year, including the Moving Straight Ahead unit (Lappan et al. 2006). In addition, a few students had taken an online course during the summer to qualify for the honors algebra course. The project, which might have been assigned at the end of a unit in a traditional class, was assigned *before* the start of the unit. It is detailed in **figure 1**.

By assigning the project early in the unit, students' prior learning and initial understanding became apparent, offering both a subject for mathematical reasoning and the construction of mathematical arguments. The objective of the project was to determine students' understanding of proportional relationships regarding ratios, proportions, and equation representations that are more formal and typically part of an algebra 1 curriculum. The task supported the development of an essential understanding from *Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning for Teaching Mathematics in Grades 6–8* (Lobato and Ellis 2010): "A rate is a set of infinitely many equivalent ratios" (p. 42). Further, it aligned with the Common Core's algebra standards (CCSSI 2010) related to


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creating equations that describe numbers or relationships.

OVERVIEW AND RELATION TO *PRINCIPLES TO ACTIONS*

The lesson rests on the eight mathematics teaching practices described in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014). (See the list of practices in **fig. 2**.) Since the purpose of engaging students in the task before instruction was to elicit and use evidence of student thinking to guide further instruction, both the projects and the discussion that followed submission provided evidence to direct future lessons on the subject. Consistent with Wiliams' (2007) strategies, gathering evidence of student thinking and interpreting the information were the work of the teacher during the whole-class discussion.

Students assembled near the front of the room, and the discussion began. Soon after, the enormous variety in the depth of understanding within the class became clear. Although some students spoke eloquently about linear relations, making connections easily among tables, graphs, words, and equations, others conveyed only a partial understanding about proportional relations and their representations. Teachers may expect that students entering an honors algebra class would hold similar understandings of concepts; however, this task and the related discussion uncovered the varying depth of understanding among the students and an assortment of misconceptions related to the topic. These differences allowed (1) students to engage in a rich discussion that included challenging the thinking of their peers, and (2) their teacher to focus instruction in subsequent classes.

Fig. 1 This project was intended to determine seventh graders' prior understanding of proportional relationships.

Algebra Project

1. Choose a context for your project that will represent a proportional relationship. Proportional context: ______________________________

   Choose a related context that is a nonproportional relationship. Nonproportional context: ______________________________

   (For example, there is a proportional relationship between the number of gallons of water and the number of minutes when irrigating a garden from a hose that flows at 2 gals./min.; there is not a proportional relationships between the number of gallons of water and the number of minutes when dumping a 5-gallon bucket of water on the garden, followed by watering with a hose that flows at 1 gal./min.)

2. Make a table of data containing 5 coordinate pairs for each context.

3. Graph your data using graph paper.

4. Write the formula for your relationship.

5. Write a problem that could be solved using the information.

6. Make a poster with all the information in parts 1–5.

Fig. 2 The lesson built on the foundation of the mathematics teaching practices, found in *Principles to Actions*.

- Establish mathematics goals to focus learning.
- Implement tasks that promote reasoning and problem solving.
- Use and connect mathematical representations.
- Facilitate meaningful mathematical discourse.
- Pose purposeful questions.
- Build procedural fluency from conceptual understanding.
- Support productive struggle in learning mathematics.
- Elicit and use evidence of student thinking.

Source: NCTM 2014, p. 10

PREPARING DISCUSSION QUESTIONS

Between the project submission and the whole-class discussion of the projects, the teacher analyzed student work. She carefully scrutinized it for instances of partial understanding, parallel and divergent thinking, and opportunities to advance student thinking. Teacher notes regarding indicators of student thinking guided the *planning of questions* that were used in *facilitating meaningful mathematical discourse* during the whole-group discussion (Stein et al. 2008). The questions posed guided the flow of the lesson and ensured that the discussion included opportunities for assessing students' conceptual understanding; it also allowed students to make connections among the representations. **Figure 3** shows, in preplanned order, questions that were written in response to student work showing partial understanding. The questions had several purposes. They were designed to—

- orchestrate the discussion;
- focus on specific characteristics that highlighted concepts related to the task;
- focus student attention on trends and differences evident in students' work;
- elicit discussion; and
- qualify as "probing" in nature (Boaler and Brodie 2004).

SELECTING THE TASK

Central to implementing the teaching practices described in *Principles to Actions* was the choice of a task. This high-cognitive-demand task was meant to offer students opportunities to make connections among mathematical representations and to participate in a robust discussion during the class that followed the project submissions (Stein et al. 2000). The task also gave students the chance to make choices and gave the teacher the opportunity to share authority for learning with students (Engle and Conant 2002). Choosing the context of the problem demanded that students deliberately consider the definition of a proportional and nonproportional relationship. The request for multiple representations (words, table, graph, and equation) presented an opportunity for students to make connections among representations as they completed the task and, more important, to prepare students to *use and connect representations* during their engagement in the whole-group discussion. **Figure 4** illustrated student work containing multiple representations and a consistent message, but this student

Fig. 3 Planned questions were used to orchestrate a whole-class discussion.

1. You recognized that linear graphs representing proportional relationship go through (0, 0). Why is that? How did you know?
2. Breanna, Tori, Spencer, and Lauren included a table in their projects. Can you speak to the value you found in producing a table? (Error: Arya's table versus graph)
3. I notice that in Ciera's and Nathan's graphs, some lines intersect and some lines do not. Why is that?
4. Could we predict from the tables whether the lines would intersect or not?
5. What features do the equations representing parallel lines have in common? (Nathan and Lauren)
6. What features do the equations representing nonproportional equations have in common? (Kyra and Elisabeth)
7. What do you notice is different about these graphs? (The intercept is on the *x*-axis versus the *y*-axis.)
8. What is the significance of the point of intersection of the lines?

Fig. 4 This student used multiple consistent representations, but a distinction between dependent and independent variables was not evident.

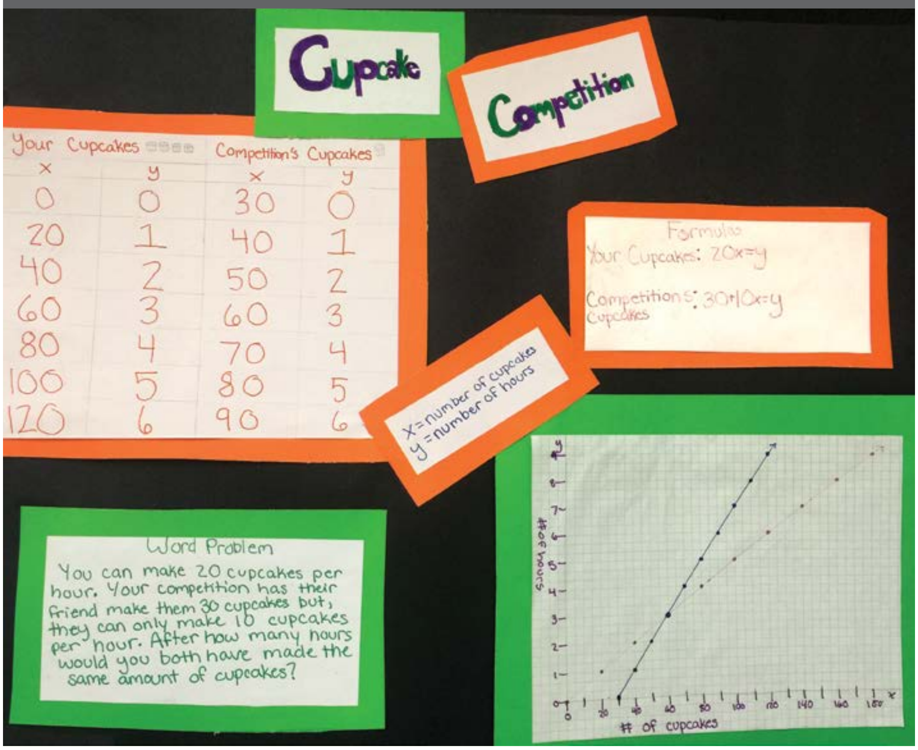

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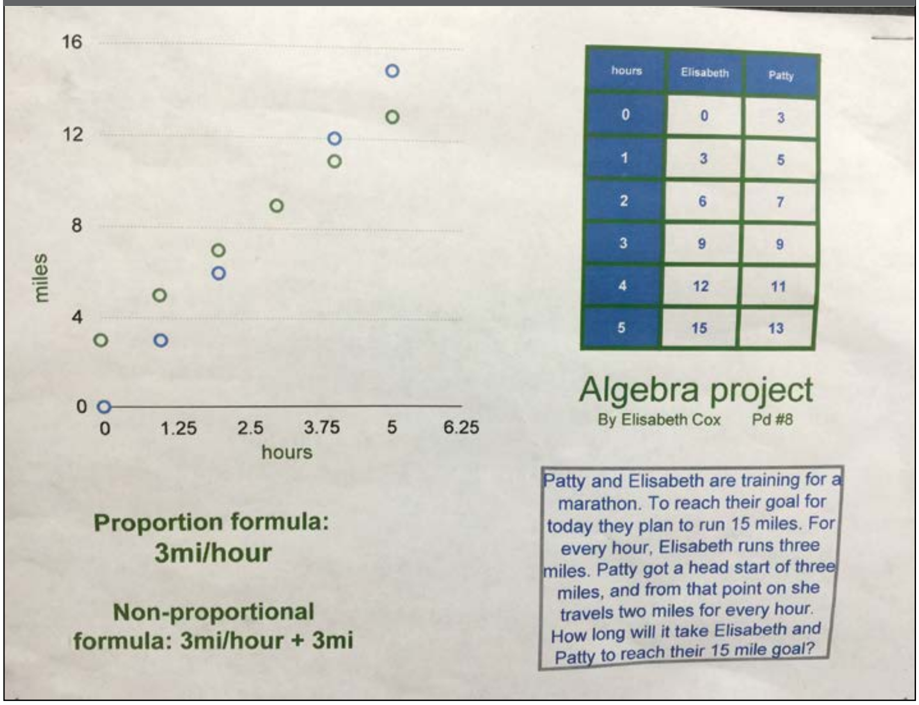
Fig. 5 Elisabeth's project did not contain an equation.



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had not yet made a distinction between dependent and independent variables, an anticipated student error.

The challenge was nonalgorithmic, requiring students to understand the nature of the mathematical concept. The purposeful *implementation of a task that promotes reasoning and problem solving* gave students multiple entry points. Students could engage with the task by completing a portion of the assignment (such as completing a table alone), by describing a context, or by attempting the entire task. Elisabeth's project (see **fig. 5**) illustrated work of a student who engaged in the task without including an equation, a representation with which she was unfamiliar.

ORCHESTRATING THE DISCUSSION

The teacher began the discussion by saying, "I notice that all of you had a proportional and a nonproportional example; I noticed that every single one of you had the proportional line go through (0, 0). So, can we have a little discussion about why that is, and how you knew it?" This discussion uncovered a misconception from one student, Julian, as well as an understanding of the features of a proportional relationship from several other students. Julian indicated that every proportional relationship would go through (0, 0) and that every y-value would be the same as every x-value. His peers disagreed, as indicated below:

Roberto: In a proportional relationship, the y-axis is the x-axis times something. So, if the x-axis is zero, the y-axis would also be zero.

Teacher: Does anybody want to add on?

Julian: If it's a proportional relationship, the x-axis always equals the y-axis.

Teacher: It equals the y-axis. So then, if x is one, y is one. Is that what you're saying? [Several seconds pass.] Rob, you are shaking your head no.

Mick: I don't think it's true, mainly because [if] something is going at a steady rate, it doesn't have to be one-and-one. It could be one-and-five and still be a proportional rate.

Teacher: So, what you're saying is that if it's proportional, it goes over one and up one every time; and Mick, you're saying no, that doesn't happen. Does anyone have a graphic example where that doesn't happen? Where it isn't one-to-one, x and y?

Ben: It goes up by a constant rate.

Lauren: Like, you're starting at zero and going up the same amount every time. You're not starting with anything, like, you're making $12 every hour.

Teacher: You're not starting with anything . . . . then it goes up a constant rate. [Hands raise.] Just talk. Jason?

Jason: Like in the one that I had, it looks like it's one-and-one, but I have it coming up at different rates on each side. Like I had it coming up two on one side and over one.

Brianna: You mean the line on the x- and y-axis.

Teacher: Julian, let's look at your project real quick. [Julian carries his poster to the front.] You had trees. How much does your tree that represents a proportional relationship grow? [Julian examines the graph.]

Julian: Ahhh [Pausing]. It grew by three inches for one year.

Teacher: So, it goes up three inches for every year. So, it's not a one-to-one ratio, but it's a constant rate.

This short excerpt described students engaging in productive struggle. Students listened carefully to the ideas of their peers to make meaning of the mathematics during discussion. Mick listened to Julian's idea regarding the values on the x- and y-axes being the same in a proportional relationship and commented in disagreement. Lauren, Jason, and Ben extended Mick's idea. Finally, Julian recognized the feature of his graph that is indicative of a constant rate of change. In this exchange, many students engaged in deep mathematical thinking through productive struggle. Purposeful questions posed by the teacher encouraged students to become fluid at the public thinking process and facilitated mathematical discourse that enabled students to shape the ideas of their peers. The teacher was no longer the single authority, and students were no longer accountable to her alone. **Table 1** summarizes the actions of the teacher that supported students' perseverance as well as the indicators of students' productive struggle.

The second planned question focused students' attention on the relationship among representations. The question was, "Breanna, Tori, Sean, and Laurie included a table in their projects. Can you bring your projects up and speak to the value you found in producing a table?"

Sean commented that the table helped him produce a graph. Tori commented on the way that the table helped him produce an equation, and he provided a personal example. Carl noted that the table helped him see a pattern of change. A follow-up question, "How would you know if the table would result in a straight line?" encouraged students to examine the four projects more carefully.

Fig. 6 Brianna's table and graph did not match. Her peers determined that she had made an error in plotting points.

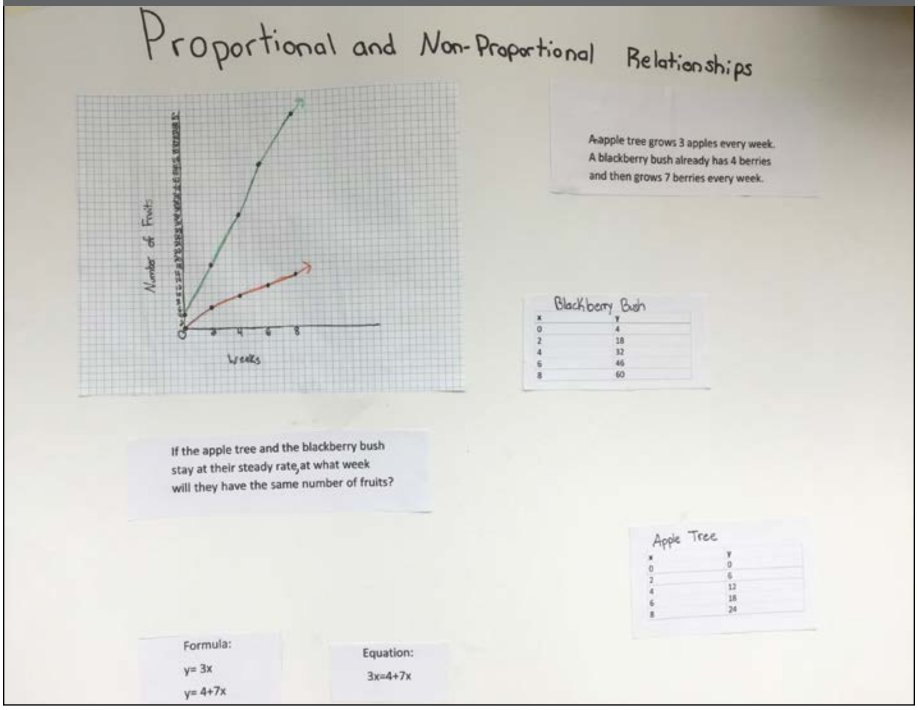

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Table 1 Teacher actions supported students' productive struggle and perseverance.	
Teacher Actions	**Indicators of Productive Struggle in the Students**
She provided a task of high cognitive demand before teaching a solution method. Both the task and its placement along the learning trajectory supported productive struggle.	
Her questions were purposeful, preplanned, and ordered to focus on specific characteristics that highlighted concepts related to the task. They focused attention on trends and differences evident in student work and were crafted to elicit discussion.	The student discussion included student explanations followed by contrasting views by peers. Peer questions focused on mathematical reasoning.
She valued the quality of student explanations, including the related vocabulary.	Students discussed their thinking in detail, which allowed others to agree and disagree. Students reflected on their own reasoning and that of their peers.
She shared authority and accountability with students. She did not determine the appropriateness of student reasoning, but shifted that responsibility to the students.	Students made their thinking public, creating a responsibility for peers to make sense of the reasoning.
During the discussion, she selected students to stand with their representations as the class focused on their work. Student work became a tool to use in comparing and contrasting mathematical ideas.	Students reexamined their own graphs and tables in response to teacher and peer questions.
She anticipated student misconceptions and planned ways to support them as they struggled.	

Brianna's project was included in the four because the information in her table did not match her graph, which was not a straight line. Students immediately noticed her error and at first attributed the "almost" straight line to Brianna's failure to use a ruler. However, within a few seconds, her error in plotting points was identified, and all agreed that plotting points from the table should produce a straight line (see **fig. 6**). The visual model produced by Brianna challenged students to connect representations and search for *conceptual understanding* regarding the meaning of having a constant rate of change, a necessary precursor to *procedural fluency* that would come later in the algebra 1 course.

Also apparent from the review of Brianna's project was that her question did not reflect an understanding of the significance of slope or rate of change. Her question suggested that the lines would intersect at some point, but in fact the y-intercepts and slope preclude them from intersecting when x- and y-values are positive.

Conversely, both Kyra and Sean's projects indicated a capacity to apply the knowledge of proportional relations to individually chosen contexts. Sean's project, shown in **figure 7**, combined with his comments during the discussion, indicated that he fluidly connected the relationship among the table, graph, and equation within the context of the problem he created.

Similarly, Kyra's project, shown in **figure 8**, indicated consistency among the representations, and her question, within her chosen context, indicated that she had some understanding of the meaning of the point of intersection of the lines. On the basis of the projects that Kyra and Sean submitted, their understanding appeared to differ widely from Brianna's conceptual understanding.

STRATEGIC INSTRUCTION AND SUPPORT

The revelation of the misconceptions held by individual students helped their teacher target additional support as the unit of study progressed. These examples of initial understanding encouraged the teacher to *establish specific mathematics goals to focus learning*. A table of anticipated understandings was created (see **table 2**) to focus teacher attention during subsequent lessons on the needs of small groups of students.

Using the information from the projects and the related discussion allowed for small-group instruction

Fig. 7 Sean's project and remarks during the class discussion showed his ability to apply proportional relationship knowledge to his problem context.

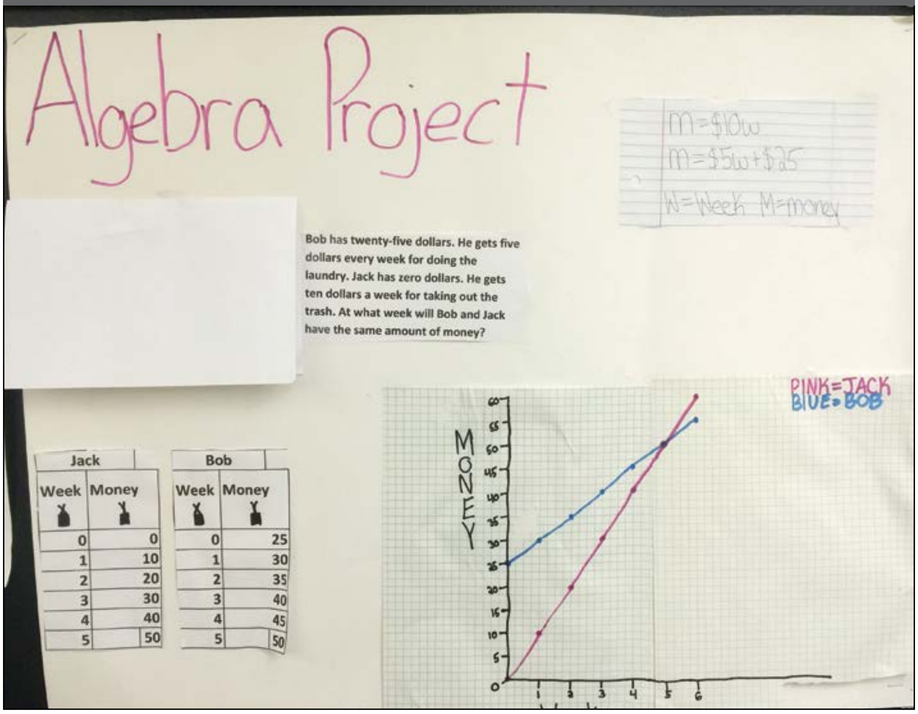

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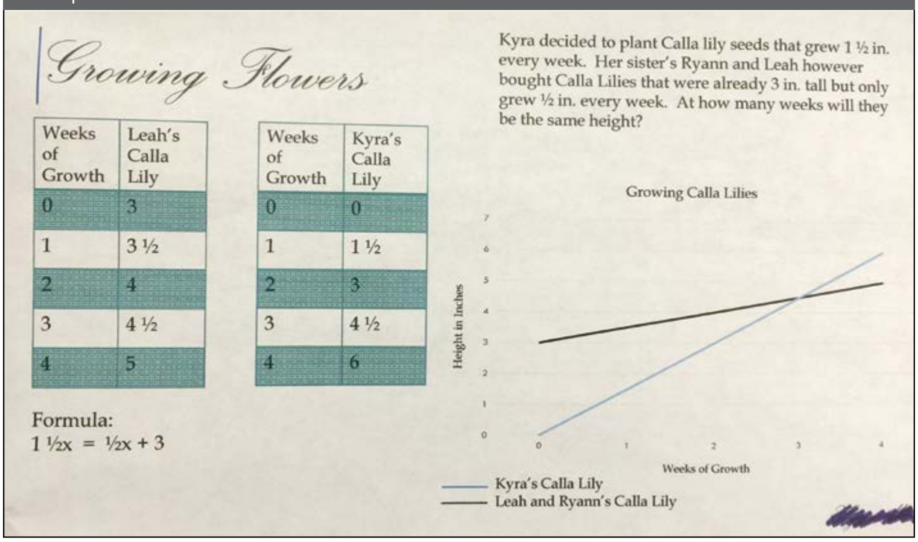
Fig. 8 Kyra's question and project showed that she understood the meaning of the lines' point of intersection.



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and additional whole-group discussion regarding proportional relations and linear functions. In addition, the information allowed for strategic choices regarding pairing of students for the completion of subsequent tasks. For example, because Brianna was unaware of the idea of dependent and independent variables, during the next task undertaken by students, she and Kyra were paired to encourage discussion regarding what variables were best placed on which axis.

Among the outstanding features of the whole-class discussion of the projects was the capacity of these students to engage with the thinking of their peers and to talk about the ideas. Despite some shortcomings with regard to the use of mathematics vocabulary, students were eager to share their thinking. Most students were engrossed in learning and were curious and interested in the project discussion.

Table 2 This table of anticipated understandings focused teacher attention on students' needs.

Anticipated/Initial Understanding	Student Names
Dependent/independent variable	
Consistent rate of change/slope relationship	
Significance of intersecting lines in the problem context	
Features of the equation that indicate a proportional/nonproportional relationship	
Features of the equation that suggest parallel lines	
Features of graphs of proportional/nonproportional relationships	
Relationship between tables of data and graphs	

Not every project was presented in the discussion, and that seemed to be just fine with students. The preplanned teacher questions allowed for a rich, meaningful discussion of concepts using commonalities and differences in the student work as focal points.

The project and the related discussion focused attention on the role that formative assessment plays in supporting learning. An important feature of the instructional processes present in the implementation of this project included establishing what students already understood related to linear functions to determine what needed to be done to move their learning toward a more sophisticated understanding of the topic. Connected Mathematics 3 seemed to prepare students well for a rich discussion and deep conceptual understanding of linear functions. Although this example includes a group of high-achieving students, designing instruction based on the way that students understand a concept can be applied to any group of students.

Let's Chat about Talk and Action

On Wednesday, October 19, at 9:00 p.m. EDT,

we will expand on the article "All Talk and More Action" (pp. 162–70), by Maryellen Williams-Candek. Join us at #MTMSchat.

We will also Storify the conversation for those who cannot join us live. Our monthly chats will always fall on the third Wednesday of the month.

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