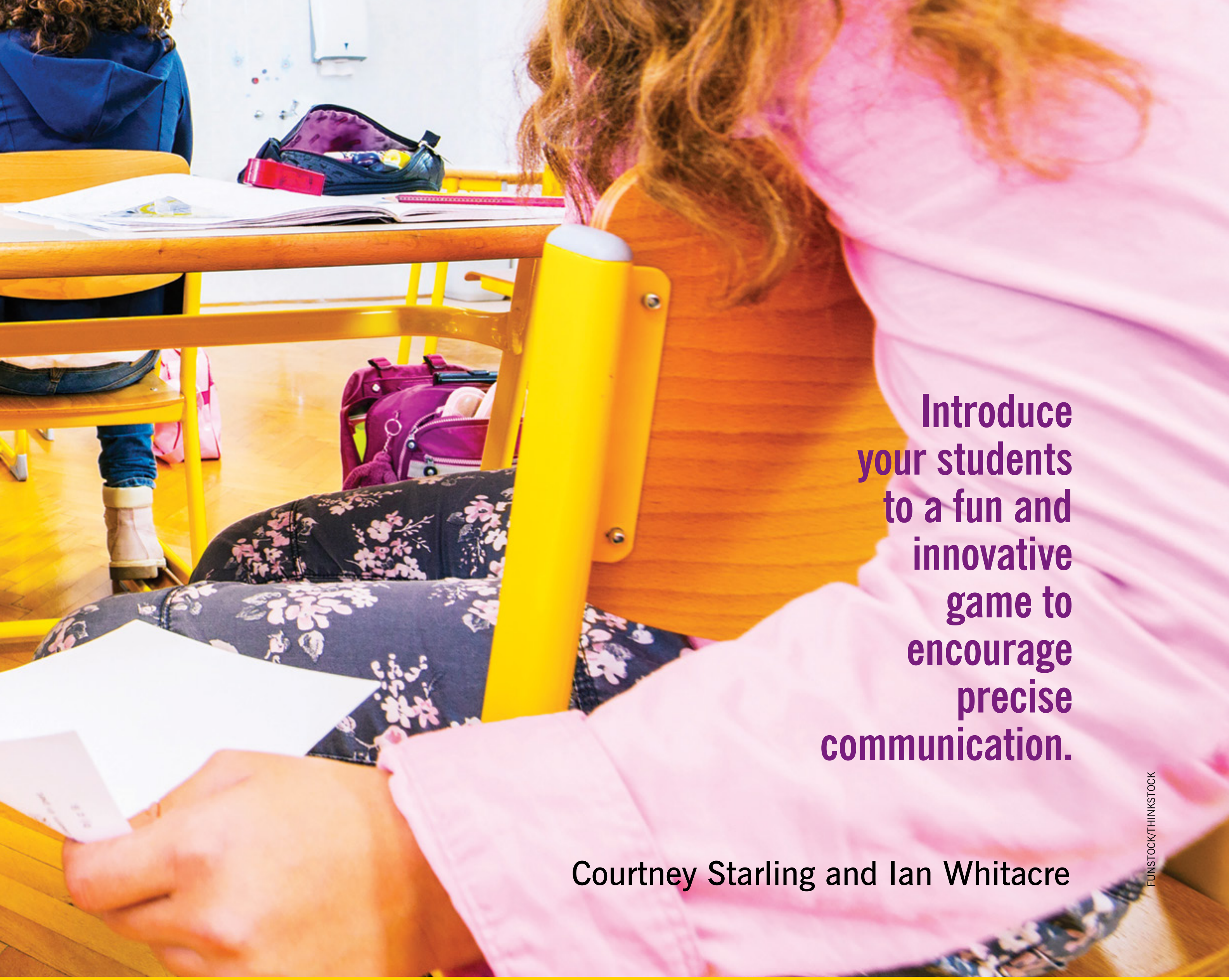
# Attending to Precision

Mathematics is a language that is characterized by words and symbols that have precise definitions (Devlin 1998). Many opportunities exist for miscommunication in mathematics if the words and symbols are interpreted incorrectly or used in imprecise ways. In fact, we find that imprecision is a common source of mathematical disagreements and misunderstandings between students, as well as between students and teacher.

To help students learn to communicate more precisely, we devised an activity called a Communication Game that involved relating trips along a number line to arithmetic equations. A key feature of this game is that it places students in situations that require precise communication with their classmates. Students find out for themselves how successfully they communicated, and they give feedback to their peers to help them communicate more precisely in the future. Communication Games, more broadly, can be incorporated in different mathematical domains.



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Introduce your students to a fun and innovative game to encourage precise communication.

Courtney Starling and Ian Whitacre

# with Secret Messages

## A CLASSROOM EXAMPLE

Our example of a Communication Game had its origins in an introductory unit on integer arithmetic. We used this unit with fifth graders who had no previous integer instruction. The unit involved reasoning about number line trips and relating such trips to arithmetic equations. A simple example of a number line trip would be to start at 5, go 3 spaces to the right, and end at 8. This trip would typically be expressed by the equation $5 + 3 = 8$. At the beginning of the unit, we restricted number line trips to the right side of zero so that we could establish foundational ideas that would be important when students began reasoning about negative integers. Although addition and subtraction of whole numbers were familiar topics for the students, relating such equations to number line trips required careful attention to the details of the correspondence. We found that students were imprecise in relating equations to trips. Therefore,

**Fig. 1** Group A's drawing for 5 + 20 = 25 is on the top; group B's question appears on the bottom.

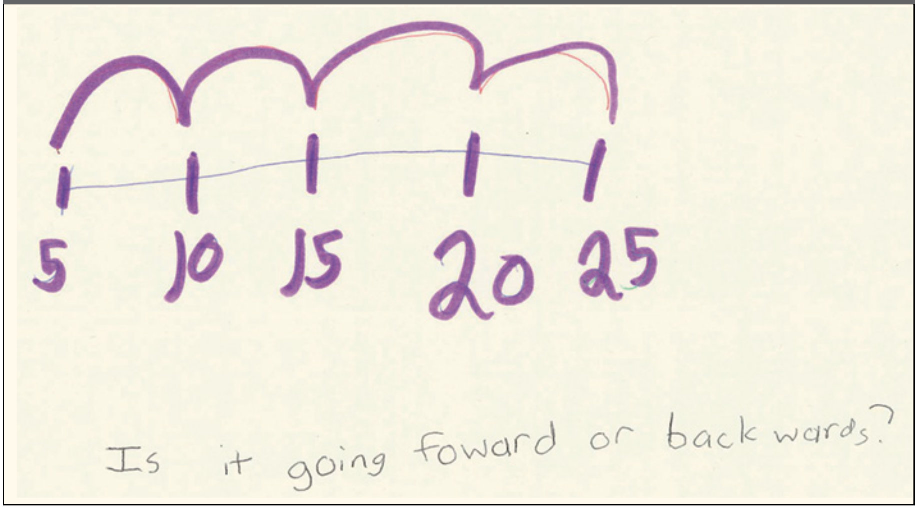
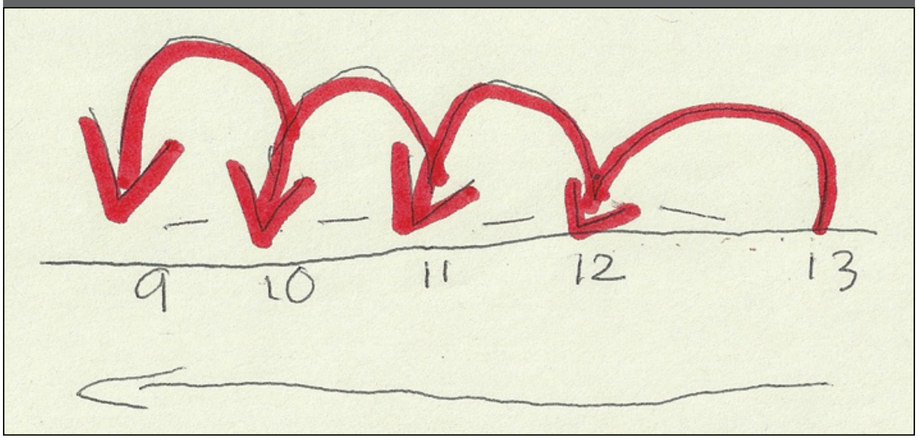



**Fig. 2** Group B produced this drawing for 13 – 4 = 9.




we introduced the Communication Game as a way to improve precision.

**Step 1.** Students were assigned to groups. Two "secret messages" (equations, in this case) were given to each group. Students had to communicate their secret messages to a partner group by "encoding" them as drawings of number line trips. This task required students to consider such issues as how to illustrate traversing distance on the number line and how the order of the addends would affect the drawings. In the first game, group A was given the messages 20 + 5 = 25 and 5 + 20 = 25; group B's messages were 9 + 4 = 13 and 13 – 4 = 9.

**Step 2.** Students worked in groups to encode their secret messages. Group A decided to distinguish its two equations by drawing the first trip from 20 to 25 and the second from 5 to 25. In this way, the students attended to one important issue in precisely relating equations to number line trips. On the other hand, they did not attend to other matters of precision, such as indicating the direction of their trips. (See **fig. 1**.) Group B, by contrast, focused on the issue of direction and decided to use arrows to show whether a trip was going to the right (as in 9 + 4 = 13) or to the left (as in 13 – 4 = 9). (See **fig. 2**.)

**Step 3.** Group A found it easy to interpret group B's messages because the direction of the trips was clear. Group B, on the other hand, was unsure about the intended direction of group A's trips. That group also questioned whether the trips were additive (as in 5 + 20 = 25) or multiplicative (as in 5 × 5 = 25). The groups did their best to decode the messages that they had received. They also prepared feedback to improve precision in future communications.

**Step 4.** The groups then met and held a conference to share their interpretations and reveal their secret messages. Group A had interpreted group B's messages as intended, whereas group B was unsure about the intended meanings of group A's drawings.

**Step 5.** The teacher helped to facilitate the conference by asking group B, "Is there anything that you would suggest changing or adding to their drawing that would make it more clear?" Student Kiyana suggested, "Adding arrows going this way or that way." She further explained, "This way to add [pointing to the right] and this way to subtract [pointing to the left] 'cause you go lower or higher." Other students agreed with Kiyana's suggestion.

In subsequent activities, it became the norm for students in the class to use arrows to indicate the direction of number line trips. As students played these games, we saw improvements in their attention to precision.

The episode above highlighted improvement in one aspect of precision:

indicating the direction of a number line trip. Other aspects were addressed in interactions and games that occurred later. For example, trips related to equations like –5 + 10 = ? were consistent with Kiyana's description of addition moving to the right. However, when students began exploring trips such as 10 + –5 = ?, they had to reason through the idea that adding a negative number would have the opposite effect as adding a positive number. In fact, this new possibility led to a discussion of the important point that two different equations could be used to represent the same trip (e.g., 10 – 5 = 5 looked the same as 10 + –5 = 5). In cases of adding or subtracting a negative integer, the class agreed that although there could be more than one correct equation, there were correct and incorrect correspondences between equations and number line trips. Thus, precision did not necessarily imply one unique answer. We used a Communication Game as a catalyst for these discussions. Whereas it was the students' responsibility to communicate effectively, we devised messages that would likely lead to miscommunication and give rise to important discussions.

In addition to the issue of direction, differences were found in how students illustrated how number line distances were traversed. Some students seemed to think in terms of representing the equation itself by showing the starting and ending points and then illustrating the distance traveled as one big jump. Other students used their drawings to represent the strategy that they had used to traverse the distance. For example, the drawing in **figure 1** showed the distance from 5 to 25 being traversed in jumps of 5 units, giving it a repeated-addition structure, as opposed to one big jump from 5 to 25. Thus, it became necessary to clarify whether the drawings were intended to simply match the original equation or to illustrate students' strategies. This Communication Game created opportunities for such matters to be negotiated and discussions to take place.

## STUDENT OWNERSHIP OF THE NEED FOR PRECISION

To appreciate the value of using a Communication Game, consider an alternative approach. Rather than giving students the opportunity to attempt to communicate secret messages to one another, the teacher could instead have emphasized the importance of precision and instructed students to always make arrows when drawing number line trips. Such an approach may have resulted in students drawing arrows; however, students would only have experienced the need to attend to precision as a rule imposed by the teacher. By contrast, when our students participated in Communication Games, they gained an appreciation for the need to be precise after receiving practical feedback from other students when they were not entirely successful in their communication. We believe that they experienced an intellectual need for precise communication (Harel 2013). When participating in Communication Games, students see the natural consequences of the precision of their encoded messages on their ability to communicate. We found that students were motivated to communicate successfully, so they worked to make their drawings more precise.

## WHAT IS A COMMUNICATION GAME?

A Communication Game is an engaging and productive activity that we have used to improve students' mathematical understandings while encouraging attention to precision. The goal is for each group of students to communicate one or more secret messages effectively to their partner group. We suggest groups of two to four students; however, group sizes may vary depending on the nature of the messages and the intended discussion. A Communication Game creates a need for precision in

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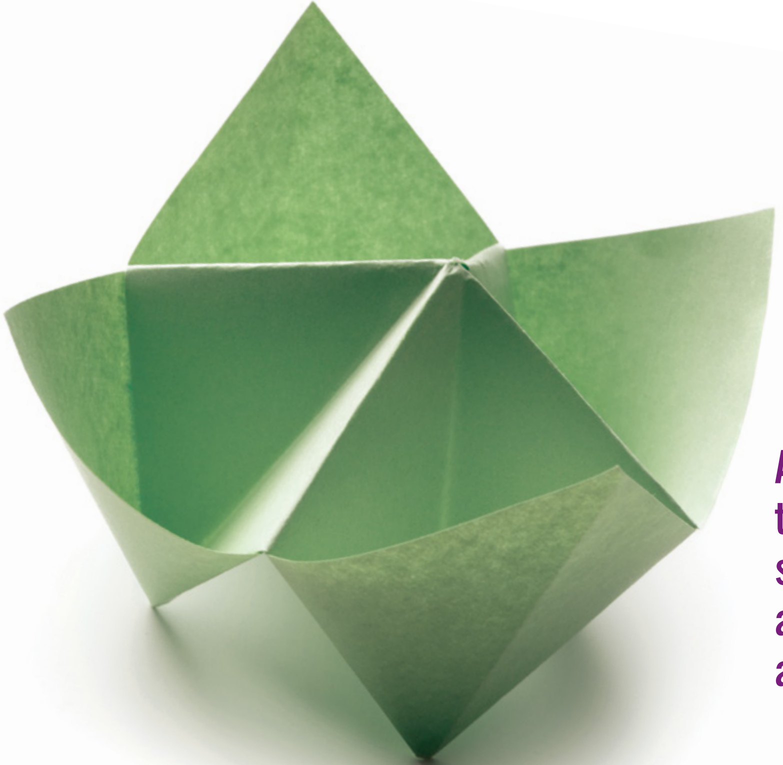

A Communication Game puts students' thinking on display in ways that highlight salient features, specifically those aspects of representational relationships about which students might disagree.

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student-to-student communication, rather than the teacher having to play an evaluative role. The process of facilitating these games involves several of the Mathematics Teaching Practices described in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014). Students' participation in these games also creates opportunities for them to engage in mathematical practices and to develop their mathematical communication skills.

We devised a Communication Game to consist of five steps. We describe these steps with connections to the Mathematics Teaching Practices and the Common Core's Standards for Mathematical Practice (SMP) (CCSSI 2010):

1. *A teacher hands out previously devised secret messages to student groups.* A secret message may be presented as a solution, equation, definition, expression, or other representation. Each group receives different secret messages that the teacher deliberately chooses, designing the messages to orient students to reason about particular issues (Mathematics Teaching Practice 2: Implement tasks that promote reasoning and problem solving).

2. *In groups, students encode their secret messages through illustrations or other written descriptions.* A Communication Game involves small groups of students using and connecting two distinct types of representations: the message and its encoded form. Students encode each message into a different form, such as a drawing, graph, or story, which their partner group will attempt to decode. The teacher typically specifies the type of representations to be used for encoding but can also allow for students to choose their type of representation. By focusing students' attention explicitly on relationships between representations, a Communication Game creates opportunities for students to clarify their understanding of the nuances of those relationships (Mathematics Teaching Practice 3: Use and connect mathematical representations). A Communication Game puts students' thinking on display in ways that highlight salient features, specifically those aspects of representational relationships about which students might disagree. While students are working to encode a message, the teacher has opportunities to ask questions to clarify how students are thinking (Mathematics Teaching Practice 8: Elicit and use evidence of student thinking).

3. *Partner groups exchange and decode messages.* Partner groups trade papers and attempt to decode the messages that they have received. The task of decoding messages should not be trivial or procedural. Like the task of encoding, it should promote mathematical reasoning and problem solving (Mathematics Teaching Practice 2). The task of decoding is also intended to create situations in which students struggle to make important distinctions and grapple with how to interpret other students' representations (Mathematics Teaching Practice 7: Support productive struggle in learning mathematics). This phase provides opportunities for students to share and discuss their interpretations with their partners and to justify their interpretations based on the available evidence (SMP 3: Construct viable arguments and critique the reasoning of others).

**Table 1** These examples show how Communication Games can occur in various domains.

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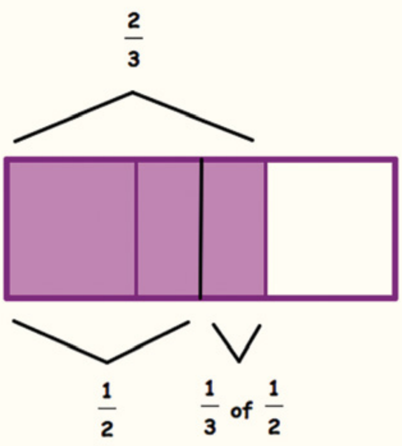
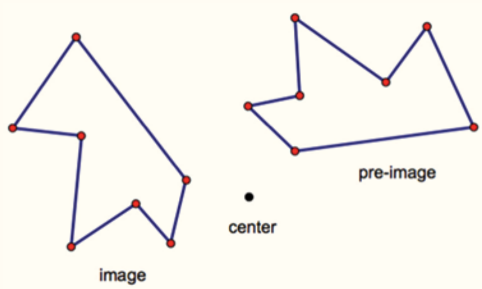
|---|---|
| Ratio and Proportion: Students encode proportion problems by contextualizing them as story problems (SMP 2: Reason abstractly and quantitatively). Students write story problems, thus giving students the opportunity to be creative and to personalize their mathematical work while requiring them to attend precisely to proportional relationships between quantities. | Secret message: $\frac{3}{5} = \frac{9}{x}$<br>One possible encoded message:<br><br>3/5 of the players on a team are right handed. There are 9 right-handed players. How many players are on the team? |
| The Number System: Students encode expressions or equations involving rational numbers into story problems or drawings. Such activities afford a focus on meaningful connections between real-world situations and operations involving rational numbers. Students can develop a better understanding of operations involving rational numbers by creating their own representations. | Secret message: $\frac{2}{3} \div \frac{1}{2} = 1\frac{1}{3}$<br>One possible encoded message:<br><br>Our drawing shows that there are<br>$1\frac{1}{3}$ copies of $\frac{1}{2}$ in $\frac{2}{3}$. |
| Functions: Students encode functions by evaluating them at values of their choosing and encoding them as coordinates of points. Students receiving the encoded message will then have to work backward to find the equation. Relating multiple representations can help students deepen their understanding of functions (Brenner et al. 1997). | Secret message: $y = 2x + 9$<br><br>One possible encoded message:<br><br>Our message is the line that passes through the points (–3, 3) and (1, 11). |
| Geometry: Students encode transformations into drawings. The secret message may take the form of the description of a transformation, which students encode by drawing a figure and its preimage. This activity requires precision and also affords students the opportunity to see a transformation as both an object (here, a message to be communicated) and a process (rotate the figure 120 degrees) (Sfard 1991). | Secret message: 120 degree rotation<br><br>One possible encoded message:<br> |

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**We believe that the better a teacher and fellow classmates understand students' thinking, the more conducive the classrom environment will be to student communication and learning (Horn 2008).**

4. *Partner groups meet to share their interpretations and find out how successfully they have communicated.* In conferences between partner groups, students share their interpretations of the encoded messages that they received. Then the secret messages are revealed, and students find out how successfully they communicated. It is important for the teacher to model appropriate, respectful ways of interacting during these conferences. In particular, the emphasis is not on right or wrong, but on clarity of communication. The teacher celebrates details of successful communication and orients students to focus on improving communication. Both within groups and in conferences between groups, Communication Games encourage students to attend precisely to meaning to communicate effectively (Mathematics Teaching Practice 4: Facilitate meaningful mathematical discourse).

5. *Partner groups offer feedback to facilitate improvement in future communications.* In the conference between partner groups, students have the opportunity to provide feedback to improve future communications. This step is crucial because it provides the opportunity for students to take ownership of precise communication. The more precise that students are in using words and symbols in correct or conventional ways, the more successful their communication becomes. A Communication Game can position students to respond to one another's ideas and provide constructive feedback (Mathematics Teaching Practice 4).

The broad purpose of Communication Games is to encourage attention to precision (SMP 6). This purpose is especially emphasized in steps 2 and 5. In step 2, students make decisions related to precision when they encode their secret messages. In step 5, they give and receive feedback to improve precision.

## CREATIVITY IN COMMUNICATION GAMES

Communication Games give students opportunities to learn and understand the language of mathematics. Ideally, they also allow opportunities for creativity. At first glance, it may appear that precision and creativity would be incompatible; however, we have found the capacity for creativity to be advantageous in Communication Games. In the classroom episode above, our students drew number line trips, but a variety of representational forms may be used. Story problems are a great example. A wide range of contexts is possible, allowing many different story problems to be written to correspond to a single equation. However, with the help of precision, a story problem may be decoded as corresponding to the intended equation or to an equivalent equation. (See **table 1**.) In this way, Communication Games touch on Mathematics Teaching Practice 2 from *Principles to Actions* in that they "allow multiple entry points and varied solution strategies" (NCTM 2014, p. 10).

## VERSATILITY IN COMMUNICATION GAMES

Communication Games are versatile enough to be adapted to all domains of mathematics, as well as to other content areas, to promote understanding and precision in multiple topics. We regard Communication Games as a regular classroom activity that can be used to help students relate concepts and representations. (See **table 1** for a few examples.)

## INTERPRETATIONS AND FEEDBACK

Communication Games provide authentic opportunities for students to be precise and creative in sharing their interpretations of mathematical representations and in providing feedback to one another. It is important that teachers pay attention to students' interpretations and ideas to support their learning (Jacobs, Lamb, and Philipp 2010). We believe that the better a teacher and fellow classmates understand students' thinking, the more conducive the classroom environment will be to student communication and learning (Horn 2008). The opportunities created by Communication Games allow for students to actively participate in mathematical practices and to experience ownership of the mathematics that they are learning.

We find that students enjoy Communication Games and benefit from them. Participation in the games gives a class access to the details of students' thinking, which helps both students and teacher better understand one another and can help students learn to speak the language of mathematics. As with learning any language, this process is gradual and can be aided by multiple experiences using the language for practical communication. We hope that this article gives readers a new tool that can be used in classrooms. If you use Communication Games with your students, please share your experiences with us.

## Let's Chat about Attending to Precision

**On Wednesday, November 16, at 9:00 p.m. EDT,**

we will expand on "Attending to Precision with Secret Messages" (pp. 208–15), by Courtney Starling and Ian Whitacre. Join us at #MTMSchat.

We will also Storify the conversation for those who cannot join us live. Our monthly chats will always fall on the third Wednesday of the month.


## ACKNOWLEDGMENT

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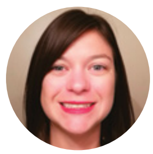


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