palette of problems solutions

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(*continued from pp. 272–73*)

ANSWERS

1. 12/13 hour
2. A circle of radius 2
3. 11
4. 1/4
5. 7.2 percent
6. $6(12^2) + 21(\pi/2)$ in.^{2}
7. \$24,800.00
8.–50
9. 9
10. Please see the solutions section for this answer.
11. 4
12. 29/44
13. 11
14. 58,240 ft.^{2}.
15. Student estimates will vary; 32 years
16. 24 minutes

SOLUTIONS

1. From the times given, we know that Alyssa can paint at a rate of 1/2 room per hour; Bryce, at a rate of 1/3 room per hour; and Chris, at a rate of 1/4 room per hour. Working together, Alyssa, Bryce, and Chris can paint at a rate of

$$1/2 + 1/3 + 1/4 = 13/12 \text{ rooms per hour.}$$

The trio needs to paint 1 room, not 13/12 of a room. So if we let h represent the total amount of time they spend painting, we solve $h \times 13/12 = 1$ for h; therefore, $h = 12/13$.

2. Recall that the circumference of a circle is equal to $2\pi r$, where r represents the radius of the circle. Because 2π is greater than 6, a circle of circumference 12 will have a radius less than 2. Because the area of a circle is equal to πr^2, the circle of radius 2 will have a larger area.

3. Because the mean of the set of 5 numbers is 12, the sum of those 5 numbers is $5 \times 12 = 60$. Removing the number 16 from the set gives a set of 4 numbers whose sum is 44. The mean in this case is $44/4 = 11$.

4. To find the probability of having 2 girls, look at the sample space of outcomes.

First Child	Second Child
Boy	Boy
Boy	Girl
Girl	Boy
Girl	Girl

The sample space includes 4 possible outcomes, only 1 of which results in 2 girls. The probability of having 2 girls is 1/4.

5. Scoring exactly 7 points in 3 throws would require hitting the inner circle two times and the outer circle one time. There are three different ways to accomplish this:

- Inner-inner-outer;
- Inner-outer-inner; and
- Outer-inner-inner.

The probability of hitting the inside circle is .2. The probability of hitting the outside circle is equal to the probability of hitting the board minus the probability of hitting the inside circle, or .6. Thus, the probability of hitting the inner circle twice and the outer circle once in 3 throws is equal to

$$(.2 \times .2 \times .6) + (.2 \times .6 \times .2) + (.6 \times .2 \times .2)$$
$$= 3(0.024) = 0.072, \text{ or } 7.2\%.$$

6. By drilling the holes, George has added only the lateral surface area of each hole to the surface area of the cube. Therefore, we need to find the lateral surface area of 1 hole, multiply by the number of holes, and add that to the surface area of the cube, which is 6×12^2. Since each hole is a cylinder with radius $r = 1/2$ and height $h = 1/2$, the lateral surface area is

$$2\pi rh = \pi/2 \text{ in}^2.$$

In all, there are $1 + 2 + 3 + 4 + 5 + 6 = 21$ holes drilled in the cube. The total surface area of George's die is

$$6(12^2) + 21(\pi/2) \text{ in}^2.$$

7. If X is the original value of the car, we are told that $(1 - 0.08)^3X = \$19,311.46$, so $X = \$24,800$.

8. Consecutive pairs of odd numbers followed by even numbers add 1 to the total: 2 – 1, 4 – 3, and so on. The last complete odd-even pair is –97 + 98, providing a total of 98/2 = 49. Subtracting 99 makes the total –50.

9. Try filling in the boxes. You will quickly discover that the two central squares must contain the digits 1 and 8 because these squares each have 6 neighbors. Then you will see that the squares on the ends must be filled with 7 and 2. After that, there are two places where 3 can go. Having chosen that, there is only one place that 4 can go that leaves places for 5 and 6. Any way you place 3, the boxes X and Y end up with the digits 4 and 5 (or 6 and 3). The sum is 9 either way.

10. Write $a = 10x + y$ and $b = 10y + x$ where x and y are the digits of a. Then

$$a^2 = 100x^2 + 20xy + y^2,$$

whereas

$$b^2 = 100y^2 + 20xy + x^2.$$

The difference is

$$a^2 - b^2 = 99(x^2 - y^2).$$

The coefficient 99 is divisible by 9 and 11, whereas

$$x^2 - y^2 = (x + y)(x - y)$$

is divisible by the sum of the digits $x + y$.

11. Using the given relationship with $n = 7$ lets us know that $121 = a_6 + 36 + 20$; $a_6 = 65$. Using the relationship with $n = 6$ lets us know that $65 = 36 + 20 + a_3$, so that $a_3 = 9$. Then, $36 = 20 + 9 + a_2$, so $a_2 = 7$. Finally, $20 = 9 + 7 + a_1$, so that $a_1 = 4$.

12. There are ${}_{12}C_3 = 220$ ways to select 3 marbles from the 12 in the bag. There are

$${}_5C_2 \times {}_7C_1 = 70$$

ways to select 2 blue marbles and 1 marble of a different color, so

$$P(\text{2 blue marbles}) = 70/220.$$

There are ${}_4C_2 \times {}_8C_1 = 48$ ways to select 2 white marbles and 1 marble of another color, so

$$P(\text{2 white marbles}) = 48/220.$$

There are

$${}_3C_2 \times {}_9C_1 = 27$$

ways to select 2 red marbles and 1 marble of another color, so

$$P(\text{2 red marbles}) = 27/220.$$

The probability that 2 of the 3 marbles will have the same color is

$$\begin{aligned} &70/220 + 48/220 + 27/220 \\ &= 145/220 \\ &= 29/44. \end{aligned}$$

13. The longest side of any triangle cannot contain more than 4 toothpicks. The Triangle Inequality theorem states that the sum of any 2 sides of a triangle is always greater than the third side. This theorem will not hold if 1 side is 5 or greater. The triangles with sides of 4 toothpicks or fewer are

1, 1, 1; 2, 2, 3; 3, 3, 4; 4, 4, 2;
2, 3, 4; 2, 2, 2; 3, 3, 3; 4, 4, 1;
2, 2, 1; 3, 3, 2; and 3, 3, 1.

14. 26 inches represents a length of 208 feet; 35 inches represents a length of 280 feet. The area of the rectangle is 58,240 ft.^{2}. Alternately, 26 in. × 35 in. = 910 in.^{2}. According to the scale, 1 in.^{2} is 64 ft.^{2}; 910 × 64 = 58,240 ft.^{2}.

15. Counting to a billion would take 1 billion seconds, which is 16,666,666 and 2/3 minutes, or 277,777 and 7/9 hours, or just over 11,574 days, which is almost 32 years (assuming 365 days in a year). It will take at least 32 years, without stopping to sleep or eat.

16. Steve shovels 1/60 of the driveway per minute. Greg shovels 1/40 of the driveway per minute. Together, they shovel

$$1/60 + 1/40 = 100/2400 = 1/24$$

of the driveway per minute. Working together, it will take them 24 minutes to shovel the driveway.

This problem is similar to the first problem in this Palette. Can you make a generalization about how to solve such problems based on your work on these two?