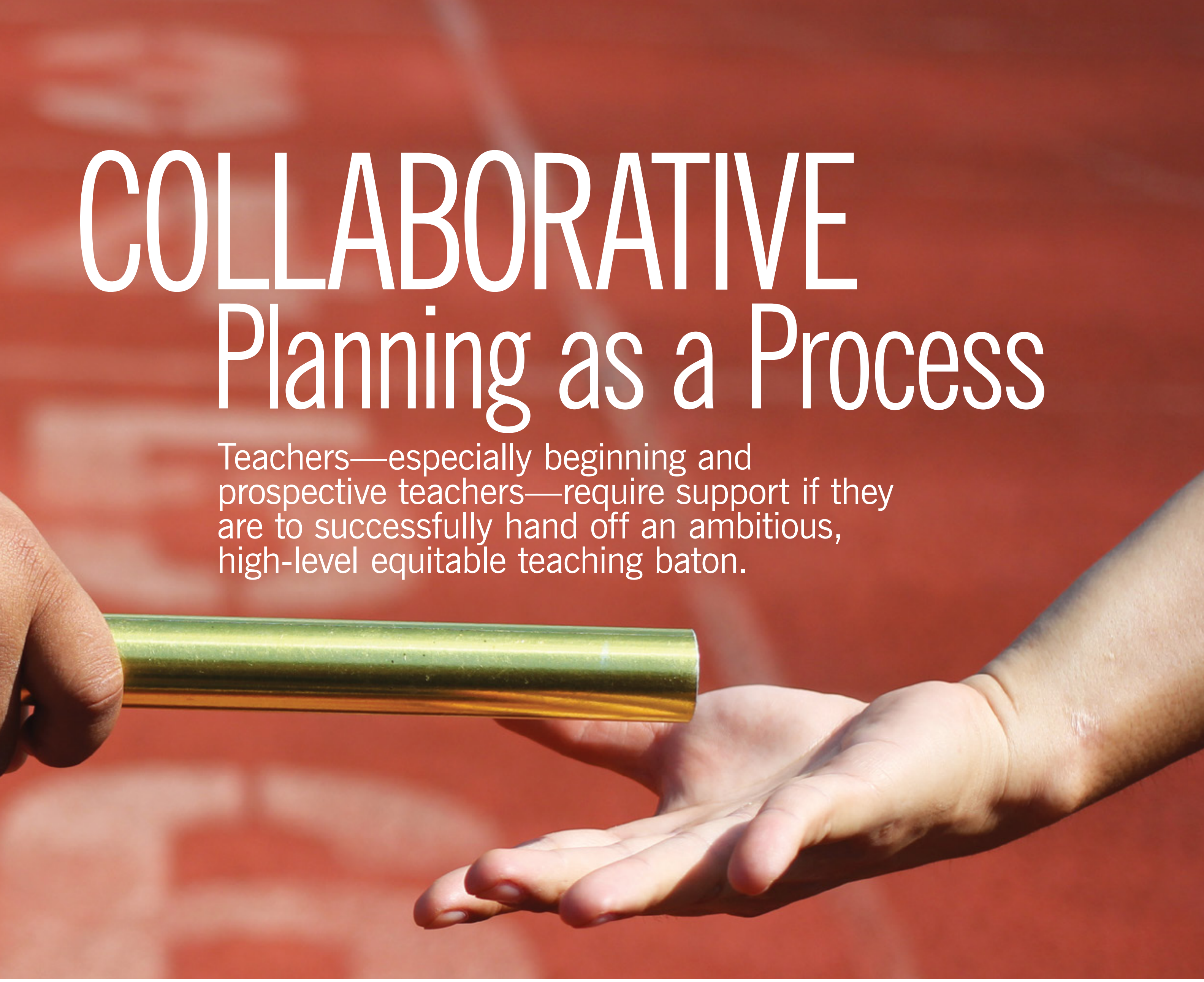
Justin D. Boyle and Sarah B. Kaiser

All students should be provided with opportunities to develop conceptual understanding prior to procedural fluency (NCTM 2014; CCSSI 2010). To develop students' conceptual understanding, teachers must learn such skills as how to select, plan, and enact cognitively demanding tasks (CDT) (Lambert and Stylianou 2013; Smith, Bill, and Hughes 2008) and to evaluate evidence of student learning (Hiebert et al. 2007). Therefore, teachers need opportunities to develop these skills to maximize their students' learning outcomes. Starting with a well-designed CDT is essential. In other words, before planning the enactment of a task, teachers should analyze the task and make revisions to align it with student learning goals that promote conceptual understanding (Hiebert et al. 2007; Smith and Stein 2011).

In this article, we share an example of a prospective secondary mathematics teacher designing, planning, and reflecting on the enactment of a CDT before addressing procedural fluency. The setting is a school in which 92 percent of the students scored below proficient on the 2014–2015 eighth-grade state test. Our perspective is that if extra attention is dedicated to the task design, then all students, including those who are low performing, can access a new mathematical concept and develop understanding. In other words, a lesson plan is important, but a lesson plan based on a task that does not support a conceptual learning goal limits students' learning opportunities. We believe that if teachers spend time collaborating and providing critical feedback on their tasks with a goal of conceptual understanding, then their students have a better chance of developing the mathematical understanding needed to build procedural fluency, increase interest in mathematics, and make greater gains on state tests.



COLLABORATIVE Planning as a Process

Teachers—especially beginning and prospective teachers—require support if they are to successfully hand off an ambitious, high-level equitable teaching baton.

CONTEXT OF THE TEACHING SITUATION

Student teacher and author Sarah Kaiser is enrolled in a mathematics education course; her assignment is to design and implement a CDT (e.g., Smith and Stein 1998). Author Justin Boyle is the course instructor and collaborator in the planning process. Ms. Lane is Kaiser's mentor teacher and has given her a potential task to teach her eighth-grade students, in what will be the first time that Kaiser has planned to teach a full school day.

The eighth graders in the school are tracked into four different class groups (advanced, high, average, and low performing) on the basis of past academic performance. Seventy-five percent of students in the school qualify for the free or reduced lunch program. The low scores on state tests at the school create a culture of remediation to improve basic skills. Lane typically solves problems on the board, and Kaiser's role is to support students by reproducing the modeled procedure with similar problems. Although Lane has enacted tasks with her students, Kaiser has never been present when she did so. In this context, Kaiser was both nervous and excited to design, plan, and enact a lesson to develop conceptual understanding.

COLLABORATIVE PLANNING

Boyle's mathematics education course follows a four-part process to design and study the effects of teaching on student learning:

1. Design or modify a cognitively demanding task.
2. Develop the lesson plan using the Thinking Through a Lesson Protocol (TTLP) (Smith et al. 2008).
3. Enact the task so that all students are given an opportunity to

participate (Lambert and Stylianou 2013).
4. Collect student work, analyze it, and reflect on evidence of learning.

Design or Modify a Cognitively Demanding Task

Lane explained to Kaiser that the task she would supply involved teaching square roots. In Boyle's course, the class developed a set of principles to modify tasks (see **fig. 1**), revised Lane's task, and shared the set of principles with Boyle (see **activity sheet 1**). Boyle reviewed Kaiser's draft and offered feedback that aligned with the general principles (see **fig. 1**). In addition to the framework developed in class, Boyle had to provide explicit feedback to support the novice teachers in learning what the principles mean in the context of their specific task before they could generalize their understanding of the principles.

Scaffolding. Scaffolding allows students entry to the problem and helps to advance their thinking toward the conceptual learning goal. In the case of the first task, the term *perfect square* should be removed from the first question because students will not know what it means. Also, adding a question that connects to prior knowledge is needed so that students can feel success as they get started. Boyle also suggested that squares and roots both be included in the task, so that students can make connections between prior knowledge of squares and develop a connection to the new concept of square roots.

Provide access and promote understanding. Boyle explained that the task should include more and different opportunities to explore patterns. For example, since the number 7 is prime, it allows for only one rectangular formation (1×7), so a different number should be chosen. Also, to examine more examples, students will require additional tiles. Although 40 tiles (see **activity sheets 2–4**) as opposed to 25 tiles (see **activity sheet 1**) allows for only one more square (6×6), the number 36 has many factors or ways to arrange the tiles. Finally, Boyle suggested that Kaiser include a question about the relationship between a perfect square and its roots. This question was meant to support students with generalizing their understanding, abstract reasoning, and advancement toward a generalized understanding of the relationship between perfect squares and square roots beyond the 40 physical tiles.

Fig. 1 A class developed this set of principles to govern task design.

1. Scaffolding
   A. Take away/remove
   - Questions that promote recall
   - Fill-in-the-blank questions

   B. Add
   - Accessibility
   - Connection to prior knowledge
   - Opportunities to brainstorm
2. Provide access through exploration to develop understanding.
3. The overall structure should balance promoting student struggle and helplessness.
4. Inductive-to-deductive reasoning
   A. Reasoning and proving (Stylianides 2008)
   - Look for patterns
   - Make a generalization
   - Develop an argument (proof or nonproof)

Inductive to deductive reasoning. Traditional textbooks start with a general rule, which is often inaccessible and falls short of giving students opportunities to reason and construct viable arguments. For example, in the case of square roots, a traditional textbook might state that

$$\sqrt{x^2} = \pm x,$$

which has little meaning to most students and fails to give them the chance to explore when or why this generalization is true. A task designed to promote reasoning should present opportunities to explore examples and nonexamples. Giving students the opportunity to identify patterns supports the development of making generalizations and constructing arguments (Stylianides 2008).

After this first round of feedback, Kaiser revised the task a few more times (see **activity sheets 2–4**) before reaching agreement with Boyle on the final version (**activity sheet 4**). One major issue that they discussed regarding the task versions in **activity sheets 2** and **3** is that the questions prompt students to state a definition before engaging in the activity. Kaiser articulated that her goal is for students to learn what a perfect square is and how a perfect square relates to a square root. Boyle explained that launching a task should promote student thinking, so teachers do not just want to

tell students new information. They also discussed the wording of several questions. For example, in **activity sheet 2,** question 3 in the "Square Roots" section reads, "Can 24 have a square root?" Students will learn about nonperfect squares, so Boyle and Kaiser revised the wording. The task in **activity sheet 4** allows the teacher to learn what students know about squares. It also supports students as they generalize their understanding of the relationship between the area of a square and the side length of the square (square root), thus aligning with the conceptual learning goal. After a task is designed, then it is time to think through the enactment.

Develop the Lesson Plan Using the Teaching through a Lesson Protocol (TTLP)

Setting up the task. The setup establishes expectations about how the students and teacher will interact with one another. Kaiser decided to have students work in the same groups as Lane had assigned. She planned to distribute the task to each student and wait until after they had completed part 1 individually before distributing the envelope of 40 tiles to each group. To give each student time to make sense of the task on his or her own before any group discussion, she would ask students to explain their thinking to the part 1 questions as she circulated around the classroom during the first seven minutes. Kaiser wanted to be sure that all students understood the context of the problem (Jackson et al. 2012)—that each square tile is one square inch—and to have a couple of students share what they had written for question 4 on **activity sheet 1**. For instance, students might conjecture that the side length is half the number of unit squares. Kaiser anticipated accepting various interpretations of the relationship, because in part 2, students would have opportunities to explore, revise, and justify their thinking about the relationship.

Supporting exploration of the task. In the exploration phase, the teacher monitors student thinking. The teacher's role during exploration is to ask questions that balance students' current thinking with intended learning outcomes. Planning the exploration requires a teacher to first anticipate students' solutions, including those that may be incorrect. The teacher then plans oral questions based on anticipated solutions to assess and advance students' thinking toward the mathematical goal.

Kaiser prepared a variety of solutions and planned questions to align with each anticipated solution path. For instance, regarding the question "Is 12 a perfect square?" Kaiser anticipated that a student might create a 4×3 rectangle and say that it is not a square. However, instead of saying, "Good," she planned to ask, "How do you know that you can't make a square?" If the student answers, "Look," she would respond, "But that is just one arrangement. How do you know that there is no other way to arrange the 12 tiles to make a square?" In other words, Kaiser wanted students to provide arguments that could convince others. Since teachers should also plan questions for early finishers, Boyle suggested that Kaiser consider "Could two perfect squares be consecutive numbers?" and "Find a perfect square between 50 and 60." Such questions will make students continue to think about and expand on their understanding of squares and square roots beyond the written questions.

Sharing and discussing the task. The final part of the TTLP is for the teacher to plan how to orchestrate the whole-class discussion, which stems from sharing strategically sequenced student solutions. The whole-class discussion addresses connections across the solutions while advancing students' thinking toward the mathematical goal. The teacher must plan questions to promote mathematical connections across their anticipated solutions.

Kaiser planned to have four students share perfect squares that align with task questions 5 and 6. The sharing would start with perfect squares that students had drawn or had created with tiles and would move toward sharing perfect squares greater than 36. The class would record a list of statements relating squares and square roots. Kaiser also planned to ask students about nonexamples, such as 99 and 50. Finally, the class would return to the listed statements from the launch phase to underscore the relationship between a perfect square and square root.

KAISER'S REFLECTION ON THE ENACTMENT

My overall experience was positive and productive. Most important, I believe students achieved the learning goal. They worked together throughout each class period, were engaged, and learned about perfect squares. During the whole-class discussions, students reached agreement about what makes a perfect square and related it to the side length of a square, which is when I introduced the term *square root*. It was an exciting experience for me since the students learned new mathematics while developing conceptual understanding, without me first defining a square root.

Based on what the students said and wrote in class, it is clear that they reached my goal. In class, students were able to use the tiles to produce a perfect square, recognize a nonsquare number, generate examples of perfect squares beyond the set of 40 tiles, and explain connections between a square and side length. For example,

during the lesson, many students constructed 2 × 6, 4 × 3, and 3 × 4 figures with the tiles and explained that 12 is not a perfect square because "it's a rectangle." When I asked what they meant by this, student responses included that "It is a 3 × 4 rectangle or a 2 × 6 rectangle" and "It couldn't be a square, since the side lengths weren't the same and could not be made into a square." Additionally, I analyzed their written work. Students drew tile formations, and a few students wrote convincing phrases, such as "too many tiles for a 3 × 3 and too few for a 4 × 4 square." Less convincing but developing arguments included explanations that a 3 × 3 square or a 4 × 4 square was impossible with 12 tiles. Some students wrote that they tried different shapes and that the side lengths were never equal.

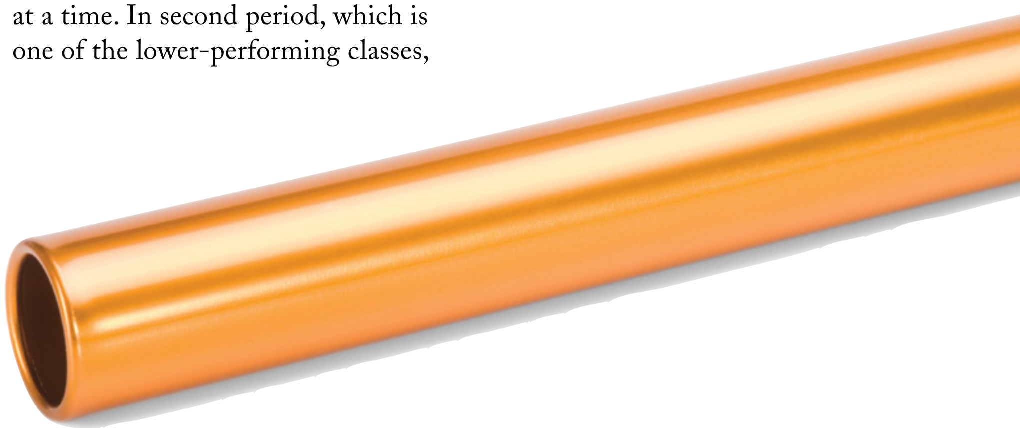
As students progressed to questions 3–6, they were convinced that 36 is a square number. Some students wrote that "there are six down and six across" or that "the sides are equal 6 × 6." Students generated many different-size squares for question 5. Some students drew 8 × 8 squares, and a student even drew a 30 × 30 square with all of the grid lines to show all 900 unit squares. I wish I had asked the group in which a student drew the 30 × 30 square questions to move beyond concrete examples. Several students did not draw a diagram but wrote a product of two numbers that were the same, such as 3,600,000,000 (60,000 × 60,000) and 2,809 (53 × 53). This showed that some students moved beyond a concrete diagram toward a generalized understanding.

Finally, I asked every class if 99 is a perfect square, and after I finished the question, every single class was completely quiet as the students considered the question. Then a number of students in each class shared their thoughts. Overall, students decided that 99 could not be a square. For instance, one student said, "Nothing multiplied by itself is 99."

Although I felt well prepared and students reached my goal, Ms. Lane helped me with a few issues, such as time management, students who were not used to working independently, and the wording of a few of the written questions. For instance, I initially set the timer to 30 minutes for the explore phase during my first period class, and found that most students were spending a lot of time on the first two questions. Lane made the productive suggestion that I set the timer at 10 minutes for students to complete two questions at a time. In second period, which is one of the lower-performing classes, the students had a hard time moving from exploring 12 to exploring 36 because they were unsure about their answers. Therefore, in sixth period, also a low-performing class, I adjusted by facilitating a short class discussion after the first two questions, which seemed to help. Finally, some of the questions on the task could have been worded better. For example, most students, after reading part 1, question 4 and part 2, question 2 on **activity sheet 4**, asked, "What are we supposed to do?" I learned that I should limit or eliminate having multiple parts to a question.

WHAT THEY LEARNED AND WANT TO SHARE

Redesigning the task before enacting it maximized students' learning opportunities. If Kaiser had planned and enacted any of the tasks on activity sheets 1–3, students would have been less likely to understand the relationship between perfect squares and square roots. Boyle and Kaiser collaborated to design the task according to a set of design principles (see **fig. 1**). Constructing a set of general task principles establishes the foundation and common understanding with which to provide critical feedback. For example, Kaiser revised her task to include fill-in-the-blank questions so that students would have the definitions to answer the questions. Her thinking was that students had to know the definitions to be able to apply them. Boyle provided feedback about redesigning the task to promote connections between squares and square roots, so that students could learn the meaning of the definitions of the concepts. Furthermore, an issue could arise in a task that is not listed as a design principle.



IMAGEDB/THINKSTOCK

Students learned new mathematics while developing conceptual understanding.

For instance, our design principles (see **fig. 1**) should include a statement about making sure the questions are mathematically sound. Finally, we suggest teachers develop task design principles (see **fig. 1**) based on frameworks such as the Levels of Cognitive Demand (Smith and Stein 1998) to provide one another critical feedback on their task design.

Boyle and Kaiser believe that the design collaboration described above could occur among a group of practicing teachers as well. Teacher groups could share a CDT task a week or two before enacting it. This would allow teachers the necessary time to solve the task themselves and offer critical feedback based on a set of task design principles. In addition, grade-level teachers in collaboration with a mathematics coach or teacher educator could take turns sharing and discussing a task on a weekly basis to cultivate skills in designing CDTs. Finally, the aim of the guidance in *Principles to Actions: Ensuring Mathematical Success for All* is to provide all students with opportunities to develop conceptual understanding prior to procedural fluency. Collaboration around task design can make this goal a reality and furnishes an enriching experience for both teachers and students.

Let's Chat about Collaborative Planning

On Wednesday, March 15, 2017, at 9:00 p.m. EDT,

we will expand on "Collaborative Planning as a Process" (pp. 406–11). Join us at #MTMSchat.

We will also Storify the conversation for those who cannot join us live. Our monthly chats will always fall on the third Wednesday of the month.

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more4U

Activity sheets are online at **http://www.nctm.org/mtms.** More4u is available to NCTM members only.

activity sheet 1

Name ______________________________

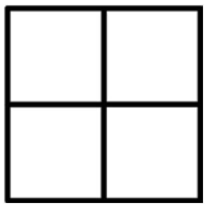
PERFECT SQUARES

Instructions: For today's activity, each group will be given a bag containing 25 tiles. Your group will use these tiles to investigate perfect squares.

Part 1: Answer questions 1–3 on your own. Once everyone in your group has answered them, discuss your answers as a group.

1. What do you believe a perfect square will be?

2. What do you notice about this square?



3. Do you believe that 4 could be a perfect square based on the picture? Why or why not?

Part 2: Work with your group, using the bag of tiles, to answer the remainder of the activity sheet.

1. Is 7 a perfect square? Why or why not?

2. Is 9 a perfect square? Why or why not?

activity sheet 1 (continued)

Name ______________________________

3. Make another square. Make your square below using the tiles.

4. How many tiles did it take? What do you notice about the square you found?

5. What are the similarities and differences between the square you found and the one on the first page?

6. Redefine a perfect square based on what you have learned today.

activity sheet 2

Name ______________________________

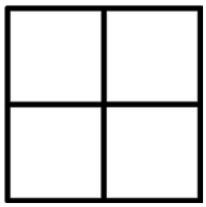
PERFECT SQUARES

Instructions: For today's activity, each group will be given a bag containing 40 tiles. Each group will use these tiles to investigate perfect squares.

Part 1: Answer questions 1-2 on your own. Once everyone in your group has answered them, discuss your answers as a group. Do not go past number 2 until instructed to do so.

1. What is a square?

2. What do you notice about this square?



Definition: A ____________________________________ is a number that can represent the area of a square having sides whose length is a whole number.

3. Do you believe that 4 could be a perfect square based on the picture above? Why or why not?

Part 2: Work with your group, using the bag of tiles, to answer the remainder of the activity sheet. Do not go on to the square roots section until instructed to do so.

1. Is 12 a perfect square? Why or why not?

2. Is 36 a perfect square? Why or why not?

3. Make another square. Make your square below using the tiles.

activity sheet 2 (continued)

Name ______________________________

4. How many tiles did it take? What do you notice about the square you found?

5. What are the similarities and differences between the square you found and the one on the first page?

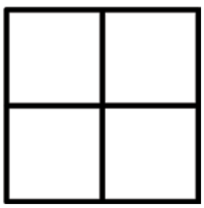
6. Give an example of a perfect square that will require more than 40 tiles. Explain how you know this is a perfect square.

SQUARE ROOTS

Definition: The square root of a number is a value that, when multiplied by itself, gives the number.

1. Based on the definition above what do you think perfect squares have to do with square roots?

2. How does the square root of 4 relate to the diagram below?



3. Can 24 have a square root that is a whole number? If yes, what is it? Explain how you know. (Hint: Use the squares in your bag to help you.)

4. Can 16 have a square root that is a whole number? If yes, what is it? Explain how you know.

5. Find a number larger than the square root of 40. Explain how you know that what you found is the square root.

activity sheet 3

Name ______________________________

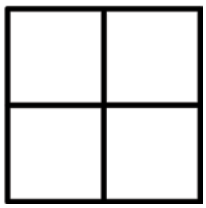
PERFECT SQUARES AND SQUARE ROOTS

Instructions: For today's activity, each group will be given a bag containing 40 tiles. You will use these tiles within each group to investigate perfect squares.

Part 1: To begin, answer questions 1–2 on your own. Once everyone in your group has answered them, discuss your answers as a group. Do not go past number 2 until instructed to do so.

1. What is a square?

2. What do you notice about this square?



Definition: A ______________________________ is a number that can represent the area of a square having sides whose length is a whole number.

Definition: The ______________________________ of a number is a value that, when multiplied by itself, gives the number.

3. Do you believe that 4 could be a perfect square based on the picture? Why or why not?

4. How do you think you could find a square root of 4?

activity sheet 3 (continued)

Name ______________________________

Part 2: Work with your group, using the bag of tiles, to answer the remaining questions.

1. Is 12 a perfect square? Why or why not?

2. If 12 is a perfect square, what do you believe the square root of 12 would be? If not, do you believe it would still be possible to find the square root?

3. Is 36 a perfect square? Why or why not?

4. Make another square. Draw your square below and find the square root.

5. How many blocks did it take? What do you notice about the square you found?

6. What are the similarities and differences between the square you found and the one on the first page?

7. Give an example of a perfect square that will require more than 40 blocks. Explain how you know this is a perfect square and find the square root.

activity sheet 4

Name ______________________________

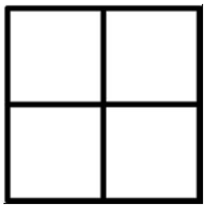
PERFECT SQUARES AND SQUARE ROOTS

Instructions: For today's activity each of your groups will be given a bag containing 40 tiles. Each group will use these tiles to investigate perfect squares.

Part 1: Answer questions 1-4 on your own. Once everyone in your group has answered them, discuss your answers as a group. Do not go past number 4 until instructed to do so.

1. What is a square?

2. What do you notice about this arrangement of four tiles?



3. Explain why you think the number 4 is called a perfect square. Use the tile diagram above.

4. How long is a side of the square in question 2? Explain the relationship between the side length of the square and the number of tiles used to create the square.

activity sheet 4 (continued)

Name ______________________________

Part 2: Work with your group to answer the remaining questions. Use the bag of tiles.

1. Count out 12 tiles from your bag. Arrange the tiles to learn if 12 is or is not a perfect square. Sketch the diagram you created below using your tiles to explain your answer.

2. Using the 12 tiles again, rearrange them to form another (different from above) four-sided shape. Is it a square? How many four-sided shapes are possible using 12 tiles? Sketch all the different possible four-sided shapes you found using 12 tiles.

3. Count out 36 tiles from your bag. Use the process you followed above to determine if 36 is a perfect square. Provide a convincing argument as to why or why not 36 is a perfect square.

4. List all of the different side lengths you found as you created four-sided shapes from the 36 tiles. Explain what you notice about the side lengths and the number of tiles used.

5. Now choose tiles from the bag to create as many different squares as you can. Draw each square below and list the side lengths.

6. Find as many examples of perfect squares that you can that will require more than 40 tiles. Explain how you know each is a perfect square and find its square root.