MAXINE'S JOURNAL

September 25

Almost all adults know how to identify place values in a multidigit number: 357 means 3 hundreds, 5 tens, and 7 ones. They also know the result of multiplying a number by 10: $357 \times 10 = 3{,}570$. You put 0 in the ones place, which shifts the rest of the digits one place to the left. But these facts are typically known in isolation, without connecting them to one another or to many other facets of the base-ten structure of numbers. The purpose of the second session of the seminar is to help participants make those connections.

Before participants entered the room, I posted the following statement:

> The value of a number is determined by multiplying the value of each digit by the value of the place that it occupies and then summing. In whole numbers, the value of the place farthest to the right is one; the value of every other place is ten times the value of the place to its right.

At the start of the session, I pointed to the poster and read the statement aloud. I said, "Every day, we work with multidigit numbers and don't have to think explicitly about what's involved in determining the value of a number from its digits. But when we look at this statement, we see that it's actually a pretty complex undertaking. Sometimes, just because we're so used to writing numbers this way, we lose sight of the complexity and the power of the system. In our session today, through the math activity, video analysis, and case discussion, we'll examine various features of the base-ten place-value system."

But before we started in on the content, I wanted to spend a few minutes to discuss group norms—the ways in which we engage and interact in a DMI seminar. This is not because anything problematic occurred in Session 1. Rather, in such an interactive setting, it is important that we all understand our responsibility to ourselves and to each other.

To lead into this discussion, I shared some of the exit cards from the previous session. I also wanted to address some questions that had come up. "Most participants said they appreciated the variety of activities, including both small- and whole-group work. Here is one comment that I think is especially significant: 'The video showed students who can solve the problems in interesting ways. What about the students who don't get it or don't know where to start?'" I explained, "It's important to pay attention to both—the thinking of students who are able to solve the problems and the thinking of students who struggle. But there's one thing I'll want us to pay attention to when we find a student who is struggling or is confused, and that is finding where the student feels solid as a place from which to build. I'd like to remind you of the discussion we had during the last session about Sarah in case 4. If you recall, simply making a judgment on her incorrect strategy was very different from following her thinking." Throughout the seminar, I want to emphasize how we can be more effective teachers when we pay attention to our students' reasoning.

Then I moved to another topic. “One comment I read on several exit cards was that it felt safe to share in the group. I agree that lots of you spoke up in the last session, and I was glad of that. But we can’t really be sure that everyone felt safe to share. Speaking up, especially in whole group, may be very difficult for some.

“So we can all learn to work productively together, it’s helpful to discuss this process explicitly. Therefore, as I do with all the seminars I teach, we’re going to spend a few minutes talking about group norms.”

Discussion: Norms for learning

For many people in the seminar, the structure of our sessions and my expectations about participation were completely new. The beginning of the second session seemed like a good time to check in and make some of these expectations explicit. It was also time to establish some guidelines for ensuring that our discussions felt productive and open to everyone.

I started out by mentioning that the last session was typical of the work we would be doing together in the seminar. That is, most of the time would be spent in group discussion—sometimes in pairs or small groups, sometimes in whole group—and, in most sessions, part of the time would be spent working on the cases and some of the time would be spent on the mathematics itself. On occasion, they would also be talking about their own students. I said that at this point, I wanted to hear how they were feeling, and that I wanted us all to think about how to make group participation more comfortable for everyone.

It took some time to get the discussion moving, but eventually we arrived at the idea that as people work in groups, they have different kinds of responsibilities to themselves and to the group. We ended up soliciting responses to two questions: What do you do to make this a good learning experience for yourself? What do you do to make this a good learning experience for others?

The thing is, once we started to generate the list, items didn’t fall neatly into one category or the other; they applied to both. This is the list we came up with:

- Ask questions.
- Indicate that you’ve heard someone else’s idea. It’s all right to disagree.
- Would like a respectful response.
- Sometimes we’re not respectful and get a little angry. We need to move beyond the anger.
- Sometimes it’s useful to establish ground rules in small groups beforehand.
- Don’t take things personally.
- When we do disagree, it’s important to clarify what we think the other person has said.
- Sometimes you think you heard something that wasn’t said.
- Sometimes I need to get my own thoughts together before I start listening to what someone else is saying.

- Notice when you're overwhelming others through your own excitement.
- Sometimes it feels like you're adrift in a rowboat with people who have speedboats.
- If necessary, say "I'm not following you."

We also listed these ground rules for the seminar:

- Start and end on time.
- Everyone should come prepared.
- Participate as an active listener.
- Turn off cell phones.

Toward the end, Marina said that everyone should contribute to discussions. But when Beatrice asked, "How can you set things up so that everyone talks?" Damaris objected, "Is it really important that everyone talk? What if it works better for some people to listen and not talk as much? Can we accept that some people might choose to be quiet?"

In the moment, I said that this was something we should all think about, and some people might choose to comment on this on their exit cards this evening. As I'm writing about this now, I'm thinking about the varied personalities—some people find it easy to talk and some find it difficult. Why do I want quiet people to speak up? Sometimes, I know, quiet people have good ideas that will enrich the group. And sometimes, I know, quiet people want to speak, and will learn more by having their ideas out there, but they need encouragement. Still, that's not always the case. In past classes, some teachers have assured me that they're learning a lot and that it's better for them not to feel pressured to speak up in the whole group. The thing is, when people are quiet, I have to initiate other mechanisms to find out what's going on with them. Exit cards and homework assignments are two such mechanisms. In any case, I'll need to be watching some of those participants who haven't been speaking up, particularly Andrea, Huong, and Nancy.

Math activity: Multiplying by 10

To begin the math activity, I explained that everyone in the room knows what happens when you multiply a number by 10. For example, $23 \times 10 = 230$; you take the number 23, and attach a zero to the right, and it becomes 230. To multiply by 100, you attach two zeros at the right: $23 \times 100 = 2{,}300$. But what's really going on? If we want our students to understand calculation as more than a set of rules, what do we want them to understand behind this rule of multiplying by 10?

Damaris interjected, "If *we* want to understand calculation as more than a set of rules!"

I smiled and nodded. In fact, Damaris expressed exactly what I was thinking but was hesitant to say explicitly. I was expecting the participants to come out of this activity with a deeper understanding of place value and what it means to multiply by 10. But I frequently find that when I ask teachers to delve deeply into a familiar topic, they don't recognize the potential for their own learning and stop short. For this reason, I framed the activity in terms of their students' understanding.

I continued, “Yes, we all want to understand calculation as more than a set of rules. But one way you might think about this activity is that, by the end, you will have several ways of helping students understand what’s going on when you multiply by 10.”

I distributed the math activity sheet and participants got to work. I gave them a few minutes to read it over before I gave further instruction. I said, “It asks you to model computations with base-ten representations. We worked with some representations in our last session. Let’s first clarify what that means.”

Carol suggested, “The representation shows ones and tens.”

Beatrice added, “The numbers will get larger than that, so the representation will need to show ones, tens, hundreds, thousands, and so on.”

I asked, “So what materials can we use?” Camisha said, “We can group cubes into tens.”

I nodded, while Elspeth said, “You can do that, but it will get pretty hard to show thousands with those cubes.”

Roberto said, “I noticed you brought in base-ten blocks. We can use those.”

Just in case not everyone was familiar with these manipulatives, I held up the different components. “This one-centimeter cube stands for one.”

Then I held up the “long,” the piece that corresponds to ten attached cubes, and asked, “What’s this?”

Several people called out, “Ten.”

I said, “I want to tell you a story that a second-grade teacher told me. She said that at the beginning of the year, she had left out these blocks for the students to play with during free time. A few weeks into the year, she brought out the blocks and explained that from then on, they would be working tools. When she explained that the small cube would stand for 1, she asked what this piece, the long, would be. Many students said that it should be 10, but one child insisted it should be 42.” I paused to let that 42 sink in, and then asked, “Why do you think the child chose 42?”

I started to distribute the pieces so that participants could take a closer look. Then Roberto started to chuckle and said, “The surface area is 42. If you count the number of squares on the surface of the piece, you get 42.” Several participants said, “Oh, yeah” in response.

I nodded, “When we say this is 10, we’re counting the number of whole cubes that would fit into the stick.”

I held up the other pieces to show 100 and 1,000. Then Louise said, “I sometimes work with those blocks, but no child has ever said the 10 was 42. But often they think the 1,000 is 600, because they’re counting the squares they can see, 100 on each face. It’s the same idea.”

I then put the base-ten blocks aside and asked if there might be any other representation. Gaye said, “I like working with graph paper, marking off tens and hundreds.” I pointed out that I had graph paper that marked off every tenth line.

Then I said, “These problems are asking you to use the representations to show multiplying by 10. What are ways to show multiplication?”

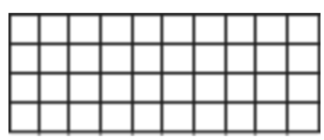
Fatima said, “Groups of.” In response to my request that she say more, she added, “If you multiply by 10, you make groups of 10. Like 4×10 is 4 groups of 10.”

I nodded and asked, “Any other ways?” Jade said, “You can use arrays.”

I said, “Yes, you can think of arrays as a way to organize the groups. For example, if you have 4 groups of 10, you can make 4 rows of 10.” I drew an array of Xs on the board.

XXXXXXXXXX
XXXXXXXXXX
XXXXXXXXXX
XXXXXXXXXX

Roberto added, “If you think of an array of squares, then you get a rectangle.” He came to the board to show us what he meant.



Roberto continued, “The lengths of the sides of the rectangle are the factors and the area is the product.”

My sense was that these models for multiplication were new to some participants. I wanted to have them made public, but I didn’t want anyone to feel overwhelmed by them. So I said, “To start, you should choose one form of base-ten representation and one form of representing multiplication to investigate the questions. If you feel that you’ve fully explored one representation, then please try another.” Then I gave participants a few minutes to get started before I began circulating among the groups.

Camisha called me over to say that she and Huong had different ways of showing 23×10, and they wanted to know which was right. She explained they were both using base-ten blocks, but she thought it should be 23 groups of 10, whereas Huong had created 10 groups of 23. They had both representations laid out on the table. Nancy, the third member of the group, was looking on.

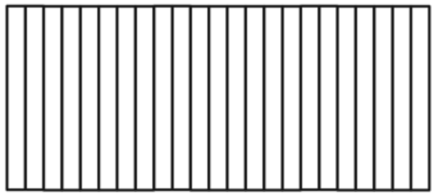


Fig. 2.3. Twenty-three groups of 10

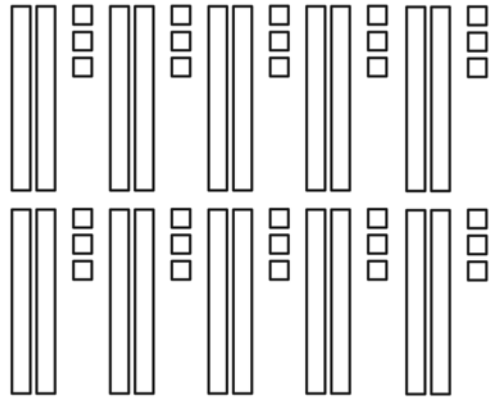
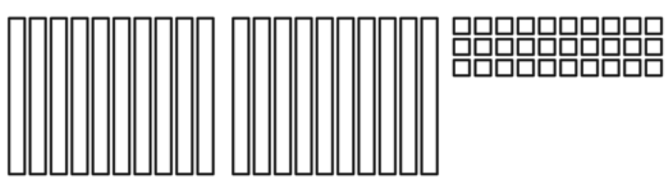


Fig. 2.4. Ten groups of 23

I said, “They look pretty different, don’t they? How do you determine the quantity represented in each?”

Camisha laid out two flat hundreds blocks on top of her tens blocks and counted, “100, 200, 10, 20, 30—230.”

Nancy had started counting Huong’s blocks, first the tens, then the ones. Before she finished, Huong rearranged them.



Huong said, “I can group the long blocks into tens. Each group makes 100, so that’s 200. There are 10 in each row of the singles, so that’s 30, or 230 in all.”

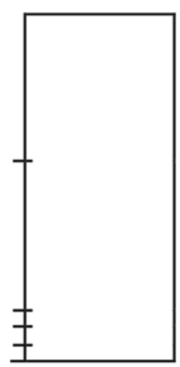
I asked, “Does it surprise you that both representations show the same amount?”

Camisha said, “I guess not, because 23 × 10 equals 10 × 23.”

I said, “I think it’s a good idea to look at both representations. How do they help you see why the rule for multiplying by 10 works?”

Moving to a different group, I saw that Gaye, Karran, and Marina had taken a sheet of graph paper to think about 23 × 10. They represented 23 as a straight line marked off as 2 tens and 3 ones.

Karran said, “To multiply by 10, you build a rectangle with a side of 10.” Gaye followed her instructions and drew the rectangle.



I noted that this group felt very comfortable using the area of a rectangle to represent multiplication.

Marina said, “You can see more clearly what’s going on if you extend those lines.” She reached over to draw what she was envisioning.

Gaye remarked, "Yeah, now you can really see that it's 2 hundreds and 3 tens."

I asked, "So what happens when you start with 23 and multiply by 100?" Karran suggested, "You do the same thing. One side of the array is 23 and the other is 100."

As Marina filled in her horizontal lines, she said, "I get it. Now you've got 20 hundreds and another 3 hundreds. Cool."

By the time I got to Troy, Penny, and Lilo, they were working on the set of questions about 462 objects that might be pennies, dimes, or dollars. Troy was saying, "462 pennies is 46.2 dimes or 4.62 dollars. It seems funny to write out the number of dimes as 46.2, but it gives a different sense of what you're saying when you write 4.62 dollars." He wrote down $4.62.

Penny said, "But if you have 462 dollars, that's 46,200 pennies."

Whole-group discussion: Sharing representations

When I called participants together for whole-group discussion, I began by writing $4 \times 10 = 40$ on the board and asked, "What would you like students to understand about this?"

Roberto said, "We often teach students to say that 40 stands for 4 tens and zero ones, which is the same as just 4 tens. We also teach them to say that 4×10 is 4 groups of 10. But those are just two ways of saying the same thing: 4×10 and 40 both mean 4 tens."

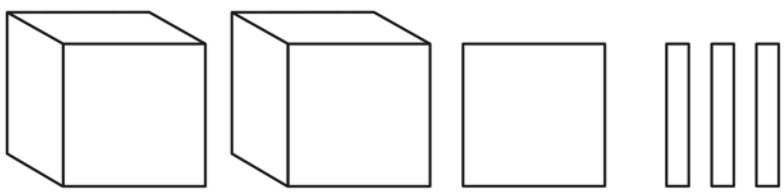
Gaye remarked, "Hey, I've never thought of that before. But now it seems so obvious."

I said, "You know, there are lots of things that seem obvious once we notice them. But the tricky thing is to notice them and to appreciate it when we do. It's precisely these things that become the glue of understanding—making all these connections." Then I asked, "What else did you notice in this activity?"

Fatima said that she wanted to show us something with the base-ten blocks, and then she went to the board to draw so that everyone could see. She started out with the following representation:

Fatima said, “This shows 213.” Then she pointed to one of the large squares. “That’s 100, and if I multiply it by 10, I get one of these,” now pointing to the cube-shaped thousands block. “I have got two big cubes, so it’s 2,000.” She paused and then pointed to the long rectangle. “I have one 10, and when that is multiplied by 10, it becomes 100,” and she held up one flat hundreds block. Finally, she pointed to the three small squares and explained that when multiplied by 10, each square becomes 10, so she represented the multiplication with 3 tens-blocks. She wrote this equation on the board and showed how it matches the blocks she has:

$$2{,}000 + 100 + 30 = 2{,}130$$



Troy said, “I want to show something I’ve never thought of before.” He came to the board and wrote out the following addition:

$$\begin{array}{r} 213 \\ 213 \\ 213 \\ 213 \\ 213 \\ 213 \\ 213 \\ 213 \\ 213 \\ \underline{+\ 213} \end{array}$$

He said, "When you start doing the addition, you add up everything in the ones column, and that's 30. So you put down the 0 and carry the 3. Then, when you add up everything in the tens column, you have a multiple of 10 plus 3, so it's like all the tens get out of the way, and the 3 falls into the tens place and you carry the 1. But the same thing happens in the next place. Since you've got ten 2s, you have another multiple of 10 plus 1, so the 1 goes into the hundreds place and the 2 in the thousands." As he talked, Troy wrote out the steps.

3	13	13
213	213	213
213	213	213
213	213	213
213	213	213
213	213	213
213	213	213
213	213	213
213	213	213
213	213	213
+ 213	+ 213	+ 213
0	30	2,130

I asked, "What would happen if you multiply 43,678 by 10?"

Troy responded, "The same thing! Whatever you carry gets added to a multiple of 10."

I wasn't sure how many participants were following what Troy saw in the addition algorithm, so I brought them back to Fatima's representation. I asked, "Would the same thing happen with this large number?"

Gaye said, "We can't show it because we would run out of blocks. There isn't a block that shows 10,000 or 100,000."

Carol said, "Even if there isn't an actual ten-thousands block, we can imagine it. It would be 10 thousands-blocks stuck together. It would look like a tens block, but bigger."

Jade added, "Our group thought about that, too. If you can imagine the ten-thousands block, then the hundred-thousands block would be ten of *those* stuck together, and it would look like a flat hundreds block, but bigger."

We didn't have that many thousands blocks to demonstrate, but I asked Jade to work with those we had and use her hands to show what the ten-thousands block and hundred-thousands block would look like. She stacked the 5 thousands-blocks we had and said, "You would need five more of these stacked on top, and that would give you a ten-thousands block." She placed her hand to show about how high it would be. While she stood with her hand at that height, I stood next to her and showed where a second ten-thousands block would be, and a third, and a fourth, up to the tenth so that participants could have an image of what the hundred-thousands block would look like.

I asked, "So let's imagine the number 43,678." Jade, who was still in the front of the room, arranged the blocks to show 3,678, and I made hand gestures to demonstrate 4 ten-thousands blocks. Then I asked, "What happens if this gets multiplied by 10?"

Elspeth said, "You need to have ten times everything you have up there. I found it easier to think about multiplying from left to right. Each of those 4 ten-thousands blocks gets multiplied by 10, so you end up with 4 hundred-thousands blocks. Then each of those 3 thousands-blocks gets multiplied by 10, so you can think about those towers on top of each of the cubes; that's 3 ten-thousands blocks. Each of those 6 hundreds-blocks gets multiplied by 10, so they all grow into thousands blocks. Each of the 7 tens-blocks becomes a hundreds block; that's 700. And each of the 8 singles becomes a tens block. So you end up with 400,000 + 30,000 + 6,000 + 700 + 80."

Roberto said, "Each time you multiply a digit by 10, the digit stays the same, but the unit it counts is ten times as large, and it goes into the next place."

Elspeth had described the steps without demonstrating with the blocks, and then Roberto described the phenomenon in general terms. In order to bring more people in, I repeated what Roberto and Elspeth said while arranging blocks, but starting at the right. I pointed to the 8 ones and said, "When you multiply a digit by 10, the digit stays the same, but the unit it counts is ten times larger. So 8 ones become 8 tens." I held up 8 tens-blocks. "Then the 7 tens," I pointed to the tens blocks in Jade's arrangement, "become 7 hundreds," and I pulled out 7 hundreds-blocks. I was running out of blocks, and so I went through the rest of the digits gesturing with my hands.

I said that it was time to end this activity, but we would continue thinking about the ideas. "In fact," I said, "some of the students in the video and print cases are working on similar ideas. So let's look at some video before we take a break."

Discussing the video: Interviews with three students

I explained that we would be watching clips of interviews with three students. In these clips, the interviewer is not trying to teach. Instead, the interviewer is trying to learn about the students' thinking. We would look at each clip and then consider two questions: What does the student understand? What does the student have yet to learn?

We viewed the first clip in which the interviewer, Jill, asks six-year-old Chris about the numbers 1,000; 1,005; 102; and 110. Then I asked the group, "What does Christopher understand about these numbers?"

Carol volunteered, “There are three digits or places in the hundreds.” Camisha said, “To write one hundred anything, you always start with 1.” Gaye said, “He knows something about zeros.”

Elspeth added, “He sees them as a place holder.”

I asked, “Does he know about zero, or does he know about the symbols used to write hundreds or thousands?”

Nobody took on my question, but instead participants continued to volunteer ideas of what Chris understood.

Beatrice said, “He has a sense of the number of places needed to write a number in the hundreds and thousands.”

I asked, “What seems to be missing for Chris? What does he still need to learn?”

Now Beatrice went back to my question. She wondered, “Does he know what zero means?”

Damaris went further, “Does Chris understand what each digit in a number stands for? If he knew that the second digit from the right stands for tens, he wouldn’t think one thousand five is written as 1,050.”

Marina commented, “Chris doesn’t seem confident in his answers.”

Iris said, “I’m concerned about what the interviewer was doing. When young children are learning math, math has right and wrong answers. You should let them know what’s right or wrong.”

It was likely Iris was expressing a sentiment that other members of the group were feeling. However, I didn’t want to move into a discussion about what the interviewer should have done instead—at least not yet. And some of what I’ve been trying to do is to open up the assumption that the work of the teacher is to keep students clear of confusion. This arose in the last session around Lucy’s case, and here it is again. I responded, “Iris is raising an important question about the responsibility of the interviewer. I’d like to hold off on the question for a few minutes. For now, I want us to focus on what we can learn about the students’ thinking. So let’s first look at the other two clips and then return to Iris’s concern.”

I then showed the clip of an interviewer, Deborah, asking nine-year-old Cole about the number of tens in 341. Afterwards, Gaye said, “At first he said there are 4 tens in 341, and then he said there are 34 tens. I’ve always said there are 4 tens in 341. Is that wrong?”

I put the question back to the participants, “What do you think about Gaye’s question?”

Marina said, “We were always taught that 341 means 3 hundreds, 4 tens, and 1 one so that’s what we’ve been teaching.”

Karran pointed out, “But when we’re working on division and we ask how many tens are in 341, then the answer is 34 remainder 1, or 34 and $^{1}/_{10}$, or 34 point 1.”

Carol asked, “So does the answer depend on what you’re currently studying? That doesn’t seem right.”

I responded, “There are different ways to decompose 341. One way is 3 hundreds, 4 tens, and 1 one. Another way is 2 hundreds, 14 tens, and 1 one. What are some other ways?”

Huong said, “You can do 1 hundred, 24 tens, and 1 one, and you can do 34 tens and 1 one.”

Roberto suggested, “You can also say 3 hundreds, 3 tens, and 11 ones.”

Several participants laughed, “There are lots of ways to decompose that number!”

Then Iris asked, “So if someone asks how many tens are in 341, what’s the right answer?”

I responded, “The most important thing is to understand that there are different ways to think about it as well as different ways to understand what those ways mean. But I think if the question is posed without indicating any other context, I would say there are 34 tens in 341.”

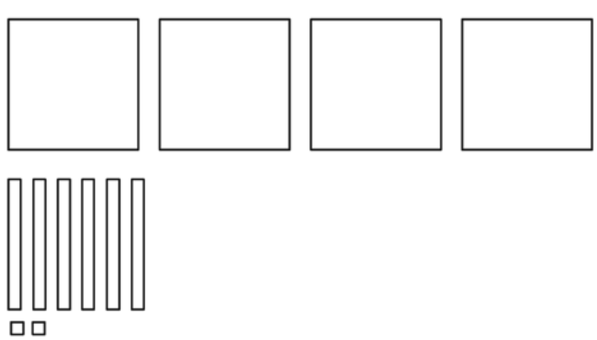
I was about to move on to the next clip when Roberto said, “But, you know, Cole didn’t first say there were 4 tens or even 34 tens in 341. He said there were 40 tens, and then 70 tens.”

Penny said, “Yeah. He first looked at the 40 and thought of it as 40 tens instead of 4 in the tens place makes 40 ones. But what’s strange is that then he recognized the 3 in the hundreds place as 30 tens and added that to 40.”

Damaris added, “He corrected himself, so he does know. It seems to me that just illustrates how complicated it is to decode exactly how the place-value system works.”

I said, “One thing we’re noticing is how challenging it is to keep track of the different units—ones, tens, and hundreds. That was something we saw in the last session with Sarah, the student in Lucy’s case. It’s not only being able to keep track of the different units but also keeping track of how they are related to each other. This is an important idea that you should keep in mind as you read cases. It’s also something important to pay attention to when you’re interacting with your students.”

Finally, I showed the clip of an interviewer, Virginia, questioning sixth-grader Jenna about the following representation:



After Jenna confirmed that the number represented 462, Virginia asked what quantity would be represented if the small cube is worth 10, and then if the small cube is worth 100.

After viewing the tape, Damaris said, “It’s interesting, she seems to know that she needs to multiply the quantity by 10, but she doesn’t just apply the add-a-zero rule.”

Penny said, “I wonder what she would do if you just gave her the problem, 642 × 10.”

I asked, “So if she’s not applying that rule, what is she doing?”

Beatrice said, “She seems to be thinking about what happens to each type of block. Like when she figures out the value when the small cube is worth 100, she says, ‘forty thousand plus six thousand plus two hundred.’ She figures out each place separately and then adds it all together.”

Roberto asked, “What was going on when she said 4,080? That was when she was determining the value when the small cube is worth 10.”

Lilo said, “I think she multiplied the ones by 10 to get 20, but didn’t multiply the 6 tens by 10, so she added 20 + 60.”

Damaris added, “But she did multiply the 4 hundreds by 10.” Roberto nodded, “I see. But then she corrected herself.”

Before ending the discussion of the video, it was important to return to Iris’s question. After all, I was going to ask participants to interview a student before the next session. I wanted them to be able to keep the interview open enough for the student to be able to think things through. So I said, “Let’s go back to Iris’s concern. Iris, do you want to remind us of what you said?”

Iris said, “Yes. I felt it most strongly about Chris, but I saw it in the other interviews, too. The interviewers never gave the students any clue about whether they were right or wrong. They just asked questions. I don’t think that’s fair to the students.”

I felt that it was important to get other points of view, so I said, “I’m quite confident that Iris isn’t alone in the room with these feelings. Does anyone feel differently?”

Penny said, “You know, I’ve been thinking all week about Lucy’s case that we discussed in the last session. At first we all thought her student, Sarah, shouldn’t have been allowed to use the manipulatives she chose because it led her to an error. But when we looked at the case more closely, we saw that it was an opportunity for her to learn. So I’m wondering if there’s something like that going on here.”

I looked around to see if anyone else wanted to speak. Marina said, “I’ve been thinking about the last session, too. When we were playing ‘Close to 100,’ it was important for me to try some things that didn’t work very well, in order to figure out what does work.”

I chose to inject an additional idea, “You’re talking about what students might learn if the teacher isn’t too quick to prevent confusion. But the purpose of the interview is for the interviewer to

learn about students’ ideas. So that’s another question to consider: What can be learned about a student’s thinking if the interviewer is quick to say what’s right and what’s wrong?”

While I wanted participants to consider these questions, I didn’t want Iris to feel isolated or put down for raising her concerns. For that reason, I said, “I also want us to take Iris’s concern seriously. Although the purpose of the interview is to learn about the student’s ideas, the ultimate responsibility of the teacher is to help students learn. So another question we might ask is given what we’ve learned about the thinking of Chris, Cole, and Jenna, what might be next steps a teacher could take? That is, what would we want the students to learn next, and how would we help them do so? You might think about what would be on the tape if the interviewer had another 15 minutes and took off the interviewer hat to become a teacher.”

It was time to end this discussion, so I called a break.

Small-group case discussion: Ideas about the number system

After the break, I organized participants into new groups according to grade level and distributed the focus questions about the print cases. I said, “Just as in the video, when we examine students’ thinking as they confront the ways numbers are represented, we learn about different facets of the base-ten structure of numbers.”

At first, as I moved from group to group, I felt disappointed. Once again, participants were thinking about how to prevent students from making errors rather than considering implications of those errors. For example, in response to Dawn’s case 6, “Number of Days in School,” there was some discussion about whether a straight number line from one to one hundred is more appropriate for kindergartners than the hundreds chart, but there was no reference to the importance of considering the ideas those kindergartners might be working on. I’m not saying that the teachers shouldn’t be thinking about the value of different representations, but there are other issues to be considered first. I want them to learn how to use the cases to consider, say, how the different representations might highlight various patterns that would help children begin to decide whether “fifty-ten” or “sixty” follows 59. They might also consider how different children in the class used the hundreds chart to argue that the number after 59 is 60. The insights the teachers would gain from such investigations could then inform the decisions they make when confronted with situations like those described in the cases.

On the other hand, when participants came to Danielle’s case 7, about how different children wrote the number one hundred ninety-five, they did discuss student thinking. To explain 10095, Camisha said, “There’s the 100 and the 95, and the child just put them together.”

Andrea asked, “Then why doesn’t he write 100905?”

Elspeth pointed out that maybe he already knows that ninety-five is written 95.

In another group, Huong asked, “What about the child who wrote 1095?”

Fatima suggested, “Maybe he already knows 101, 102, 103, … So maybe he thinks you put 10 for “one hundred’ in front of the other part of the number that’s spoken.”

When Gaye said the child who wrote 195 must understand place value, Damaris responded, “Or maybe, like we said about Chris, he knows that you use three digits to write a number in the hundreds. That doesn’t mean he knows that 195 is composed of 1 hundred, 9 tens, and 5 ones.”

I also felt more satisfied when I heard the discussion of Donna’s case 9, “Groups and Leftovers.” Donna had given her second graders an activity in which they grabbed handfuls of beans, divided them into groups of a specified size, and filled out a chart identifying the number in a group, the number of whole groups, the number of leftovers, and the total number. When I got to Louise, Marina, and Troy, they reported to me they needed to do the activity together before they could see what was going on. After they figured out the mathematics, Louise had this pedagogical insight: “Putting the beans into different-size groups—it made me realize that I leave that step out. If you just do tens, you don’t see what is unusual about tens. That made me realize that sometimes I think I’m being open and discovery oriented, but I’m really looking for my students to see this thing about ten, and so I set them up with forty activities all about groups of ten. Then I feel like, ‘Well, after all these activities, don’t you see the pattern here?’ But they never got to see when the pattern *doesn’t* hold!”

It seemed to me that Louise had figured out something about the power of the system of tens—but also that the only way you can appreciate that power is to have the opportunity to explore other amounts, such as groups of threes, sevens, and so on. Students need to think about lots of ways of grouping in order to notice what grouping by tens offers.

Yet, even though Louise is still talking about what the teacher does, why do I find her remarks more satisfying? Because she’s talking about making different decisions based on new insights and the learning gained by truly engaging with mathematics. That’s what I’m after.

Teachers of the older grades spent a lot of time on Leslie’s case 10. They were fascinated by the different number lines constructed by a class of sixth graders. They especially focused on decoding Olivia’s pictures, which were drawn to different scales.

After about fifteen minutes of small-group discussion, I asked participants to turn to questions 4 and 5 if they hadn’t already. This gave everyone a chance to create their own number line from 0 to 10,000 and figure out where to place 375 before they examined the work of a few students given the same task.

Whole-group case discussion: Number lines

When I brought participants together for whole-group discussion, I asked them to write down what surprised them or what they learned from creating their own number line or examining those of the students in Leslie’s case 10. After about two minutes, I had them share with one other person. Then I asked that we share these insights as a whole group.

Camisha started us off. “As a primary-grade teacher, I usually think of 375 as a big number. When I drew the number line from 0 to 10,000, it surprised me to see how close to 0 it is.”

Roberto added, “Or, if you look at Anthony’s work, how far away 10,000 is.”

Gaye said, “I thought Olivia’s magnifying glass was really cool.”

I asked, “What did the magnifying glass seem to indicate? What did Olivia seem to understand?”

Louise said, “There’s something Olivia understands about scale. When you look at a number line from 0 to 10,000, it’s hard to see where 375 is in relation to 0 and 500. If you blow it up a little, you are better able to see that 375 is to the left of 500, but closer to 500 than to 0.”

Damaris pointed out, “Not all the students seemed to notice how close 375 is to 0. For example, look at Shaquille and Chris.”

I asked, “What do you notice about the work of those two? Is there anything you’d like to ask them?”

Beatrice said, “I’d like to ask Shaquille about the distance between 1,050 and 10,000. From his number line, it seems that 1,050 is about two-thirds the distance to 10,000.”

Gaye said, “It seems to me he just ran out of space. He knows the order of the numbers.”

Beatrice responded, “I’m not so sure. I would want to ask him some questions to find out what he thinks.”

I used this opportunity to make a point about using cases to analyze student thinking. I said, “All the evidence we have is what is written in the case. On the basis of this evidence, we might make different conjectures. For example, Beatrice offers the hypothesis that Shaquille might think 1,050 is about two-thirds the distance from 0 to 10,000. Gaye’s theory is that Shaquille knows that 10,000 is much further from 1,050 but ran out of space. Both conjectures are plausible. But having thought about these different possibilities, if we were to encounter a student like Shaquille, then, as Beatrice said she’d like to do, we would be in a position to ask him further questions.”

Jade brought us back to the specifics of the case. “Chris wrote that he counted by hundreds. But look, he counted by hundreds until he got to 1,000. Then he started counting by thousands, and the intervals he marked are the same as for 100.”

Karran asked, “Do you think he doesn’t realize he stopped counting by hundreds?”

Roberto wondered whether Chris understands that on a number line equal intervals indicate equal distance between numbers. He said, “Maybe he knows that some of his intervals represent a distance of 100 and doesn’t see a problem with the same size intervals representing a distance of 1,000.”

This was the kind of discussion I was happy to hear. Participants weren’t talking about how to prevent Shaquille and Chris from making mistakes but were considering what the work of these students reveals about their understanding. Furthermore, they then wanted to know more about the students’ thinking. This is precisely the habit I want these teachers to develop, and they would have a chance to practice it with one of their own students in the homework assignment for the next session.

I then suggested we turn back to the kindergarten and first-grade cases. I acknowledged that not everyone in the seminar had discussed them in detail, but everyone had read the cases for homework. I said, "Just as we saw with the sixth graders, the errors these young students make point out significant aspects of our place-value system. What do you see?"

Jade said, "I had never thought about a student writing one hundred ninety-five exactly as he hears it—100905. If you add plus signs, it's right." Jade came to the board and wrote 100905 and then 100 + 90 + 5.

Huong said, "I see that kind of error pretty frequently. I just never thought before about how much sense it makes. It's not so obvious that 90 + 5 should be written as 95."

Elspeth added, "And if you learn that 50 + 9 is written 59, then why isn't 50 + 10 written 510? It was interesting to me to see the different patterns in the hundreds chart and which ones the kindergartners were paying attention to."

Lilo said, "When they get to the multiples of ten, looking at the pattern of the row suddenly doesn't help. You have to look at the pattern in the column."

I was pleased to see that although participants met in grade-level groups and focused on different questions, now the discussion spanned grade levels. Higher grade teachers were interested in the thinking of these young students as well.

Homework and exit cards

I ended the whole-group discussion with fifteen minutes left in the session to provide some time for participants to prepare for their assignment to interview a student. As I distributed the assignment sheet, I pointed out that in the last discussion, participants identified what they would ask Shaquille and Chris in order to understand more about what they were thinking. Now they would be able to engage in a similar task with one of their own students. I said, "The assignment asks you to conduct a math interview with one of your students to explore what that student understands about the content we have been studying. You might ask about the base-ten structure of numbers, number lines, place value, or strategies for adding and subtracting. Read over the sheet and let me know if you have any questions." Because there weren't any questions, I asked them to pair up with someone close to their grade level to think about some of the questions they would like to ask in their interview.

With five minutes left in the session, I distributed index cards and asked participants to address the following exit questions:

1. What was important or significant to you in the mathematics discussed at this session?
2. What do you want to tell me about how the seminar is working for you?

Two main points emerged from the first question: 1) the power of having a visual representation of multiplying by 10 and 2) numbers can be decomposed in more than one way; for example, the number 320 can be seen as composed of 32 tens or 3 hundreds and 2 tens. A few also commented on the power of the number line from 0 to 10,000.

I particularly looked for the exit cards of the participants who were very quiet in this session, Nancy and Andrea.

Nancy: Looking at math in this way (I guess a nontraditional way) is very difficult for me. At times I feel very overwhelmed and almost "stupid" that I can't wrap my brain around it. But I think it is an excellent thing to be doing, to be pushing my mind in this way.

Andrea: I am enjoying the seminar, and at times feel like my brain is taxed. This is not necessarily a bad thing, but I am aware of how much I have to work and concentrate to understand and work through the concepts we are focusing on. Perhaps I will lose my math phobia by the final session.

Reflecting on the lesson

I found the participants' engagement with the content of the lesson very interesting. Sometimes it's challenging to help them delve into familiar content to discover new insights. Perhaps my framing the lesson in terms of what we want students to understand helped us get over that hurdle. Once they were into the content, there were plenty of new insights for each individual.

Several participants are struggling with pedagogical aspects of the seminar. These teachers still view errors as something to avoid rather than an opportunity to learn. I'd rather that they see it's not always productive for a teacher to structure an activity to prevent student errors. If a student misunderstands a piece of mathematics (and all of us have some gaps or misunderstanding), that misunderstanding can come to light through an error. Either student or teacher or both might recognize that there is a problem. That very recognition allows the possibility of rooting out the error and arriving at new understanding.

I chose to group teachers in the case discussion according to grade band. I wanted the primary teachers to spend time on the first three cases even if it meant they wouldn't get to case 11. But though case 10 comes from a sixth-grade classroom, I still wanted these teachers to get there because it raises significant issues about number lines as well as the relative magnitude of numbers between 0 and 10,000.

Responding to the second homework

September 28

The homework that participants turned in for this session asked them to analyze the following two strategies for subtracting 123 – 76, which they had seen students explain in the video of seventh graders shown in Session 1, as follows:

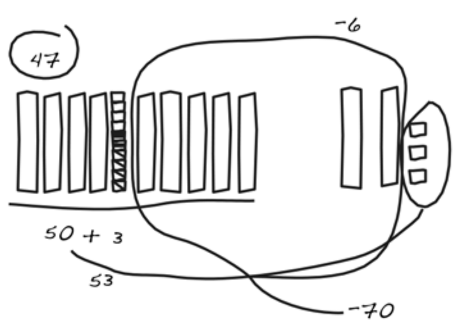
1. "I changed the 123 to 120, and the 76 to 70. Then I did 120 – 70 = 50. Then I added the 3 to the 50, and then I took the 6 away and got 47."
2. "I did the inverse operation. I did 76 + something = 123. I tried to get to 100. That would be 24 and then there is 23 more. So it is 47."

Participants were to demonstrate the first strategy with a base-ten representation and the second with a number line, apply the strategies to another pair of numbers, describe the strategies, and identify the mathematical ideas needed to understand each.

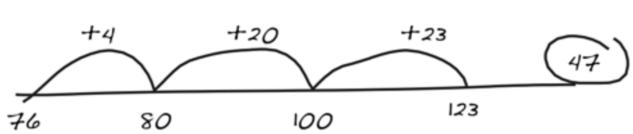
Almost everyone was able to demonstrate the strategies using the representations, both for 123 – 76 and for another pair of numbers of their choice. What was missing from most was an explanation of why the strategies worked. For example, consider Troy's work.

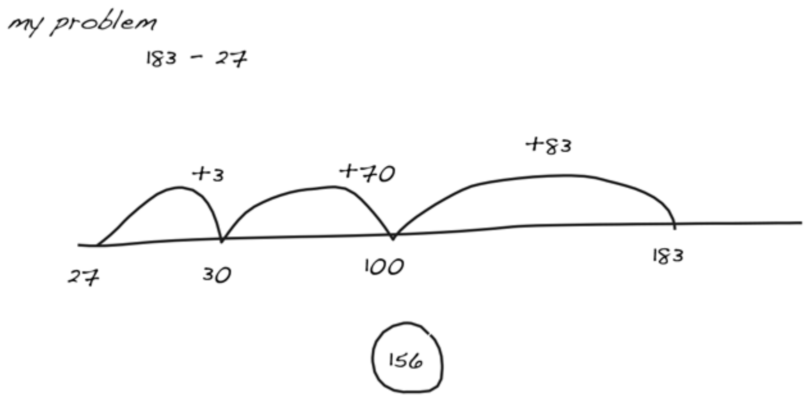
Troy's homework

Strategy 1 for 123 – 76



Strategy 2 for 123 – 76





183 - 27

(-3) (+3)

180 - 30 = 150 + 6 =
156

In the first strategy, I started with 183 and subtracted 3 to get to 180. I also added 3 to 27 to get to 30. Then I subtracted 30 from 180 and added the 6 back on to get to 156.

Math ideas:

- base ten
- similarities of addition/subtraction
- that subtracting numbers means "negative" numbers

I would like Troy to be able to explain the reasoning behind the procedures, and I'd like him to become more precise in identifying the mathematical ideas employed in these strategies. At the same time, I want to recognize the strengths exhibited in Troy's homework. I want my response to be encouraging, but also challenging. I couldn't respond to everything in his work, so this is what I wrote:

> Dear Troy,
>
> You have indicated the steps of the students' two strategies, using both a number line and a base-ten representation. Your representations are correct and clearly drawn. Nice work. Though I have several questions you might consider.
>
> When I look at the first strategy for 123 – 76, I don't have a sense of why the steps give you the correct answer. Why do you add 3 onto 50 and then take off 6 to get 47? I know that's the student's strategy, but I wonder why you think it works.
>
> The number lines that you drew don't have the property of equal intervals indicating equal distance between numbers. For example, in the number line for 183 – 27, the interval between 27 and 30 is almost as large as the interval between 30 and 100. Perhaps in this

sketch it doesn't matter. But this raises the question for me as to when it is important for number lines to be drawn to scale and when it doesn't matter.

In the number line representation of the student's second strategy for 123 – 76, you indicate a jump from 76 to 80 and then to 100, whereas the student said that she immediately aimed for 100 and added 24, without first stopping at 80. What does this tell us about the student? What does she know or understand?

You state that "base ten" is an idea used in the strategies, but "base ten" covers a lot of territory. What specific aspects of the base-ten structure of numbers are used in one strategy but not the other? What aspects are used in both?

What are the similarities between addition and subtraction, and how do they show up in one strategy or the other?

I'm not saying that you should have addressed all of these questions in your homework. Rather, as I look over your work, these are questions that come to my mind, and it may be useful for you to put some thought into them.

There is one error in your work I'd like to alert you to. You wrote the following statement:

$$180 - 30 = 150 + 6 = 156$$

I know what you intended, and it's notation that's commonly used, but it is incorrect. The symbol = means that the expressions on either side are equivalent—they equal the same amount. For example, you could write 3 + 5 = 5 + 3. What you have written actually means that 180 – 30 = 156, and of course, we know that is wrong. Even though it takes longer to write, the correct notation is to write two equations:

$$180 - 30 = 150$$
$$150 + 6 = 156$$

I appreciate your participation in our sessions and your willingness to share your ideas. I'm looking forward to working with you through the rest of the seminar.

Maxine

I said something about participation in the sessions to each participant. In particular, I was encouraging to Nancy and Andrea and mentioned how much I appreciated their honesty in their exit cards. I said that as they became more comfortable in the sessions, I hoped to hear their voices.