Mathematical background notes

Standards for Mathematical Practice

Practice 3: Construct viable arguments and critique the reasoning of others.

Mathematically proficient students at the elementary grades construct mathematical arguments—that is, explain the reasoning underlying a strategy, solution, or conjecture—using concrete referents such as objects, drawings, diagrams, and actions. … Mathematically proficient students can listen to or read the arguments of others, decide whether they make sense, ask useful questions to clarify or improve the arguments, and build on those arguments. They can communicate their arguments, compare them to others, and reconsider their own arguments in response to the critiques of others. (Illustrative Mathematics 2014)

In this session, participants discuss norms of behavior for their own work together in the seminar. During this discussion, include remarks about practice 3, particularly what it means to critique one another's reasoning. "Critique" does not mean criticize; rather it involves following each other's thinking. When participants ask questions to better understand a representation or solution strategy, they are critiquing each other's arguments.

At the end of the math activity "Multiplying by 10," ask participants to reflect on practice 3 in relation to how they shared and followed one another's explanations.

Practice 7: Look for and make use of structure.

Mathematically proficient students at the elementary grades use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (e.g., the less you subtract, the greater the difference), and attributes of shapes to solve problems. (Illustrative Mathematics 2014)

During the math activity and the case discussion, ask questions such as, "How does this representation show the base-ten structure?" or "How does this student's thinking about how to write one hundred and ninety-five with number symbols help you understand how he views the number system?" Refer to Maxine's Journal line numbers 163–202 for an example of this kind of discussion. While Maxine does not explicitly cite practice 7, you should make the connection explicit.

Practice 4, model with mathematics: Highlighting a common misconception

When given a problem in a contextual situation, mathematically proficient students at the elementary grades can identify the mathematical elements of a situation and create a mathematical model that shows those mathematical elements and relationships among them. (Illustrative Mathematics 2014)

Note that the word *model* has two different definitions that apply in mathematics education. Frequently, *model* indicates a physical representation, such as Sarah's cubes in case 4 or the models with base-ten blocks that participants use in the math activity as illustrated in lines 165–180 of Maxine's Journal. These physical embodiments of mathematics are powerful tools to support the reasoning of participants and their students. However they are not examples of practice 4.

A second definition applies to practice 4: a description of a situational context using mathematical concepts. In order to fit the meaning of practice 4, the problem begins outside of mathematics in a real-world context. Students abstract the mathematical aspects from the context, expressing those in mathematical language, diagrams, or physical representations.

In *Building a System of Tens,* the word *model* refers to the first definition, a physical representation of a mathematical situation. This is unrelated to practice 4. It might be necessary to clarify this distinction so participants do not leave with the misconception that building models of numbers using base-ten blocks is an example of practice 4.

An example of integrating content and practice standards

The Standards for Mathematical Practice are intended to be integrated with the Standards for Mathematical Content. Students can only learn how to engage in these practices meaningfully in the context of engaging with content. In this facilitator note, we provide an example of how content and practice standards are integrated in the Session 2 seminar activities. Examples based on Sessions 3 and 6 are described in those sessions. Facilitators should use these examples as guidelines for integrating the content and practice standards in the other sessions as well.

After considering how knowledge of place value is applied in addition and subtraction strategies in Session 1, participants focus in Session 2 on the structure of the base-ten system. In the math activity “Multiplying by 10,” participants multiply numbers by 10, 100, and other powers of ten. They create representations to help them describe what occurs and explain why it occurs. The cases of chapter 2 focus on how students learn to understand the place-value system across the grades. These cases include students’ attempts to coordinate and make sense of the way numbers are written and spoken, to represent small and large numbers on a number line, and to understand the structure of large numbers. Teachers consider common student errors, such as counting fifty-eight, fifty-nine, fifty-ten, and how such errors reveal students’ logic as they are learning to make sense of the base-ten system.

This work relates to Common Core content standards across the grades that focus on the following three aspects of our number system:

- The place-value system is built on powers of ten.
- The relationship between places in a multidigit number is multiplicative, a digit in any place represents ten times as much as the same digit in the place to its right.
- Written numbers and spoken numbers call upon this place-value structure in different ways.

The content standards relating to these ideas span the grades. For instance, a grade 1 standard states that students should “understand that the two digits of a two-digit number represent amounts of tens and ones” (1.NBT. 2), while a grade 5 standard states that students are expected to “recognize that in a multidigit number, a digit in one place represents ten times as much as it represents in the place to its right and $1/10$ of what it represents in the place to its left” (5.NBT.1).

Participants are also working with practice 8 (look for and express regularity in repeated reasoning) in this session as they describe the regularity of the base-ten system. For example,

in problem 1 of the math activity, they consider the set of problems 23×1, 23×10, 23×100, $23 \times 1{,}000$, and $23 \times 10{,}000$. They consider the regularity of the progression of products (23, 230, 2,300, etc.), and describe and explain why this regularity occurs. When they call upon the multiplicative structures in their representations, participants are enacting practice 7. As they move from description to explanation, they are engaging in practice 3 and practice 6, constructing and articulating their arguments. Through this activity, the teachers recognize these practices in their own learning, while the cases illustrate how teachers can help students learn to engage in these practices. For example, an instance of engaging students in practice 8 occurs in case 9, in which second graders notice that when they divide a number of beans (e.g., 67) into groups of ten and leftovers, the number of tens (6) and the number of leftovers (7) together create the numeral for the total amount; they notice that this only happens when they use groups of ten, not when they use groups of any other amount. The teacher has set up the activity and a way of recording it that highlights this regularity for students. She then encourages them to express what they are noticing.

Place value and multiplying by 10

Although many students begin working with the principles of place value before they learn to multiply, the concept of multiplication lies at the heart of understanding place value. Examining what happens when multiplying by 10 helps make explicit some fundamental ideas about the way numbers are written in the base-ten place-value system.

Breaking down a multidigit number into a sum of single-digit multiples of powers of ten is known as expanded notation. Consider the following:

$$43{,}567 = 4(10{,}000) + 3(1{,}000) + 5(100) + 6(10) + 7(1)$$

In expanded notation, the digit in each place is multiplied by the power of ten, which that place represents; these products are then added to determine the value of the number. Each place value is ten times larger than the place to its right.

For expanded notation, the multipliers are the digits from 0 to 9; however, we can decompose numbers into sums of multiples of powers of ten in different ways by dropping this constraint. In fact, when we read aloud or write out the number above, it is *forty-three thousand five hundred sixty-seven*. Consider the first part of this number: $43{,}000 = 4(10{,}000) + 3(1{,}000) = 43(1{,}000)$ as well as 43 thousands. Looking at this portion of the number in two ways points out that while the place-value decomposition would be 4 ten-thousands plus 3 thousands, the way we verbalize the number is 43 thousands.

Now consider 567. While expanded notation leads to $5(100) + 6(10) + 7(1)$, it is also correct to say this is 56 tens + 7 ones. In fact, this is also the same as 55 tens + 17 ones or 54 tens + 27 ones, and many other such decompositions. Many computational procedures are dependent on such flexibility. For instance, in applying the traditional subtraction algorithm for a problem such as $45 - 28$, we decompose the 45 into $30 + 15$ and then subtract $15 - 8 = 7$ in the ones place and 3 tens – 2 tens = 1 ten, producing the final answer of 17. (Session 3 of this seminar includes more examples of how place-value decompositions of number are called upon in computational procedures.)

The fact that the place-value system is based on powers of ten and the fact that it is possible to decompose a number without the single-digit constraint of expanded notation suggest connections between the way a number is written and multiplying by 10. Consider a situation in which there are 24 objects. If each object is worth 1, the total is 24, or 24 ones. If each object is worth 10, the quantity is 24 tens, or 240.

Examining different ways to represent multiplying by 10 can illustrate why the familiar rule “just place a zero at the end of the number to multiply by 10” works. Consider these possible ways to think about 342×10:

Arithmetic approach using expanded notation

$$342 \times 10 = [3(100) + 4(10) + 2(1)] \times 10$$

There are two ways to perform this multiplication. The steps for both procedures can be justified by calling upon the distributive, commutative, and associative properties.

One way is to multiply each of the digits by 10, resulting in 30(100) + 40(10) + 20(1), and then regrouping the powers of ten into 3(1,000) + 4(100) + 2(10), or 3,420.

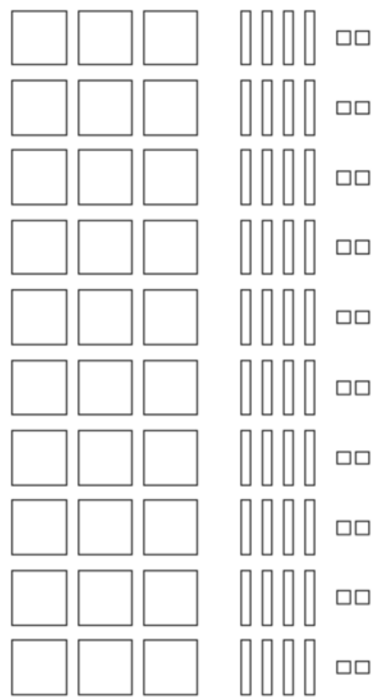
Another way is to multiply each of the powers of ten by 10, resulting directly in 3(1,000) + 4(100) + 2(10), or 3,420.

In either case, the sequence of digits is the same, but each digit is in the place to its left and there are zero ones.

Using base-ten models

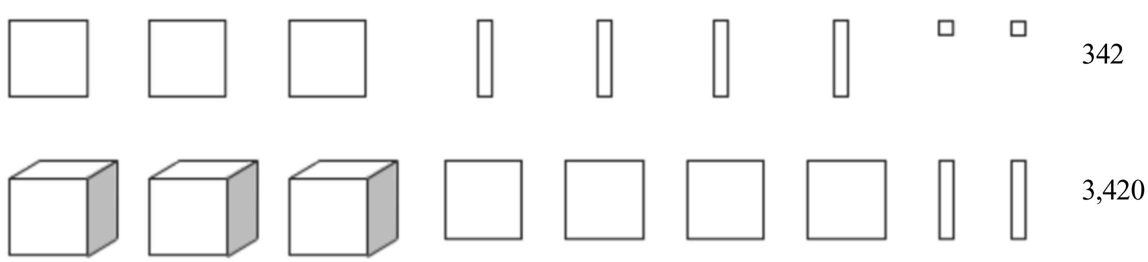
The number 342 can be represented by 3 flats, 4 longs, and 2 units.

One way to consider multiplying by ten would be to replicate this arrangement ten times.



This arrangement of blocks can be interpreted in two ways: 10(300 + 40 + 2) or as each column of hundreds representing 1,000, each column of tens representing 100, and each column of units representing 10, resulting in 3 thousands, 4 hundreds, and 2 tens, or 3,420.

A second method to do this multiplication would be to note that each of the blocks, when multiplied by ten, would become the next larger block. So instead of 2 ones, there would be 2 longs (tens); instead of 4 longs there will be 4 flats (hundreds); and instead of 3 flats, there will be 3 cubes (thousands). This matches 3(1,000) + 4(100) + 2(10).



The arithmetic and the base-ten models illustrate two aspects of multiplying a number by ten:

1. The resulting product has no ones. This explains why a zero occurs in the ones place.
2. The digits of the result are the same, but each is associated with a power of ten one greater than the original number; that is, the number of ones has become the number of tens, the number of tens has become the numbers of hundreds, and so forth. This explains why the sequence of digits remains the same.