When children are able to pay attention to how all the amounts in a problem are related to one another, when they understand which numbers represent parts and which designate wholes, they are able to select solution strategies that don't mimic the specific action of the problem. Understanding how parts and wholes are related to one another in addition and subtraction problems also allows children to combine and separate quantities more flexibly. Students often use strategies based on facts they already know when they get to this stage. They take apart numbers and recombine them to form new quantities that they find easy to work with (Fuson 1992, 2003; Verschaffel et al. 2007; Fuson and Beckmann 2012). In case 5, for example, Cecile thinks about 6 + 8 as if it were 7 + 7, saying, "7 + 7 is easier for me to think about and that makes 14, so if I move 1 from one of the 7s to the other, I have 6 + 8 and that is 14" (lines 000–000). Cecile no longer needs to represent the specific amounts in the problem, nor does she rely upon counting when she needs to add.

It is interesting that the growth from modeling all quantities and actions in a combining or separating problem to abstract reasoning with numbers is not a smooth or consistent transition for children. Some students appear to be at several levels simultaneously, able to count on when solving some problems but needing to model directly when making sense of a scenario that involves something harder, like separating or grouping, as shown by the kindergartners in case 8 when counting legs on bunnies and eggs in baskets. These children are mostly still modeling every quantity in a problem and counting to get the total. At the same time, some are also grouping amounts in an array format, constructing 3 rows of 4 cubes each to represent the number of legs on three bunnies, which is a conventional representation for multiplication. They haven't begun to actually multiply and are still modeling all of the amounts directly, counting by ones to find their answers, but they are starting to expand their concept of addition to include the idea of combining equal groups.

Section 3

Modeling multiplication and division

Although children come to an understanding of multiplication and division by modeling actions in a way similar to what they do for addition and subtraction, there are important and subtle differences.

One line of research suggests that children first encounter multiplication when having to combine several groups of the same size (Fischbein et al. 1985; Greer 1992; theory cited in Fuson 2003; and in Verschaffel et al. 2007). Students again begin by modeling directly all the quantities and actions in a given problem. When trying to find out how many cans are in 3 six-packs of soda, for example, a child may first count out 6 objects, next count out 6 more, and then create a third group of 6, finally counting all to get a result. As students move beyond the need to show every item in a group of objects, they use a more abstract counting strategy, just as they do for addition and subtraction. However, instead of representing one amount and counting up

or back from there, keeping track of the number of counts made, students might skip-count to or back from a given number while tallying the number of counts (Carpenter et al. 1996; Carpenter et al. 1999; Verschaffel et al. 2007).

In the addition or subtraction situation, students are counting individual objects, whereas in the multiplication or division scenario, they are counting both distinct objects and groups of objects. Thus, when adding the number of soda cans in 3 six-packs, a student may start at 6, count up 6 more to get 12 sodas with the second six-pack, and then count up 6 more to get a total of 18. In this case, the student is counting every soda can in each six-pack, **6**, 7, 8, 9, 10, 11, **12**, 13, 14, 15, 16, 17, **18**. When multiplying, the child might put up one finger and count, "6," another and say "12," and a third finger, saying "18." In this latter case of skip counting, each of the three fingers raised represents a group of 6.

Later, according to this line of research, students are able to keep in mind all of the amounts in a multiplication or division problem and can break up numbers to use them flexibly (Baek 1998; Carpenter et al. 1996; Fuson 2003; Verschaffel et al. 2007). Examples of students making sense of division at this level of abstraction, as well as a glimpse of how difficult it is to keep track of whether one is counting single objects or groups of objects when multiplying, are found in Janine's case 10. In this episode, three girls are trying to figure out how many packages containing 6 candy canes they would need so that each of 609 students could have one candy cane. Janine writes:

> Almost immediately the three decided that the problem had something to do with the 6 times table—and that they had to go pretty far up the table to get to where they wanted to be. They started "6, 12, 18, 24, 30," and soon realized that there had to be a faster way. (p. 38, lines 216–219)
>
> [One student knew that 6 × 9 was 54, so 6 × 90 would be 540.] From 540 they decided that they had to add up by 6s to get to 609. … Letitia, after some thought, said that 540 + 10 more would equa1 600. One of the others said that 540 + 10 is equal to 550 and not 600. Letitia struggled to explain. … She knew that 10 was important, but couldn't hold onto the idea that it wasn't 10 but 10 *groups* of 6 she was thinking about. (p. 38, lines 226–232)

Letitia works this through, and her group decides that 90 packages and 10 more packages would provide 600 candy canes. Another package would bring them to 606, but they recognize that they would need still another package to supply 609 students with candy. Here, as in addition and subtraction, the children use relationships they already know in making sense of multiplication and division situations; and they also demonstrate that "skip counting isn't a pure strategy," but is often used with addition when the most familiar sequences (2s, 3s, 5s, 10s) aren't employed or when knowledge of a specific progression of multiples, in this case 6s, gives out (Empson, personal communication, May 20, 2008).

Just as students come to understand multiplication as a way of counting groups or as a form of repeated addition, they may understand division as a form of repeated subtraction. In case

11, for instance, Vanessa divides 8 into 24 by writing 24 – 8 = 16, 16 – 8 = 8, 8 – 8 = 0, and then writes 3 for the answer (p. 40, lines 263). We can see, then, that some of these processes are similar to the modeling and counting strategies children use to figure out what addition and subtraction are.

However, multiplication and division are different from addition and subtraction in ways that aren't instantly apparent. Addition and subtraction situations involve numbers that count or measure amounts directly. Each number represents a set of something, a simple quantity. Problems are solved by joining, separating, finding missing parts, or comparing sets, and the result is a third amount of the same kind as the other two: 3 donuts plus 4 donuts gives you 7 donuts (Carpenter et al. 1999; Dienes and Golding [1966] as cited by Harel and Confrey 1994; Fuson 2003; Hiebert and Behr 1988; Schwartz 1988). Multiplication, though, frequently involves two factors that represent different kinds of things; for instance, 3 bags each contain 4 donuts; how many donuts do I have? One factor is the number of sets, whereas the other is the number of items per set; both of these amounts must be tracked. The result is the quantity of donuts in all three bags, or in other words, the items contained in a set of sets (Greer 1992; Harel and Confrey 1994; Lamon 2006; Thompson and Saldanha 2003; Verschaffel et al. 2007).

To understand the idea of a set of sets, the child must be able to conceive of a collection as a unit (Harel and Confrey 1994; Lamon 2006; Thompson and Saldanha 2003). In case 8, we see at least two children working with this idea as they represent the number of eggs there would be in 3 baskets, each containing 3 eggs.

> Jenna used the interlocking cubes. First she made a unit of 3 cubes and brought them over to show me. I asked her what she had, and she said that she had one basket.
>
> "Where are the other baskets?" I asked.
>
> "I haven't made them yet," she said, heading back to the table. Soon she was back again, this time with two units of 3.
>
> "How many baskets do you have now?" (pp. 31–32, lines 56–61)

Jenna knows she has made two baskets and needs to make one more, which she goes off to do. Although she won't know how many eggs are in her 3 baskets until she counts them, she does realize that each group of 3 comprises a unit, namely 1 basket, and that each basket contains 3 items. Junior, however, is still building this idea. His model of the same problem consists of 3 orange cubes, 3 white cubes, and 3 yellow cubes. He knows he has 3 baskets, each containing 3 individual eggs, but when his teacher asks how many eggs he has in all, he says, "3." The teacher writes:

> Junior showed me how he counted: Each unit of 3 was "one" to him. I took one unit and broke it apart. He counted them, "1, 2, 3," and agreed that they represented 3 eggs. I put them back together with the other two units of 3 and asked him how many eggs were there. Again, he said, "3." I tried several ways to help him see that I wanted him

> to count the individual cubes, but it didn't help. He was unable to see the difference or sameness between cubes once they became a single unit of 3 (pp. 32, lines 80–85).

Whereas Jenna seems to understand the idea of a set of sets, Junior is still trying to figure out how one unit can be comprised of multiple items and yet maintain its identity as both a single entity and as a collection of entities.

To summarize, then, one line of the research literature suggests that children first come to understand multiplication as a way of counting groups, as just described, and then go on to construct other meanings for it later, as we shall see. Much of this research tries to define types of multiplication and division problems and examines student problem solving as it relates to those types. We will look at these ideas more closely in the next section of this essay. However, there is another group of researchers who find that students' ideas about multiplication and division develop not out of counting but rather from experiences with sharing, in which equivalent portions must be created. It is in dealing out these "equal shares" that students come to understand ratio, multiplication, and division as inextricably related concepts, inherent in the same situations. They are doing this, in many cases, while still working on higher number counts and while sorting out addition and subtraction concepts (Confrey 1995; Confrey and Harel 1994; Empson 2001; Empson et al. 2005; Lamon 2006; Steffe 2002; Thompson and Saldanha 2003).

Consider, for example, the problem fourth-grade students were working on in case 22: "I invited 8 people to my party (including me) and I only had 3 brownies. How much did each person get if they had fair shares?" (p. 84, lines 12–13). Maribel still directly models the actions of the problem, using pictures to create and allocate her portions:

> For example … she drew 8 faces for the 8 people at the party and drew 3 brownies that she cut into eighths. She then began distributing the pieces to the people. Each time she distributed 8 pieces, she crossed out the brownie they came from. After she finished distributing the pieces, she counted them up. "They each get 3/8," she wrote. (p. 84–85)

Maribel understands that this method results in equal groups, and she knows that the people in the problem have equivalent shares without having to count the portions. Although she uses counting to verify her solution, the actions she uses to solve the problem seem neither to be rooted in counting nor, by extension, in repeated subtraction. Instead, they could derive from an independent action which Confrey and her colleagues call *splitting*, the basic structures of which are halving and doubling (Confrey 1988, 1995; Confrey and Harel 1994).

Although we don't see Maribel halving and doubling quantities in order to solve multiplication and division problems, thereby working out an understanding of what those operations mean, there are examples of students doing just that. In case 11 we see Cory working on a problem in which someone is building bookshelves that require 4 boards each; the person has 36 boards with which to work. Cory reads the problem and then comments to his teacher:

> "I thought it was times." Then he reread the problem aloud. "See, that's why I changed it to divided by. If it was 4 divided by, I would probably use 2 first so it would be easier. First I would do 2 divided by 36, and that equals 28, and 2 divided

> by 28 equals 14 … Because I knew divided by is half of whatever the number is, like 2 divided by 100 is 50.” (p. 41, lines 283–287)

To Cory, division means splitting a number in half, regardless of the structure of the problem. According to Confrey and a growing number of researchers, splitting numbers in half is a cognitive structure—an idea in terms of which other ideas are organized (Confrey 1995; Confrey and Harel 1994; Empson 2001; Empson et al. 2005; Lamon 2006; Steffe 2002; Thompson and Saldanha 2003). If we put aside the calculation error Cory’s teacher helps him sort out moments later, we notice that Cory sees multiplication and division as interchangeable and that his knowledge of both of them is developing simultaneously, which supports Confrey’s contention that multiplication, division, and ratio co-evolve (Confrey, personal communication, February 3, 1997).

Section 4

Making meaning for multiplication and division

As students sort out the various contexts that can be modeled by multiplication and division, they continue to work on understanding the different types of units in these problems, deepening their knowledge of how the operations work and what they mean.

As mentioned earlier, one branch of the research literature (the one which suggests that children first come to understand multiplication as a form of repeated addition and division as repeated subtraction) examines student problem solving as it relates to the type of problems being explored. Three distinct types of multiplication situations have been categorized, each of which is related to a division situation as well. We discuss the three types of multiplication and their related division separately, the same way this research typically examines them.

We have already described one type of multiplication, in which the two factors represent different kinds of things—one, the number of sets, and the other, the number of items per set. Greer (1992) and Kouba and Franklin (1993) call this type of multiplication *asymmetrical* because the multiplier and multiplicand play different roles and cannot be interchanged. As mentioned earlier, with the multiplication problem “3 bags, each containing 4 donuts; how many donuts do I have?” the two factors can’t be used in place of one another. Three bags containing 4 donuts each (4 + 4 + 4) is a different situation from 4 bags of 3 donuts each (3 + 3 + 3 + 3), even though both involve 12 donuts. If one person gets each bag, in the first case fewer people eat more donuts; in the second, more people eat fewer donuts. Because the two factors represent distinct types of units, the related division situation can be defined in two ways (Fischbein et al. 1985; Greer 1992; Kouba and Franklin 1993; Thompson and Saldanha 2003).

In one type of related division, known in the research literature as *partitive* and commonly called *dealing*, an amount is split evenly between a certain number of groups. An example of this sort of “equal sharing” problem (Greer ’s term, 1992) might be this: “You have 12 donuts to share equitably among 3 people. How many could each person have?” The result is the number of items in each group or portion. In the other kind of related division problem, known as *quotitive* or *measure* and

commonly called *grouping*, a quantity is split into shares of a certain size and the result is the number of groups obtained: “You have 12 donuts. You want to make bags containing 3 donuts each. How many bags can you make?” Though both problems can be solved using the same division equation, 12 ÷ 3 = 4, the solution to the first problem would be 4 donuts for each of 3 people, whereas the answer to the second would be 4 bags with 3 donuts in each.

Students usually represent and solve these two types of problems differently. For a quotitive situation like the one in which 12 donuts are packed in bags of 3, the number of donuts in each group is known and the number of groups is sought. To solve, students typically count out 12 objects and then put them in piles of 3 until they run out (Carpenter et al. 1996; Kouba and Franklin 1993). Beyond the direct modeling stage, students might skip-count (3, 6, 9, 12) to find out how many groups of 3 are in 12, or rely on their knowledge of multiplication facts (Carpenter et al. 1996; Fuson 2003). Here, our classroom narratives contribute even greater detail to the research picture. In case 11, the students come up with other solution methods: repeatedly subtracting 3 from 12 until there isn’t anything left to subtract from; adding 3 to itself to get 6 and then adding 6 to itself to get 12 (also documented in Verschaffel et al. 2007); setting up a chart showing that 1 bag contains 3 donuts, 2 bags contain 6, 3 bags contain 9, and 4 bags would contain 12. Though the first method could be based on counting, the last two might be considered splitting strategies, among possible interpretations.

When working on the partitive version of that same situation—“If 3 people are sharing 12 donuts, how many donuts does each get?”— students use several direct modeling methods. The most common is to select 12 items and then deal them one by one into 3 piles until they run out, counting the number in each group to get the answer. Another approach is to count out 12 objects to represent donuts, guess at a number that could go to each person, then deal out that many to see if any are left, and modify the initial guess if needed. A third method is not to count out the total with which to begin, but simply to create 3 groups of an arbitrary size, keeping track of how many objects are used up, until the total of 12 is reached (Kouba and Franklin, 1993; Verschaffel et al., 2007). There are, of course, more abstract strategies as well—guessing an amount to skip-count by and seeing if it reaches 12, and then adjusting the guess accordingly so that 3 counts are obtained or using known multiplication facts (Carpenter et al. 1996; Verschaffel et al. 2007).

The second type of multiplication described through this line of research is called *symmetrical* by Greer (1992) and by Kouba and Franklin (1993) because both factors play the same role, represent the same type of unit, and can be interchanged. One of the most common instances of symmetrical multiplication is area; the length of a rectangle might be 4 feet and the width 3 feet until you rotate the figure 90 degrees, when the length then becomes 3 feet and the width 4 feet. The same symmetry is true of the array model. If you have a box of chocolates arranged 3 rows and 4 columns, you can turn the box at a 90 degree angle and then have 4 rows and 3 columns. Another symmetrical form has been called *cross-product* or *Cartesian-product multiplication*, an example of which is “If you own 4 shirts and 3 pairs of pants, how many different outfits can you put together?” (Greer 1992; Huinker 2002; Verschaffel et al. 2007).

Because the factors are interchangeable in these scenarios, there is only one type of related division, that is, one in which a missing factor is determined. For example, if the area allotted for

the rectangular third-grade garden is 12 square feet, and we've already put 3 feet of fence along one side, how much fence will we need for a side perpendicular to the first? Although the factors in a symmetrical multiplication problem are the same kind of unit (items of clothing, linear feet of fence), the result of such multiplication is a unit different from that of the factors. *Items of clothing* times *items of clothing* become *outfits*, and *linear feet* times *linear feet* become *square feet* (Greer 1992; Kouba and Franklin 1993; Verschaffel et al. 2007).

The third type of multiplication involves comparison. For example, consider this problem: Today I walked twice as far as yesterday. Yesterday I walked 3 miles. How far did I walk today? Whereas additive comparisons involve adding to one quantity to result in the other, multiplicative comparisons involve multiplying one quantity to get the other. In this problem that compares the distance I walked over two days, the number of miles I walked yesterday is multiplied by 2 to find the number of miles I walked today: 2×3 miles = 6 miles.

In case 29, Selena gave her students this multiplicative comparison problem: A piece of elastic can be stretched to $5\frac{1}{2}$ times its original. When fully stretched, it is 33 meters long. What was the elastic's original length? In this problem, $5\frac{1}{2}$ is multiplied by the initial amount to get 33 meters, and the students were charged with finding the initial amount. Selena's students created different models to represent the problem, agreed the answer was 6 meters, and by the end of the class, they used the context to consider what is the same and what is different among the following equations:

$5\frac{1}{2} \times 6 = 33$ and $6 \times 5\frac{1}{2} = 33$

$33 \div 5\frac{1}{2} = 6$ and $33 \div 6 = 5\frac{1}{2}$

$5\frac{1}{2} + 5\frac{1}{2} + 5\frac{1}{2} + 5\frac{1}{2} + 5\frac{1}{2} + 5\frac{1}{2} = 33$

$33 - 5\frac{1}{2} - 5\frac{1}{2} - 5\frac{1}{2} - 5\frac{1}{2} - 5\frac{1}{2} - 5\frac{1}{2} = 0$

Section 5

Encountering fractions in sharing situations

As students model and solve problems that involve sharing quantities equally, they encounter a new type of number: fractions.

There is much evidence to suggest that children explore sharing problems and begin to build fraction concepts before they reach school (Cwikla 2014; Empson 1999, 2002a, 2002b; Hunting and Davis 1991; Hunting and Sharpley 1988; Mack 2002b; Piaget 1965; Sophian, Garyantes, and Chang 1997). Children encounter sharing situations early in their daily lives where they deal out a number of items and there is material remaining as well as where the number of sharers is greater than the amount to be shared (Charles and Nason 2002; Empson 1999, 2002a, 2002b; Flores and Klein 2005; Hunting and Davis 1991; Lamon 2006; Sophian et al. 1997).

An example of the first situation, in which children figure out what to do with the leftovers after dealing out items to share equally, is provided in *Building a System of Tens* (Schifter, Bastable, and