Mathematical background notes

Standards for Mathematical Practice

Practice 2: Reason abstractly and quantitatively.

Mathematically proficient students at the elementary grades make sense of quantities and their relationships in problem situations. They can contextualize quantities and operations by using images or stories. They interpret symbols as having meaning, not just as directions to carry out a procedure. Even as they manipulate the symbols, they can pause as needed to access the meaning of the numbers, the units, and the operations that the symbols represent. (Illustrative Mathematics 2014)

The cases in this chapter provide examples of students actively connecting their number work to story contexts to develop a deeper understanding of the mathematics. The video clip of the third-grade student, Ebony, working with the problem, "How many legs on 4 elephants?," illustrates how the story context can support a student making sense of the math sentence that represents it. Use questions like, "How does the context support the child's thinking?" "How does this illustrate practice 2?" to help participants see these connections.

In the Math Activity: Problems for Division, participants are given an abstract expression, 32 ÷ 5, and create contexts for it, concluding that the contextual situation requires interpretation of the mathematical solution. During this work, participants will be engaged in practice 2. Conclude the math discussion with a question such as, "How has this illuminated practice 2?" or "What questions does it bring up for you about practice 2?" For an example of a seminar discussion that connects contexts to the interpretation of symbols, review the passage of "Maxine's Journal" titled Math activity: Story problems for division, lines 326–410. While Maxine does not cite practice 2, you should highlight the links between participants' work and practice 2.

Practice 3: Construct viable arguments and critique the reasoning of others.

Mathematically proficient students at the elementary grades construct mathematical arguments—that is, explain the reasoning underlying a strategy, solution, or conjecture—using concrete referents such as objects, drawings, diagrams, and actions.…Mathematically proficient students can listen to or read the arguments of others, decide whether they make sense, ask useful questions to clarify or improve the arguments, and build on those arguments. They can communicate their arguments, compare them to others, and reconsider their own arguments in response to the critiques of others. (Illustrative Mathematics 2014)

Session 2 includes a discussion about behavioral norms for the seminar group. Use this opportunity to discuss what it means to construct and critique mathematical arguments. "Critique" does not mean criticize. Rather, it means work to understand another's reasoning. When you don't follow, you pose questions. At times, this challenges one to reconsider or revise one's own argument. Later in the session, toward the end of the math activity, ask participants to identify when they were critiquing one another's arguments, and refer to practice 3.

Situations modeled by multiplication and division

Just as there are different situations that get condensed into addition or subtraction expressions, there are different situations that can be represented by multiplication as well. Consider these two situations:

(a) There are 5 elephants. Each elephant has 4 legs. How many legs are there on all the elephants?

(b) I have 4 packs of gum. Each pack has 5 sticks of gum. How many sticks of gum are there?

The elephant situation represents 5 groups with 4 objects in each group; the packs of gum situation represents 4 groups with 5 objects in each group. In our work, we consider both of these situations to be modeled by 5×4 or 4×5.

In some communities, people establish conventions about which factor in the multiplication sentence refers to the number of groups and which refers to the number in the group. For example, in Japan, the elephant problem would be represented as 4×5, the gum problem as 5×4. On the other hand, in most textbooks in the United States (though not all), the convention is reversed: the elephant problem is modeled as 5×4, the gum problem as 4×5. For our purposes, what's important is the notion that each of the factors represents something different, regardless of which occurs first or second in the expression. After all, the result of the multiplication is the same, no matter what the order of factors is.

These story situations for multiplication also point out a very significant difference between the operations of multiplication and addition. While the arithmetic expressions for $5 + 4$ and $5 + 4$ appear very similar when written out, applying each to a situation illustrates that in an addition problem, the two addends and the sum all stand for the same kinds of quantities (5 apples + 4 apples = 9 apples). This is not true for a multiplication expression. In a multiplication situation such as the legs on the elephants or the packs of gum, one number indicates the number of groups; the other indicates the number of items in each group.

Division expressions are also mapped to different story contexts. For instance, $12 \div 3$ provides the answer to both of these situations:

(a) There are 12 apples to share among 3 people. How many apples will each person get?

(b) There are 12 pears. I am placing them into bags that hold 3 pears. How many bags will I fill?

In the problem about apples, the number of groups is known and the number in a group is to be determined. In the pear situation, the number in each group is known and the number of groups is to be determined. Drawing a model for each of these situations illustrates these distinctions. Even though the answer to both is 4, in one case the 4 represents each person's share, and in the other, the 4 represents the number of bags.

	Apple Problem	Pear Problem
Begin with 12 items	X X X X X X X X X X X X	X X X X X X X X X X X X
	XXXX XXXX XXXX	XXX XXX XXX XXX
	Make three groups	Make groups of three

The Apple problem is technically labeled as *partitive* or, more informally, as *dealing*—this term is suggestive of dealing out the objects. Similarly, the Pear problem is labeled as *quotitive* or *measurement*—a reference to the idea that examining the question, “How many 3s in 12?,” is a way to measure 12 in units of 3.

These examples illustrate interpretations of multiplication and division of whole numbers. Later sessions will introduce multiplication and division of fractions, which will require extending the interpretation of the operations.