MAXINE'S JOURNAL

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As participants entered the room, they seemed to be excited to share their student-thinking assignments. Even before I brought the group together, Odette spoke up to tell her story.

She said that she had set up her class to work on the subtraction problems from the Second Homework assignment. When it was time to go to lunch, one boy, who wasn't done, asked whether he could stay in the classroom to finish his work. He then remarked to Odette, "This is so much better than when we did math!" She was shocked. What did he think math was? Doing pages of problems? It definitely was *not* talking about his own ideas. "What do the other children think math is?" she wondered.

Elspeth reported that a parent and child had recently visited her class because they were going to be moving into the school district. Initially, the child was nervous, but soon became comfortable enough to sit at a table with other children while his mother sat at the back of the room. As the children were working on a problem individually, Elspeth said she overheard the visiting child ask one of her students what subject he was doing. When the student replied that he was doing math, the visitor wrinkled up his face and asked, "That's math?" The student responded, "What's math like in your class?" When the visitor said they do things like 3 + 5 = 8 and 2 + 3 = 5, Elspeth's student exclaimed, "That's *all* you do in math? We do much more than that!"

Elspeth's story is like Odette's; both are about children's sense of what math is. However, Elspeth's youngster recognizes that mathematics is much more than rehearsing math facts. Odette's students still need to revise a view of mathematics identified with years of doing worksheets. I'm pleased the teachers are hearing and noting their students' ideas about what mathematics is, and I'm glad they are reporting this to the seminar.

Once Odette and Elspeth told their stories, I oriented participants to this session's work. I said, "Last time we met, we considered the meanings of addition and subtraction. First we looked at the work of young children who were counting to solve problems, and then we examined the work of older students, up through seventh grade. We thought about the variety of contexts that are represented by addition and subtraction, the relationship between those two operations, and how they can be represented with discrete objects like cubes and with a number line."

"Today we'll be doing similar work with multiplication and division. We'll view students' work through print and video cases, and we'll do some mathematics for ourselves. You have noticed that the cases in this chapter are from kindergarten to fourth-grade classrooms, but the ideas in them have important implications for later grades."

"Before we get to the cases, we'll take a few minutes for you to discuss your student-thinking assignments with another participant, and we'll also take a few moments to talk about how we function as a seminar group."

To the last point, Nadra asked, "Did we do something wrong?"

Gaye assured her, "Don't think about it that way. We did the same thing in *Building a System of Tens*. It's really helpful to talk about how to work together."

Sharing student thinking

For homework, participants had given their students the two problems from the video clip from Session 1: "I have 375 candy bars. I ate 90 of them. How many do I have left?" and "I am taking a trip to my sister's house. I drive 90 miles and then stop to rest. The total distance is 375 miles. How much farther do I have to go?" Teachers working at the lower grades modified the numbers to be appropriate for their class. I organized participants so each was paired with someone who taught at a different grade level.

Student responses were often quite similar, but some pairs were intrigued by the differences. For example, most of Beatrice's students wrote 375 – 90 for the first problem and 90 + __ = 375 for the second. On the other hand, Karran's students saw both problems as subtraction and initially said there was nothing different about them. Eventually, one student said, "They are both subtraction, but different kinds of subtraction. One is taking something away, and the other is finding how many more you need to get to the other number."

Discussion: Norms for learning

Near the beginning of the seminar, it is important to take a few minutes to consider group norms—how we can interact together to support both our own and one another's learning. The trick here, as Nadra's comment at the opening of the session points out, is to avoid making people feel chided. Discussion of behavior often occurs in classrooms only when something is amiss, but in this case, participants in the seminar have been doing fine. Even so, it's useful to articulate some of the behaviors and attitudes that make the seminar function well.

I started out by reading some of last session's exit cards. "Many people are writing about things that are working well for them," I said. "For example, someone wrote, 'This way of working in small groups helps me to practice and understand better.' Someone else said, 'For me, the cooperative learning process is a very positive way to learn.'"

I went on to say that the way we are engaging in the seminar often requires negotiation. "It's useful for us to talk together about how to work so that everyone can learn. What kinds of things should we be thinking about?"

Participants seemed reluctant to speak until Camisha, who had participated in *Building a System of Tens*, got us started. "I'm not thinking about rules for us," Camisha said. "It's more like we need to pay attention to each other's thoughts. That's different from other courses I've taken. I've never been in a class before DMI where we needed to listen to each other so carefully."

Nadra said, "You know, we really come in at all these different levels. I teach young children; I don't teach middle school. I had to remind myself that it's OK to be the slowest person in a group."

Marina (who had been to a DMI seminar before) said, "I come here to ask questions that I never could ask before. Some of my questions I never even knew I had. Some things I just accepted and hadn't ever realized you could ask about them."

Odette said, "I'd like to mention the tug between thinking things out for yourself and moving on with the group. In the last session, I wasn't ready to move on when the group was."

"But Odette," Celeste objected, "the questions you kept asking about number lines turned out to be important for us all to be thinking about."

That felt to me to be an extremely useful insight, but before I could underscore it for the group, Elspeth spoke up, "Sometimes it's not. Sometimes, you just have to feel like you're in a muddle, but you don't know whether or not it's useful to the group to stick with an idea."

I said, "I do understand what you're saying. This is what I mean by some things needing negotiation. There are times when you might take a step back and ask the group if it's useful."

I let this idea sink in until I judged the pause had gone on long enough. Then I asked about reading the cases. "This is a different kind of reading. What kinds of things do you do as you read to prepare for class?"

This time, Marina responded first. "Usually I find one case that piques my interest. I generally have to stop reading and do the math the kids are talking about."

Gaye offered, "I have to read the cases more than once. First I just read the whole chapter to get a general sense. Then I reread the introduction carefully, and read the cases slowly again with those questions in mind. When I read slowly, I start to realize that some of the things happening in the cases are like my own class. Or sometimes, after I read the cases, later in the week I notice the same thing with my students."

Elspeth agreed. "Something like that happens to me, too. After I read the cases, when I'm in my own classroom, questions pop into focus as I'm listening to my students."

"It's like all of a sudden you're alert to things," Beatrice observed, "and so now you start to hear them. Maybe the kids were always saying this stuff, but I never noticed it before."

Gaye, Elspeth, and Marina are among those participants who took *Building a System of Tens* before this seminar, and so they have more experience reading cases. I was glad that the new participants had a chance to hear their perspective. But I wanted to make sure others knew they could speak up, too. I asked if anyone new to reading cases had something to add.

Spencer said that he enjoyed reading the cases, but he hadn't been in an elementary classroom since he graduated from fifth grade, and so it's hard for him to picture what it looks like.

Nadra laughed and said, "Last time, that's how I felt reading the middle school case. But seeing the video helped."

Gaye said, "I teach elementary school, but I felt the same way when I read the first cases for *Building a System of Tens*. It really helped me to see the video. It changed the way I read cases."

I said that in this session, we would get a chance to see elementary classrooms on video so maybe that would help Spencer read the kindergarten to grade 5 cases.

Video and case discussion: Early multiplication and division

I played the video, which shows children in three classrooms (first, fourth, and third grades) working on preliminary ideas for multiplication and division. After watching the clips I said, "We're not going to have a whole-group discussion about the video now. As we just said, it's helpful to have images of students in classrooms. As you discuss the print cases, you might refer back to the video because some of the same issues arise."

Small-group case discussion

As participants settled into their small groups, Maalika didn't hesitate to start talking. "I've been thinking about what it means to count by something. Kids learn chants: "2, 4, 6, 8" or "5, 10, 15, 20." But we can ask them questions to help them think about what they mean."

"Yes," Carol said, "I was thinking about that, too. It's not just about little kids, either. We saw fourth graders showing what it means to count by 25s, and then by 100s."

Joseph, Andrea, and Labeeba were discussing Bella's case 8 about the kindergartners who were asked, "If you have 3 rabbits, how many legs are there?" Joseph noticed that Kevin represented the problem using 15 keys: 3 stood for rabbit bodies and 12 stood for legs. He found it surprising that Kevin could keep track of which keys to count and got the right answer. Andrea suggested that it was probably important that it was Kevin's own representation.

Labeeba said, "In our last session, we asked the question about whether the children in the cases about counting were adding or just counting. Well, it seems to me that Bella's students are doing multiplication. They're counting groups of things, and isn't that what multiplication is?"

I sat down with An-Chi, Camisha, and Iris as they turned to case 10, "Candy Canes in Packages." An-Chi started explaining that what the teacher was talking about at the end of the video was the same issue that the student Letitia was struggling with in the case. Later, when I stopped by Karran's group, they were discussing case 12, and Karran was saying, "Su-Yin is confused about what are seeds and what are kids like the teacher on the video was saying. And wasn't that Junior's problem in case 8?" I was pleased that many participants were noticing this important point.

Whole-group case discussion

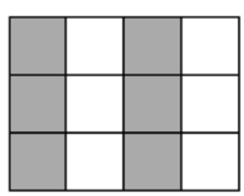
There were several points I wanted to highlight in this discussion; first among them were the various representations of multiplication. I said, "In the print and video cases, we see a number of ways to represent multiplication. I think it would be a good idea for us to consider these together. What do we see?"

As participants began flipping through the cases, Andrea said, "We can start with the kindergartners in case 8. Jason drew a picture of bunnies and showed four legs on each bunny."

Iris suggested, "We can look at Rashad's and Carlita's arrangements of keys. Rashad used a key for each bunny and also used keys for legs. It's almost like he created a model of a bunny out of keys. But Carlita arranged her keys into three rows of four. It actually looks like an array."

I stopped to clarify the meaning of the term *array*: an arrangement of items with equal rows and equal columns. Spencer added that you could think of the items as arranged in a rectangle.

Karran brought our attention to Kenya's picture, which depicted cubes arranged into an array. I drew an array on the board, a little more carefully than Kenya could manage.



I said, "This is an important image to represent multiplication. You can think of it as 3 rows with 4 cubes in each row; or as 4 columns with 3 cubes in each column; or you can think of it as a rectangle with one side of 4 units and the other with 3 units. You can see all of that in one representation."

Carol said, "The buildings in case 9 were arrays."

Celeste mentioned that Flora (in case 8) arranged her rocks into a row. "She might have counted out 3 groups of 4, but her representation doesn't show the groups, so it isn't very illuminating."

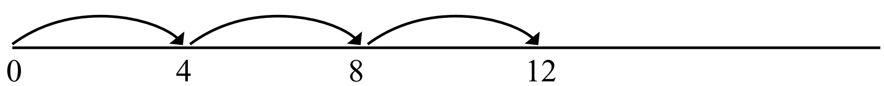
I acknowledged that, although this representation might have helped Flora solve the problem, once the rocks are arranged in a row, it's hard to see what was multiplied.

Gaye said, "What we saw on the video was a little more abstract. The girl there drew 3 circles with 4 lines in each. Each circle stood for an elephant, and each line was a leg."

Maalika added, "The fourth graders on the video showed multiplication with groups of cubes. They first had 25s arranged in arrays, and then they grouped four of those to show groups of 100."

I said, "At our last meeting, we spent quite a lot of time looking at addition and subtraction on a number line. Can you show multiplication on a number line?"

Cathleen said, "Sure, this shows 3×4," and came up to show us.



Now I was ready to move to a second topic. "Three children in this chapter—Junior in case 8, Su-Yin in case 12, and Letitia in case 10—were struggling with something I'd like to talk about. I heard several people talking about it in small groups, but I want to make sure it's something we all recognize."

Dofi said, "What was happening with Letitia was like what the teacher said on the video. It says in the case, 'she kept confusing 10 and 10 groups of 6.' It was hard for her to keep straight when she needed to count candy canes and when she needed to count packages."

Nadra said, "And it was the same thing with Su-Yin. She was getting confused between what's a kid and what's a seed."

I asked, "What about Junior?"

Celeste responded, "In our small group, we couldn't get what was happening with Junior. We thought he was just out to lunch or wasn't paying attention. But maybe it was just overload for him to think about baskets and eggs at the same time."

Spencer said, "I'm looking at these three kids and how old they are. Junior is in kindergarten, and, like Celeste said, maybe it's just overload for him. He can't think about it at all. In second grade, Su-Yin can think about seeds and kids at the same time, but loses track and doesn't seem to be able to get back on track. Letitia is a fourth grader. She also loses track, but when the teacher asks her a question, she gets back on track."

Gaye pointed out, "But there are other children in the cases who can figure this out."

Spencer responded, "Yes, I know. But maybe these kids illustrate something that is developmental.

It's not that they stand for every student. But maybe for kids who have trouble with this, as they get older they have the ability to sort it out."

Spencer was making an interesting point. In contrast to the discussion in Session 1, when I was concerned that the phrase "it's developmental" would put an end to thought, here Spencer was looking closely at what distinguishes the kindergartner from the second grader and the fourth grader.

Then I raised the next issue. "Let's take a moment to contrast this aspect of multiplication with addition. What happens in addition?"

Beatrice said, "In addition everything is the same."

Cathleen added to clarify, "The units are the same for both addends. In multiplication, the units are different for the two factors."

I nodded, "That's true of all the multiplication problems in the cases. And that's exactly what those three children are struggling with." Then I said, "Let's turn to division. The students in Georgia's case 11 were working on division, weren't they? What did you see in that case?"

It turned out my question was too general. The teachers could see that although Georgia had given her students division problems, they had used every operation except division to solve them. Even though they found the case engaging when they met in small groups, now as they talked, there was a ho-hum feeling. "So the children use different operations. That's good. I don't see why the teacher thinks there's a problem."

Well, I do want teachers to appreciate children's different solution strategies, but I think there is still something else to think about. So I asked, "Do you think there's anything important for children to understand about division as a distinctive operation? What makes division different from addition, subtraction, or multiplication?" And that got more people thinking.

Beatrice said that she asks her students to solve problems in more than one way so they can see linkages between operations. If they solved the same problem with addition and multiplication, then they could see what was the same and what was different about adding and multiplying. I was pleased to hear this remark. She wasn't just talking about a pedagogical strategy that she had been told to use; she was talking about what students learn when she employs that strategy.

Marina remarked that Vanessa used addition for one problem and subtraction for another. Why did she subtract to solve $24 \div 8$ and add to solve $36 \div 6$?

Odette reiterated the conjecture Georgia had offered, that it depended on the size of the number. If it's easier to visualize the number 24 in her head, then Vanessa is more likely to subtract. If 36 is harder to visualize, then she'll build up to it.

I know that's what Georgia said she was thinking, but because I didn't find that conjecture plausible, I asked the group if there might be something in the wording of the problem, the actual situation described, that makes one of them seem more like addition and the other seem more like subtraction.

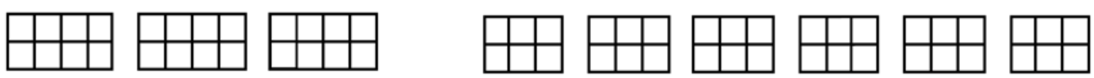
There was a buzz in the room, so I suggested that the teachers spend a few minutes talking to a partner. When we came back together, Dofi explained how she saw it: "For the first problem, you have 24 shirts in front of you and you need to put them in the drawers. So first you pick up 8 and put them away; that leaves 16. Then you pick up the next 8 and put them away; that leaves 8. And

then you put the last group of 8 in a drawer. That makes 3 drawers. Each time you took away 8, and that feels like subtraction. In the second problem, you need 36 cans; that's what you're trying to get to. So, first you put one 6-pack in your shopping cart and then you put another 6-pack in the cart. That makes 12 cans. You keep adding 6-packs to your cart until you have enough for everyone—36. That feels like addition."

After Dofi spoke, there was a long pause. It was almost as though the teachers were savoring the clarity of Dofi's explanation. Then Spencer continued: "You know, as an adult, I would never have paid attention to those things. I would just know they are both division problems. Now I see that if children are actually looking at the problem and are engaged in acting out the problem—in their heads or with pictures or manipulatives—they would think these problems are totally different. It's amazing to me that children would think that these are different kinds of problems, and I never noticed that."

After another pregnant pause I said, "We don't see in the case whether or not the students in case 11 used manipulatives or drew diagrams. But let's think about this for a moment. How can we represent division?"

Dofi spoke first. "We don't see a picture of it, but Vanessa might have acted it out with cubes, just like what I said. She gets 24 cubes for 24 shirts and then starts putting away groups of 8. Or she can start collecting groups of 6 until she has 36 cans." As Dofi spoke, she held up collections of cubes for us to see.



Iris stared at Dofi's cubes and exclaimed, "Oh, my gosh! I can see why division never made any sense to me!"

When I asked Iris to explain, she said, "We say '8 into 24,' but the 24 is what you have. You have 24 shirts. The 8 is an idea, an imaginary thing. It isn't real. And neither is the 3. So neither is the answer, nor what you are operating with. Neither the 8 nor the 3 are real objects, the way you have 3 apples and 8 apples."

After Iris finished, a number of teachers started speaking at once, so I gave them a few minutes to talk to their neighbors. I'm not sure everyone understood what Iris was saying—I'm not sure I did—but everyone seemed aware that she was saying something powerful.

After a few minutes, I brought the group together again and asked if anyone could paraphrase what Iris had said. Beatrice gave it a try. "When you see 24 ÷ 8, you might expect to see 24 things and 8 things. But it's not like that. You have groups of 8 and see that you have 3 groups."

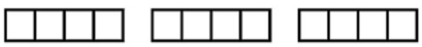
Karran added, "It's the same thing we talked about in multiplication. When you do 3 × 4, the 3 and the 4 aren't the same thing. When you do 3 + 4, you can talk about 3 apples and 4 apples. But if you do 3 × 4, you have 3 elephants and 4 legs for each elephant—they're not the same thing."

Karran's right. Iris was addressing exactly the same thing we discussed earlier. Junior, Su-Yin, and Letitia were all struggling with coordinating multiple units. Similarly, division involves multiple units. Apparently, Iris had sorted out this idea with multiplication, but had never realized this is what division was about.

Before we moved on, there was one more point I wanted to make. I said, "Here are two problems, and I want you to draw a diagram for each."

1. Janice has 12 tulips to distribute equally among 3 vases. How many flowers go in each vase?
2. Michael needs 12 pencils. Pencils come in packages of 3. How many packages does he need?

I asked participants to work alone to sketch their diagrams and then turned around to write out the problems on the board. When I turned back, Spencer was chuckling, and several participants were leaning over to talk about what they had drawn. Coming together, we agreed that both problems could be solved with 12 ÷ 3, but were represented by different diagrams:

1  2 

An-Chi declared, “I always thought that 12 ÷ 3 meant ‘How many 3s are in 12?’ That’s like the second problem. But now I see that the first problem is also 12 ÷ 3.”

Nadra said, “I always thought that in division you were told how many groups you had and you had to figure out how many in each group.”

I then related a story a fifth-grade teacher once told on herself. She said she had been working with a special needs teacher in her classroom, and they asked the class to come up with a diagram for a division problem. When the two teachers met after class, they were both dismayed that half the class drew an incorrect diagram. But as they began talking, they discovered that they each thought a different half got it wrong. The two teachers had different pictures of what division was. They had a long discussion until they concluded there were different ways to think about division and all of the students had drawn correct diagrams.

At this point, several of the teachers commented about how much more time in their elementary curriculum is spent on addition and subtraction than multiplication and division. “All of kindergarten, first, and second grade is about addition and subtraction, and then all of a sudden in third grade, you do multiplication.” I wasn’t sure just what they meant, but I didn’t think it was simply the amount of time spent on different operations, so I asked them to explain.

Odette said, “When they want kids to learn about addition, they have the children putting out 5 things and putting out 3 things and then seeing how much there is all together. There’s also lots of work on what subtraction is. But with multiplication, they just do that quickly—3 × 4 is 3 groups of 4—and the rest of the year is learning to say multiplication facts. Very little time on what multiplication *is*.”

They were making an important point. I said, “One thing the primary-grade teachers might think about is how you can start introducing ideas about multiplication into your teaching. And higher-grade teachers, you might think about how to help your students get to the meaning of multiplication.” But I still wanted to push a little on the participants’ ideas about what multiplication is, so I shifted the topic. I pointed out that they saw multiplication as groups of things, like 3 groups of 4. “What would you make of the following problem? Jacob has 3 pairs of shorts and 4 T-shirts. How many outfits can he make with those items of clothing?”

I gave them a few minutes to work on this problem, and everyone came up with 12. Odette said, “I can see that you get the right answer by multiplying, but I don’t know why it’s a multiplication problem.”

In response, Marina explained, “Look, you can think of one pair of shorts with each of the 4 shirts—that’s 4 outfits; then the next pair of shorts with 4 shirts—8 outfits; and the third pair of shorts with 4 shirts—12 outfits.”

Marina was showing how you could think of the outfits problem as 4 + 4 + 4, but it did give the teachers a sense that when you write 3 × 4, it could have more than one meaning.

Then Amber said, almost plaintively, “Why didn’t anyone ever teach us this before? Why didn’t I ever get a chance to think about this?”

The conversation got emotional as folks testified to their own experiences as math students. They had always felt this pressure to do things fast, they said; students were made to feel stupid, and if they couldn’t do multiplication, they got labeled as remedial.

“You know,” Camisha confessed, “I’ve done that to kids, too. I’ve taught that way. I didn’t know there was another way to do it.”

“Yeah,” Elspeth agreed. “But I’ve been trying to make it emotionally safer. The children need to know their facts, but it doesn’t have to be done with flash cards and timed tests.”

After a pause, I pointed out that they *are* getting a chance to think about this stuff now. Then I said that after break, I would have some division problems for them.

Math activity: Story problems for division

I began the math activity by asking every teacher to come up with a word problem for 32 ÷ 5, and then compare their problem with a neighbor’s. When they had done that, I asked Iris to share hers.

In the past, when I’ve assigned a similar task, the first suggestion usually involved something like 32 cookies to be shared among 5 people. Each person got 6 cookies, there were 2 left over, and maybe someone would think of dividing up the remainder.

Iris’s problem was different: “There are 32 children in the class and 5 tables in the classroom. If the children are arranged as evenly as possible, how many will be seated at each table?” When I asked the group what the answer was, none responded. They talked about the situation, but no one actually said, “6 or 7.”

I decided to let that go and handed out the math activity sheet, pointing out that they already had a start on problem 1. I also mentioned that if, when we came back together, they hadn’t finished all the work, we would concentrate on solutions for problem 1, discussing problem 2 only if there was time.

Everybody could get started, but it was interesting to see that different groups got stuck on different solutions. For example, some groups could find a problem whose answer was 6 but not 7; others found problems whose answer was 7 but not 6. Neither could see how the other answer was possible.

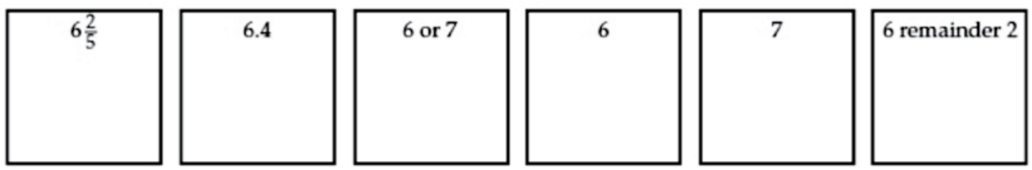
Other groups found it easiest simply to write a bunch of problems without aiming for a particular answer and then see what answers they gave. They came up with problems for most of the items that way.

But most groups had trouble inventing a problem whose answer was “6 or 7” until they looked up at the board and realized that Iris had already given it to them.

Some of the groups also had difficulty writing a problem for 6.4. They said that it was hard to think about contexts where you use decimals, except for money, and in that case the answer would be \$6.40, not 6.4. “Decimals are for money and fractions are for food,” according to Marina. “Isn’t that weird?”

When the teachers were working in small groups, I noticed that a few had written math sentences reading $32 \div 5 = 6$ and $32 \div 5 = 7$. I took the time to point out that, even though the answer to the word problem was 6 or 7, the mathematical notation means something else. You can say that $32 \div 5 = 6.4$ or $6^2/_5$, but it’s incorrect to say $32 \div 5 = 6$.

Then I put posters on the wall, each with one of the six answers as its head:



I asked participants to write their group’s problems on the appropriate posters.

It was important for participants to see that some of the items they thought were easy, others found difficult, and vice versa. This counters the notion that in mathematics there is an absolute standard of this is harder than that. “Hardness” is not necessarily a characteristic of the problem. When they’re having more difficulty solving one problem than another, it’s not just that the problem is harder. There’s something else going on.

When Marina compared the story problems on the posters for 6.4 and $6^2/_5$, she saw that some of them were the same and asked, “How can that be?”

Spencer explained that 6.4 and $6^2/_5$ are simply two different ways to write the same amount. He said, “If I run 6.4 miles and you run $6^2/_5$ miles, we’ve run exactly the same distance. It doesn’t matter!”

Spencer was right. In fact, $6.4 = 6\ ^2/_5$, and so, technically, any story problem whose answer is 6.4 also has an answer of $6\ ^2/_5$. But Marina had a point, too. In some contexts, decimals are used more frequently, and in others, fractions.

Celeste said, “There are other answers that can use the same problem, too. For example, say you have \$32 dollars to share among 5 people; the question is ‘How much money does each person get?’ If you actually have 32 dollar bills and don’t have any way to get change, then each person gets \$6 and you don’t distribute the other \$2. Or you can say three people get \$6 and two get \$7. Or, if you can get change, then each person gets \$6.40.”

Celeste’s point was different from Spencer’s because, or course, $\$6.40 \neq \$6$. Celeste has taken the same basic problem with slight variations to produce different answers.

The discussion of 6 remainder 2 was interesting. Beatrice declared, “There isn’t a problem whose answer is 6 remainder 2! If you have such a problem, you have to ask two questions. Like you might say, ‘I need 5 yards of material for drapes and I have 32 yards. How many drapes can I make?’ The answer is just 6. But what if I ask, ‘How many drapes can I make and how much material do I have left over?’ There are two answers: 6 drapes answers the first question; 2 yards answers the second.”

Beatrice's remarks created a stir, and I was pleased. I'm not so concerned about whether or not participants think you can have a problem whose answer is 6r2 but rather that they see the complexity of that answer. I think that was what surprised the group, since students are often told that 6r2 is the correct answer, especially if they haven't yet worked with fractions.

Toward the end of our activity time, I asked people to consider problem 2 on the sheet, which asks them to think of word problems for 5 ÷ 8. At first Iris blurted out, "What? You can't divide a smaller number by a larger number."

Karran said, "Sure you can." Iris turned red.

"Iris," I said, "You've been having all kinds of new discoveries about division today, and it might be that you have enough to think about already. Still, I'd like us all to think about why it might *seem* that you can't have 5 ÷ 8."

"Let's go back to the problems we made up for 32 ÷ 5," Spencer offered. "Like you had 32 cookies for 5 kids. Now let's say you have 5 cookies for 8 kids."

"It's just like the other problems," Carol declared. "You can have different answers."

"What do you mean?" I asked.

"It depends on whether you can cut up your cookies. If you have 5 cookies for 8 kids, you might just say forget it. The answer is 0—there's not enough to share. But if you cut them up, each kid gets 5/8 of a cookie."

Participants had a few minutes to get into the problem in small groups, and then the time was up. I explained, "This is the question you'll work on for homework. Just as we did for 32 ÷ 5, I'd like you to write out different story problems for 5 ÷ 8. As you work on this, think about Spencer's suggestion, that you use the problems you created for 32 ÷ 5. Before you leave, make sure you have written down at least one story problem for each of the answers on the posters. For homework, you are asked to substitute the numbers 5 and 8 into those problems, and then make up an additional problem for each answer. You'll see that this assignment is relevant for the cases you'll be reading."

Homework and exit cards

I distributed the Third Homework sheet and gave participants a chance to read it. Then I posted the exit-card questions.

- What was important or significant to you in the mathematics discussed at this session?
- What do you want to tell the facilitator about how the seminar is working for you?

The exit cards give me a sense of what participants are taking away from the session. Some of the cards express key issues related to teachers' practice.

Odette: What was significant for me today was realizing that the context in which a problem is presented could help kids think about more complicated math ideas.

Joseph: I am impressed at the level of thinking we ask of students on a daily basis. Having to create problems to fit the answer helps me understand the complexity involved.

Maalika: The problems about 32 ÷ 5 were challenging; they really highlighted how important context is. Sharing these examples with parents would help them understand that math is not all about quick recall of memorized facts. Children need to understand the problem.

Some of the middle school teachers wrote about what they got from examining cases from elementary classrooms.

Spencer: Seeing how multiple processes can be used to solve relatively simple math problems makes me realize that multiple processes can and should be used to solve more complicated, multi-step algebraic problems. That is, what I see in the cases with younger students applies to my middle school students working on more complex content—they shouldn't simply be memorizing set procedures for solving particular problem types.

Celeste: When I examine the thought processes of younger students, I see that sometimes I overlook or take for granted the skills my students have. I need to check this out.

I do appreciate it when participants let me know when they are intimidated by the mathematics.

Nadra: The fractions in word problems make me nervous. I was never strong with them to begin with.

I'll need to help Nadra face her fear of fractions.

In contrast to the depth of the comments about what teachers got from the session, the comments about the practices were rather superficial. Many participants said explicitly that they don't understand practice 2, reason abstractly and quantitatively. But the comments of Odette and Maalika, which I've quoted above, speak directly to this practice. They wrote about the importance of keeping mathematical symbols and concepts connected to contexts in order to deepen their understanding. As the seminar continues, I think they will discover how they, as well as their students, can use contexts and various representations to understand the mathematics more deeply. This will be especially clear when they work on dividing with fractions.

Responding to the second homework

February 15

For this homework assignment, participants gave students two story problems to solve, problems that many adults would identify as subtraction. Most of the participants wrote about how different students had different ways of thinking about the two problems. The teachers, who after Session 1 already were surprised as they began to see the complexity of subtraction, now could see that complexity revealed in their students' thinking. For example, Dofi discovered that the second problem was much more challenging to her first graders than the first.

Dofi's homework

I gave three of my students the following problems:

Problem 1: Max had 15 candy bars. He sold 8 of them. How many candy bars did he have left?

Problem 2: Mom was taking a trip to the store. She drove for 8 miles and then had to stop to rest. The total distance to the store is 15 miles. How much farther does Mom have to go?

All three students used the same equation for problem 1: 15 – 8 = 7. I had expected that this would happen because we have previously solved and talked about problems similar to this in class. Two of the students used symbols to solve the problem (they both drew 15 Xs and crossed out 8 of them), while the other used cubes. Again, this is what I expected. When asked why they subtracted, Student 1 said, "He is giving away stuff," while Student 2 said, "He didn't have as many candy bars as he started with." After examining their work, I learned that all three students understand that after you "take away" part of a group, you will have less than what you started with.

The surprises came with problem 2. I was not sure what to expect from the students. All three of them used a different method to solve it, and all three needed more time than with the first problem.

Student 1 used the equation 8 + 15 = 23, and decided that Mom had 23 more miles to go. He solved the equation by first taking out 8 cubes, then 15 more, and then counting up all the cubes. When I asked him about why he chose to add 8 and 15, he said that 8 and 15 were the numbers in the problem, and he added them together because "Mom was going to the store and that's not going back." I reiterated to him a couple of times that the total distance to the store was 15, but this did not change his thinking in any way. I believe this is because in the majority, if not all, of the problems he has solved in the past, he has had to use both numbers in the equation to find a sum or difference rather than a missing addend.

Student 2 used the equation 8 + 7 = 15. I asked him why he used addition, and he said he added because "She had to go more miles … she had to take a rest at 8 miles and she had 7 more miles to go." He also explained, "I knew 7 + 7 = 14 and 8 + 8 = 16, so I added 1 to 14 and took away 1 from 16, and that equaled 15."

Student 3 used the equation 15 – 8 = 7. To solve this equation, she drew 15 circles on her paper, crossed off 8 of them, and then counted the circles that were left. I asked her why she subtracted, and she said, "It is 15 miles to the store and she already drove the 8." I was surprised at the accuracy and level of her thinking because we have not solved problems like this in the past. It was interesting that she was the only one out of three to view this as a subtraction problem.

Mathematical practices: All of the students wanted to be precise (SMP 6) and one of my students used cubes (SMP 4).

I was pleased that Dofi has begun to explore the complexity of subtraction with her first graders and isn't backing away from it. Her writing about her students conveys the richness of their thinking. In my response, I wanted to give her more ideas to consider—approaches that might help Student 1 interpret the problem, particular hurdles many students in her class may need to overcome, and other questions that might be posed to further the thinking of all students.

As for discussion of the mathematical practices, all of the teachers who referred to practice 4 thought it meant using manipulatives. I corrected this in my responses and will discuss it in our next session.

Dear Dofi,

What an interesting description of the work that three of your students did. Particularly fascinating is that they all solved the first problem essentially the same way, but did very different things with the second.

It sounds like you worked with each of the three children individually. I wonder what would have happened if they'd had a chance to discuss the second problem together. Would Student 1 have been able to make sense of either of his classmates' reasoning? Would Students 2 and 3 have been able to recognize that both their ways fit the problem and result in the same answer?

There is more to think about with Student 1. He isn't thinking about the problem in a way that actually matches the context. I wonder whether this context is too abstract for him to work on a new idea or whether another context would make the ideas more accessible. What if he had 8¢, but needed 15¢ for something he wanted to buy? Or maybe he had 8 objects, but needed 15 for something he wanted to do? You said that he was the one student who used cubes to solve problem 1 rather than drawing symbols on a page. Perhaps he still needs to be thinking about something more concrete than units of distance.

You might want to check out whether Student 1 (as well as other students in the class) believes the equal sign means "here comes the answer," so whatever he does has to result in a number that goes after the equal sign. Perhaps, once he decided the problem involves addition, adding the two numbers is the only possibility; he didn't consider that the appropriate equation could be 8 + __ = 15.

I also wonder whether any of the three children noticed that the two problems involved the same numbers—especially Student 3, who wrote the same number sentence (15 – 8 = 7) to solve both problems. Could they talk about what is the same about the two problems and what is different?

There is so much to pay attention to in making sense of subtraction.

You noticed that all students were engaged in practice 6. Even though Student 1 solved the second problem incorrectly, once he decided it was 8 + 15, he added those two numbers correctly.

As for practice 4, note that the word model has two different meanings. We might talk about using manipulatives to "build a model." But practice 4 uses the word model in a different way—to abstract the mathematical elements from a context and represent them in relation to each other. In fact, in solving the second problem, Students 2 and

3 came up with two different mathematical models: 8 + 7 = 15 and 15 – 8 = 7. That is, an equation can be seen as a mathematical model in that the mathematical elements have been abstracted from the context. Student 1's difficulty is that, in seeing the problem as 8 + 15, he modeled the second problem incorrectly.

I know this can be confusing. We will continue discussing the mathematical practices in our sessions.

In any case, have fun working on subtraction ideas with your class.

Maxine