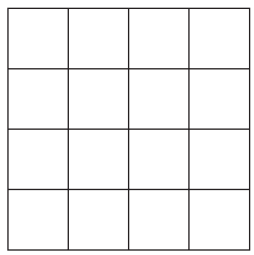
Was Pythagoras Really a Square?

What famous formula was used from as far back as 1600 BC to construct temples and the Pyramids, to find the lengths of right triangles, to compute the distance between points, and to determine the equation of a circle? The Pythagorean theorem is a fundamental mathematical topic that you, as a student, are likely to study. Let's explore what the Pythagorean theorem is all about.

1. This square has an area of 16 square units. What is the side length of the square?



You can find the area of a square by multiplying the side length by itself, which is called *squaring* the side length.

If you know the area of a square, you can work backward to find the length of the side. For example, suppose a square has an area of 9 square units. To find the length of a side, you need to find what positive number multiplied by itself is equal to 9. Since $3 \times 3 = 9$, the side length is 3 units. We say that the *positive square root* of 9 is 3. In symbols, it looks like this: $\sqrt{9}$.

In mathematics, a *square root* of a number x is the number y whose square (y^2, the result of multiplying y by itself) is the same as x, which can be written as $x = y^2$. We know that -3×-3 is also 9, but side lengths cannot be negative.

2. What is the side length of a square with an area of 25 square units?
3. What is the side length of a square with an area of 81 square units?
4. What is the side length of a square with an area of 225 square units?

What is the side length of a square with an area of 3? No whole number can be squared to equal 3. Will the side length be larger than 1? It seems that the answer is yes, since $1 \times 1 = 1$. Will the side length be larger than 2? The answer to that question must be no, since $2 \times 2 = 4$, a number larger than 3. So the side length must be a number between 1 and 2. Try some numbers to see which side length will result in an area of 3 square units.

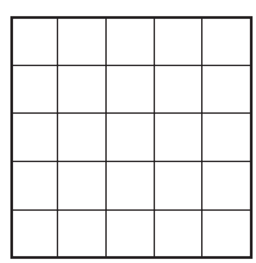
5. What is the area of a square if the side length is 1.5?
6. What is the area of a square if the side length is 1.6?
7. What is the area of a square if the side length is 1.7?
8. What is the area of a square if the side length is 1.8?

The side length of a square with an area of 3 must be a number between 1.7 and 1.8. No matter how many numbers we try between 1.7 and 1.8, we will not find a product of exactly 3. Calculators show that

$$\sqrt{3} \text{ (square root of 3)} = 1.7320508.$$

That number is rounded by the calculator because no rational number is equal to $\sqrt{3}$. (A *rational number* is a terminating or repeating decimal.) We usually round numbers like this to three decimal places; therefore, we can write $\sqrt{3}$ as 1.732.

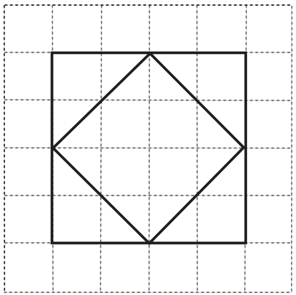
9. Between which two whole numbers is the side length of a square with an area of 5? __________
10. Between which two whole numbers is the side length of a square with an area of 30? ____________
11. Between which two whole numbers is the side length of a square with an area of 90? ____________
12. Find the area of the square outlined below (in number of smaller unit squares). Then find the side length of the outlined square.



Area = ________________

Side Length = __________

We can also create squares that do not have whole-number sides. We can show these as "tilted" squares. To find the area of a square in a "tilted" position relative to the grid paper, use this strategy: Enclose the figure in a "upright" square so that each vertex of the tilted square lies on a side of the upright square as shown on page 2.

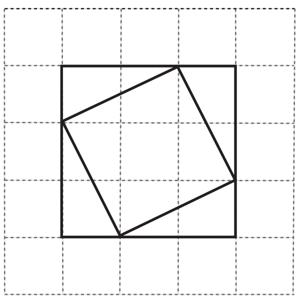


Now find the area of the upright square and find the areas of the triangular regions. Notice that these regions are not part of the original tilted square. Subtract the areas of the triangular regions from the area of the upright square. That difference equals the area of the tilted square.

13. Area of “tilted” square = ___________________

Use the area of the tilted square to find the length of one side of the tilted square = _________________

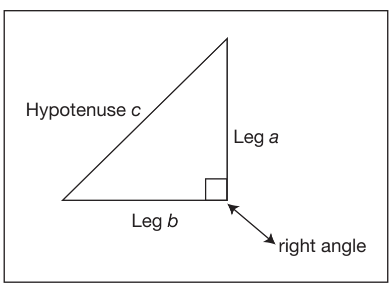
14. Find the area and side length of the “tilted” square below.



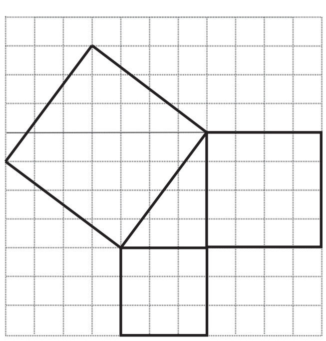
Area = _________________

Side Length = ___________

Recall that right triangles are triangles with one right, or 90°, angle. The sides of the right triangle that form the 90° angle are called the *legs* of the triangle. The side opposite the right angle is called the *hypotenuse.*



Notice that the right triangle below has squares built on the legs and hypotenuse. The area of each square can be identified. Using the areas of the squares, you can calculate each side length (see sample below).



15. Use grid paper to build right triangles with the following leg lengths. Then draw squares on each leg and hypotenuse. Fill in the table below. Use the table to answer questions 16–17.

Length of		Area of			Length of
leg #1	leg #2	leg #1	leg #2	hypotenuse	hypotenuse
6	8				
9	12				

16. How do the lengths of the legs help you determine the area of the squares built on those legs?

17. What relationship is there between the area of each square built on the hypotenuse and the areas of the squares built on the legs?

The Pythagorean theorem states that in a right triangle, the *sum of the squares of the lengths of two sides (legs) is equal to the square of the length of the hypotenuse*. Algebraically, if a and b are the lengths of the legs and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.

Use what you have learned about this theorem to find the missing side of each right triangle in the chart below.

	Length of leg #1	Length of leg #2	Length of hypotenuse
18.	12	16	
19.	5	12	
20.	7	24	

Pythagorean Triples

Any set of three whole numbers that satisfies the Pythagorean theorem is called a *Pythagorean triple*. Examples of Pythagorean triples include {6, 8, 10}, {5, 12, 13}, and {7, 24, 25}. Let’s try to find another Pythagorean triple. In other words, find a set of positive integers a, b, and c such that $a^2 + b^2 = c^2$.

"Guess and check" is one strategy to find another Pythagorean triple, but this method is time consuming. There is, however, an algebraic way to create Pythagorean triples. Take any two whole numbers, m and n, such that $m > n$. First, find the values of $m^2 - n^2$, $2mn$, and $m^2 + n^2$. These values form a Pythagorean triple. Name them a, b, and c, the side lengths of a right triangle such that $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$. To prove that this process will always result in a Pythagorean triple, we can show that $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$ by expanding and simplifying both sides of the equation:

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$$

$$m^4 - 2m^2n^2 + n^4 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4$$

$$m^4 + 2m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4$$

Since both sides are equal, it follows that $a^2 + b^2 = c^2$ and the three numbers form a Pythagorean Triple.

Example: Suppose that $m = 10$ and $n = 6$. Then $m^2 - n^2 = 64$, $2mn = 120$, and $m^2 + n^2 = 136$. Use your calculator to check that 64, 120, and 136 form a Pythagorean triple.

21. Complete the first five columns of the table below by choosing different values of m and n to determine four different Pythagorean triples.

22. Complete the remaining columns by placing checks to indicate divisibility where appropriate. How many contain at least one number that is divisible by 3? By 4? By 5? By 6? By 7?

23. Compare your answers to question 22 with a classmate's answers. Describe any patterns you see.

24. Prove that at least one number in a Pythagorean triple must be even.

Can You …

- explore this relationship with equilateral triangles built on the legs and hypotenuse of right triangles?
- explore this relationship with semicircles built on the sides and hypotenuse of the triangles?
- explore this relationship with regular hexagons built on the sides and hypotenuse of the triangles?
- use the Pythagorean theorem to find the lengths of one leg when you know one leg and the hypotenuse? Try it with the length of a leg of 24 and a hypotenuse of 30.
- use the Pythagorean theorem on triangles that are not right triangles?

Did You Know That …

- ancient Egyptian land surveyors had a method to form right angles? They tied twelve evenly-spaced knots in rope. Then they stretched the rope to form a triangle around the three stakes. One side had three spaces between the knots, another had four spaces between the knots, and the longest side had five spaces. The angle opposite the longest side was always a right angle.
- the ancient Greek philosopher Plato (c. 380 BC) used the expressions $2n$, $n^2 - 1$, and $n^2 + 1$ to produce Pythagorean triples.
- when the scarecrow gets his brain in the Wizard of Oz movie, he tries to recite the Pythagorean theorem, but he says it wrong. The scarecrow actually says, "The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side."

Mathematical Content

Pythagorean theorem, functions, proofs, area measurement, square roots

Resources

Jones, Phillip S. "Pythagoras," World Book Encyclopedia, Vol. 15, 1986.

———. "Pythagorean Theorem," World Book Encyclopedia, Vol. 15, 1986.

Mahoney, John F. "NUMB3RS Activity: Pythagorean Triples." www.education.ti.com/go/NUMB3RS

Lappan, Glenda, James T. Fey, William M. Fitzgerald, Susan N. Friel, and Elizabeth Difanis Phillips. Connected Mathematics: Looking for Pythagoras. Palo Alto, CA: Dale Seymour Publications, 1998.

					At least one of the numbers is divisible …					
m	n	a	b	c	by 2	by 3	by 4	by 5	by 6	by 7
10	6	64	120	136						

Student Math Notes Correction Notice

The checkerboard puzzle in the September 2007 issue had a extra block in the puzzle pieces, but the online version is correct.

Answers

1. A square with an area of 16 units has a side length of 4 units.
2. A square with an area of 5 units has an area 25 units.
3. A square with an area of 9 units has an area 81 units.
4. A square with an area of 15 units has an area 225 units.
5. The area of a square with a side length of 1.5 is 2.25 units.
6. The area of a square with a side length of 1.6 is 2.56 units.
7. The area of a square with a side length of 1.7 is 2.89 units.
8. The area of a square with a side length of 1.8 is 3.24 units.
9. A square with an area of 5 has a side length between 2 and 3.
10. A square with an area of 30 has a side length between 5 and 6.
11. A square with an area of 90 has a side length between 9 and 10.
12. Area = 25 square units, side length = 5 units.
13. Area = 8 square units, side length = $\sqrt{8}$ or 2.828 units.
14. Area = 5 square units, side length = $\sqrt{5}$ or 2.236 units.
15.

Length of		Area of			Length of
leg #1	leg #2	leg #1	leg #2	hypotenuse	hypotenuse
6	8	36	64	100	10
9	12	81	144	225	15

16. The area of the square is the square of the length of each leg.
17. The sum of the areas of the squares of the legs is equal to the area of the square of the hypotenuse.
18. 20
19. 13
20. 25
21. Answers will vary. Sample answers: {8, 15, 17}, {20, 21, 29}, {12, 16, 20}, {9, 40, 41}
22. Each of the Pythagorean triples contains at least one number divisible by 2, 3, 4, and 5. Not all of them will have numbers divisible by 6 or 7.
23. Each of the Pythagorean triples contains at least one number divisible by 2, 3, 4, and 5. Not all of them will have numbers divisible by 6 or 7.
24. Informally, if a is odd, then a^2 is odd. If b is odd, then b^2 is odd. If both a and b are odd, then $a^2 + b^2$ represents the sum of two odd numbers which is even. So c^2 must be even, and therefore, c must be even.

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