Research into Practice

Finding the Number of Cubes in Rectangular Cube Buildings

Michael Battista and Douglas H. Clements

Elementary school students have considerable difficulty determining the number of cubes that are contained in three-dimensional rectangular buildings like the one shown in **figure 1** (Battista and Clements 1996). The reasoning required to complete such tasks is important because it builds the cognitive framework for understanding the measurement of volume and the formulas for determining volume. This article describes typical student strategies for enumerating cubes in cube buildings and illustrates why these problems are so difficult for students. It also describes instructional tasks that can help students develop more powerful ways of thinking about such problems.

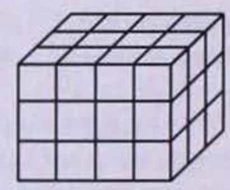
A Lack of Coordination

After Randa, a fifth grader, had built several three-dimensional rectangular buildings with interconnecting cubes, an interviewer asked her how many cubes she would need to make the building shown in **figure 2a.**

Randa: It has 4×3 and 4×4. $12 \times 4 = 48$, $+ 16 = 64$. There's some you can't see in the picture. [She multiplied the twelve cubes in each lateral face by 4, the number of lateral faces. Then she added the sixteen from the top, but not from the bottom. Note that Randa's strategy double-counted the cubes along most of the building's edges.]

FIGURE 1

A three-dimensional rectangular cube building



Randa then spent about twenty minutes trying to construct the building with cubes. She built the configuration shown in **figure 2b** several times, making the 4-by-4 top and the 3-by-4 front and then joining them. Each time, however, she stopped and started over when she noted that the front of her configuration did not match the front of the building pictured in **figure 2a.** The interviewer then asked Randa to find the number of cubes in the configuration shown in **figure 2b.**

Randa: Four rows of 4 on this side [the top] and 4 on this side [the front]. So it's $16 + 16 = 32$. [Note that she still double-counted the cubes along the top front edge.]

This episode illustrates the major source of difficulty that many students have when enumerating cube buildings. Both in counting and in building, Randa showed a striking inability to coordinate different orthogonal views of the cube configurations.

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An orthogonal view is one in which only one face of the building is visible. For the configuration shown in **figure 2b,** she constructed the top with cubes, then the front, but she could not figure out how those two parts fitted together. To *coordinate* those views, Randa would have had to decompose the top into its constituent cubes, do the same with the front, then establish a spatial relationship between the views that recognized that the four edge cubes were part of the front and top faces.

We designed the problem in **figure 3** to investigate further the students' difficulties with coordinating orthogonal views of cube buildings. Neither enumerating cubes nor constructing the building depicted by the three orthogonal views could be accomplished without some type of coordination of the views. Less than a month earlier, the fifteen fifth graders whom we interviewed on this task had received traditional instruction showing that the number of cubes in a cube building could be found by multiplying the length times the width times the height. Even so, not one of them correctly determined the number of cubes in the building described in **figure 3.** Furthermore, like Randa, seven of the students attempted to construct the building by making, then connecting, two or more intact views—clearly showing a lack of coordination. Only two of these students eventually realized that the views contained common cubes and properly coordinated them to construct the cube building correctly.

FIGURE 2

Randa's cube configurations

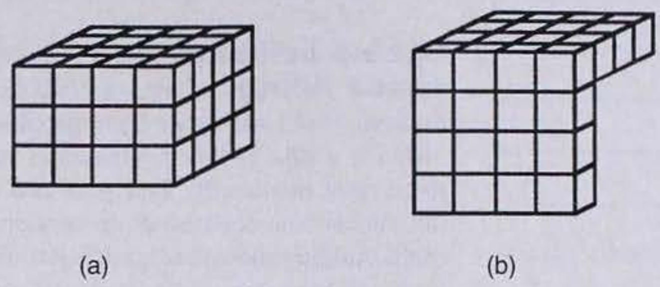
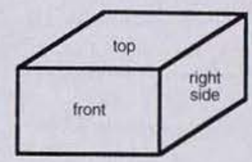


FIGURE 3

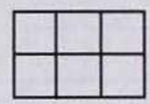
The interview problem

Suppose that we completely fill the rectangular box with a rectangular cube building. The box is transparent, so you can see the building through the box's sides.

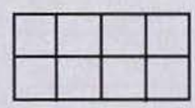


After we fill the box, we look straight at the building from its front, top, and right side. [Indicate orthogonal viewing lines with a pencil.]

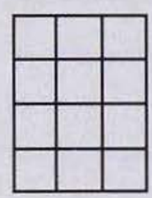
From the *front,* the building looks like this.



From the *right side,* the building looks like this.



From the *top,* the building looks like this.

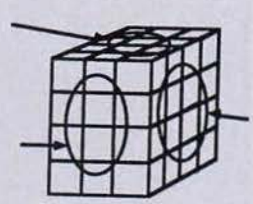


A. How many cubes does it take to make the building?

B. Can you make the building with cubes?

FIGURE 4

Seeing buildings in terms of their sides



Students' Mental Models and Enumeration Strategies

A student's mental model of a cube building determines how the student imagines or "sees" the cubes in the building (Battista 1994). Elementary school students use several types of mental models as they attempt to enumerate cubes in rectangular cube buildings. We next describe these models and how they affect students' enumeration strategies, progressing from the least to the most sophisticated strategy.

Seeing buildings as unstructured sets of cubes

Students act as if they see no organization of the cubes in the building. They usually count cubes one by one, and they almost always lose track of their counting.

Seeing buildings strictly in terms of their sides or faces

Students think only about a cube building's faces or sides. They count all or some of the cube faces that appear on the six sides of the building. As the example in **figure 4** shows, this strategy is prob-

lematic because cubes in the interior of the building are ignored and edge cubes are often counted more than once.

Seeing buildings as space filling

Students try to enumerate both the outside and the inside of a cube building, sometimes correctly but more often incorrectly. Typical of this category is the student who counted all the outside cube faces of the building shown in **figure 1**—twelve each for the front, back, top, and bottom and nine each for the right and left sides—getting sixty-six. He then said that two cubes were in the middle, arriving at a total of sixty-eight.

Seeing buildings in terms of layers

Students determine the number of cubes in a cube building layer by layer, as shown in **figure 5.** The layers can be vertical or horizontal, and students often use one of the sides of the cube building as a representation of a layer. Although some students who use layering count the cubes in a layer one by one, most enumerate the cubes by multiplying or using repeated addition.

Strategies Used by Third and Fifth Graders

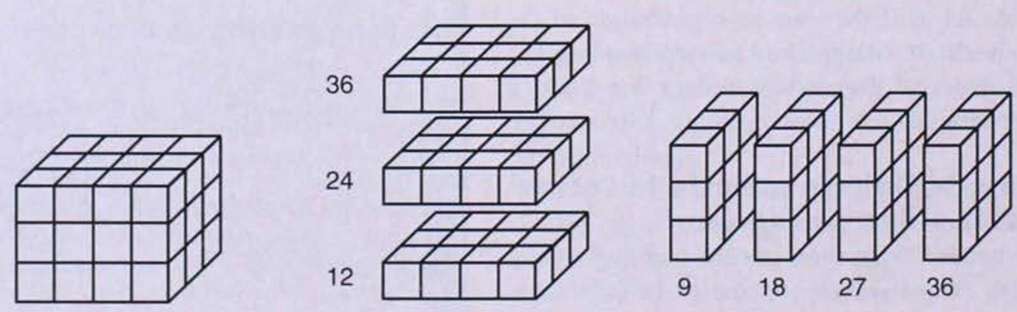
In individual interviews, we asked numerous above-average students in grades 3 and 5 to determine the number of cubes in cube buildings (Battista and Clements 1996). We ascertained that about 60 percent of third graders and 20 percent of fifth graders saw a building as consisting only of its outer faces, and 64 percent of third graders and 21 percent of fifth graders counted some cubes more than once. We also learned that fewer than 20 percent of third graders and about 60 percent of fifth graders used layering strategies, with only 7 percent of third graders and 29 percent of fifth graders correctly using a layering strategy for all three interview problems. Because these students were above average, these figures most likely underestimate how often unsophisticated strategies and how seldom layer strategies are used by students in general. For example, Ben-Chaim, Lappan, and Houang (1985) reported that about 39 percent of fifth through eighth graders counted cubes more than once. Students' use of strategies was not greatly affected by whether we gave them diagrams of cube buildings or actual cube buildings—which, for our interviews, students were not allowed to take apart—so we cannot attribute students' difficulties to their failure to understand diagrams.

Use of Formulas

Our research also illustrates the inadvisability of teaching students a formula or set procedure for determining the number of cubes in cube buildings, a practice that is found in mathematics textbooks for students in as low as grade 3. Just 7 percent of the third graders and 29 percent of the fifth graders were consistent at mentally constructing cube buildings in terms of layers, a construction that seems absolutely necessary, but certainly not sufficient, for understanding such a procedure. In fact, only students who have already constructed such a layering conceptualization seem "ready" to begin formulating their enumeration procedure more abstractly in terms of a formula. But even for them, it is questionable whether the formula $l \times w \times h$ meaningfully describes their procedure for finding the number of cubes in a box. Indeed, most of these students conceive of their personally constructed procedure as having two distinct steps: first, they determine the number of cubes in a layer, which they may or may not accomplish with multiplication; second, they account for the number of layers by using multiplication, repeated addition, or skip counting.

FIGURE 5

Seeing buildings in terms of layers



Fostering the Development of Proper Mental Models

Karen, a third grader, drew on grid paper the pattern shown in **figure 6a.** To determine the number of cubes that would fill the box made by the pattern, she counted nine for each of the four side flaps and concluded that the box had thirty-six. To check her answer, she cut out her pattern, made a box from it, constructed three intact 3-by-3 layers of cubes, and placed them inside the box—one on top of the other—then said that thirty-six cubes were inside. To encourage Karen to reflect further on her enumeration strategy, her teacher asked her to take the cubes out of the box and count them. Karen removed the intact layers from the box and placed them on top of one another. She counted the cube faces that appeared on the four lateral sides and again obtained an answer of thirty-six. In an attempt to get Karen to see her error, the teacher separated the layers and laid them flat, side by side.

Teacher: How many cubes?
Karen: There should be the same number.

Karen counted twenty-seven cubes by ones, then looked puzzled. The teacher asked her how many cubes would fit in the box. Stacking the layers, Karen said thirty-six. The teacher again laid the three layers flat and asked Karen to count the cubes. She got twenty-seven, then concluded that twenty-seven cubes fit in the box.

During this episode, Karen did not seem capable of correctly enumerating the cubes organized by layers. Even when she removed the cubes from the box, cubes that she herself had arranged into three layers of nine, she maintained her belief that the strategy of counting cube faces seen in the four lateral faces of the cube building was correct. It was as if she could hold in her mind only one orthogonal view of a cube building at a time—she could not coordinate two or more such views.

Karen and her teacher then discussed how many cubes would fit in a 2-by-2-by-2 open-box pattern that Karen had made (see **fig. 6b**).

Karen: 4 + 4 + 4 + 4 [pointing to the four side 2-by-2 flaps attached to the base].
Teacher: How many cubes are in the bottom layer?
Karen: Four.
Teacher: How many layers?
Karen: Four [pointing to the four squares on one of the sides].

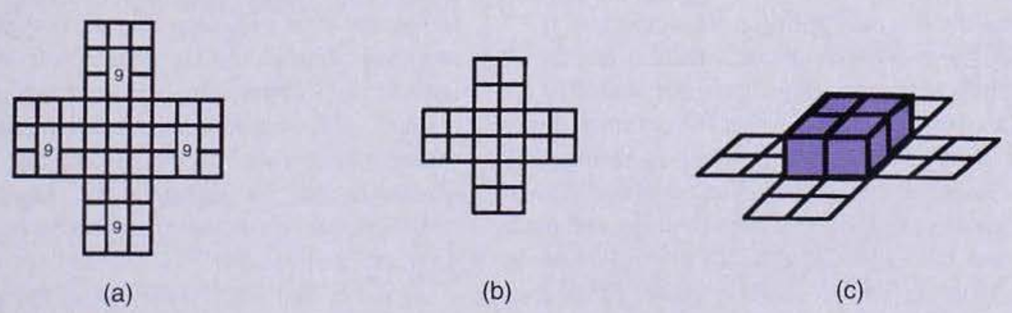
Karen's teacher placed a 2-by-2 cube layer on the bottom of the pattern (see **fig. 6c**) and again asked how many layers would fit in the box. Karen was puzzled. The teacher asked where the bottom two squares on the side of the pattern would go on the cubes when the side was folded up. Karen correctly indicated where on the cubes. He asked her where the next two squares on the pattern would go. Karen thought about it for awhile, then she pointed and said that they would go above the two cubes that she had previously indicated.

Teacher: How many layers?
Karen: I think two.
Teacher: So how many cubes would there be?
Karen: Eight.

In Karen's initial dealings with cube buildings, she enumerated cubes by focusing strictly on the buildings' lateral faces. While counting, she seemed unable to keep track of the cubes in layers. However, by giving sufficient guidance, her teacher helped her see a layering structure for the 2-by-2-by-2 building, which enabled her to enumerate its cubes correctly. However, Karen's conceptualization of cube buildings was quite fragile at this point; she needed more experiences with using a layering structure before she would be able to apply this conceptualization reliably to other cube buildings.

FIGURE 6

Karen's patterns and counting



Instructional Tasks

We next describe several instructional tasks that we have developed to help students construct an understanding of the spatial structure of cube buildings. Problems should be presented in a way that allows students to construct their own personally meaningful solution strategies. A teacher can facilitate this strategy construction not by "giving" solution procedures to students but by encouraging students to invent, reflect on, test, and publicly discuss strategies in a spirit of inquiry and problem solving.

For all grade levels, students first make predictions then check their predictions by making boxes out of square grid paper and filling them with interlocking cubes that are the same size as the squares on the grid paper. We encourage students to make predictions because students' predictions are based on their current mental models of cube buildings. Making predictions encourages students to reflect on and refine those mental models. Indeed, it is students' mental models that we are trying to develop. Having students merely make boxes and fill them with cubes does not promote nearly as much student reflection because *(a)* opportunities for cognitive conflict arising from discrepancies between predicted and actual answers are greatly curtailed and *(b)* students' attention is focused on physical activity rather than on their own thinking.

Grade 3

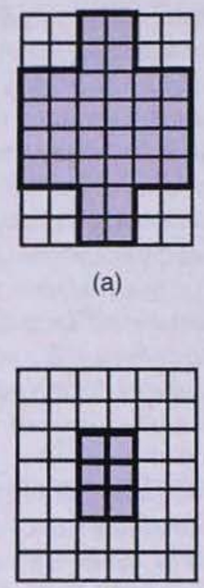
The goals of our grade-3 activities are for students to *(a)* explore the structure of rectangular boxes and the cube buildings that fill them and *(b)* develop strategies for determining the number of cubes in buildings that they are constructing (see Battista and Clements [1995a]). For reasons that we have already explained, we have students predict numbers of cubes. However, we do not expect third graders to master making such predictions.

After being introduced to the idea of a box pattern by making and exploring patterns for boxes that contain one cube, then patterns for boxes that contain two cubes, students investigate patterns for boxes that contain larger numbers of cubes. For example, after students predict how many cubes fit in the box made by the pattern shown in **figure 7a,** they determine answers by cutting out the pattern, building the box, and filling it with cubes.

In our next set of problems, students are shown the pattern in **figure 7b,** for example, and they are asked to draw the sides so that the completed pattern will make an open box that contains exactly twelve cubes. Some students can complete the pattern only after they make a cube building and place it on the base. These activities are followed by many more in which students make boxes then determine how many cubes are needed to fill them.

FIGURE 7

Grade-3 box-pattern tasks



(a)

(b)

Grade 4

Consistent with the notion that for students to construct proper mental models for cube configurations they must coordinate different orthogonal views, in grade 4 we ask students to do just that (Battista and Clements 1995b). We give such problems as the following:

(a) On square grid paper, draw the front, top, and side views of the cube configuration in **figure 8a.** Check by building the configuration with cubes.

(b) Use interconnecting cubes to make a cube configuration that has the three views shown in **figure 8b.**

Grade 5

At fifth grade, we focus on having students develop strategies for accurately *predicting* how many cubes or packages will fill a rectangular box (see Battista and Berle-Carman [1995]). Students check their predictions by making the boxes from paper and filling them with cubes. These activities encourage students to develop mental models of boxes and cube buildings that support viable enumeration strategies. Our goal is for students to learn to think about cube buildings in terms of layers and to use appropriate numerical strategies to enumerate the cubes on the basis of this layer conceptualization, for example, "A layer contains twelve cubes, there are five layers, so there are sixty cubes altogether."

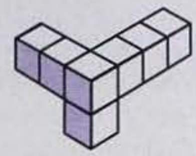
Students start with problems like those in **figure 9,** in which they are to predict how many cubes fit

in a box, then check their answers by building the box and filling it with cubes. Several problems of each type should be given. Students should check their prediction on one box before moving on to the next box. This sequence of problems gives students a chance to construct personally meaningful strategies for predicting the number of cubes in a box.

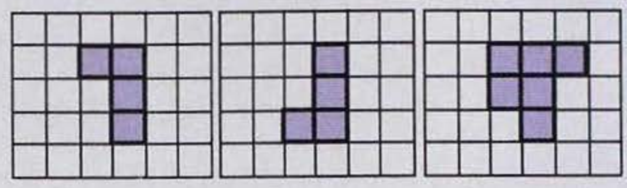
Students then predict how many rectangular packages of various sizes fit in boxes, again checking their answers by building the boxes and filling them with packages (see **fig. 10**). The goal is to ensure that students are actually imagining the organization of cubes or packages in boxes instead of using a numerical procedure that they do not understand.

FIGURE 8

Grade-4 tasks



(a)



Front view   Top view   Side view

(b)

FIGURE 9

The grade-5 "How many cubes?" tasks

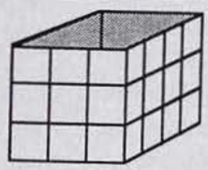
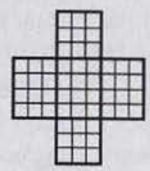
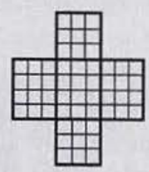
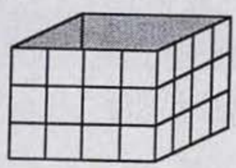
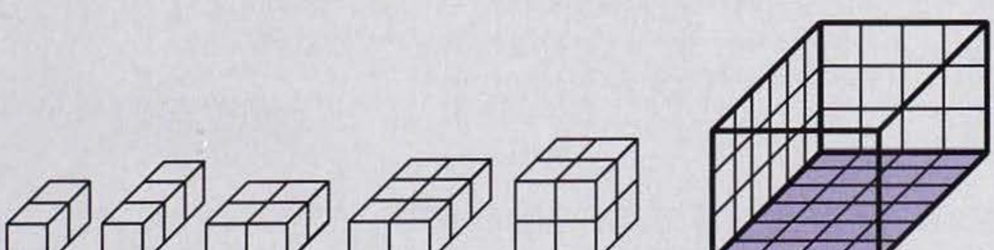
	Box	Pattern
Problem type 1: A picture of both the box and a box pattern are given.		
Problem type 2: A picture of only the box pattern is given.		
Problem type 3: A picture of only the box is given.		
Problem type 4: Only a verbal description of the box is given.	The bottom of the box is 6 cubes long and 5 cubes wide. The box is 4 cubes high.	

FIGURE 10

The grade-5 "How many packages?" tasks



Conclusions

Our research suggests that the development of students' strategies for meaningfully enumerating cubes in cube buildings, a fundamental notion in understanding the measurement of volume, is far more difficult than has previously been believed. We have seen that for many students, the source of their difficulties is an inability to coordinate and integrate the views of a building to form a single coherent mental model of it. For students to construct appropriate mental models and personally meaningful strategies for enumerating cubes in cube buildings, they must be involved with appropriate instructional tasks—tasks that encourage and support a personal development of enumeration strategies that are based on appropriate mental models of the cube buildings.

Action Research Ideas

1. Assess your students' strategies for enumerating the cubes in cube buildings by *(a)* asking them to determine how many cubes they would need to make the building shown in **figure 1,** *(b)* having them describe in writing exactly how they figured out their answers, and *(c)* having them share their results and their strategies. Classify each student's strategy by referring to the strategies described in the section of this article on students' mental models.

2. Continue to ask your students to enumerate the cubes in various cube buildings over several months. Have them write descriptions of their strategies and then share them with the class. Record each student's strategy, and note any changes from previous assessments.

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