More of the king’s really big bowl

Teacher reflection
It seems like mangoes and misconceptions go hand in hand, especially after one hears the “big bowl” discussion. We wrote this Back Talk as observers at a lesson study that brought out some enlightening facts about how student thinking can be—

- easily misguided by other students;
- helped or harmed by manipulatives; and
- ruled by misconceptions until someone (like a teacher or a fellow student) or something (like an effective teaching practice, such as probing or questioning, or such multiple problem-solving approaches as pictorial, hands on, and so on) helps not only to clarify floating misconceptions but also to solve the problem efficiently.

This investigation is from Classic Middle-Grades Problems for the Classroom on NCTM’s Illuminations website: http://illuminations.nctm.org/LessonDetail.aspx?id=L264.

The Mangoes problem involves finding how many mangoes were in the original bowl, given that five different people take turns eating a fraction of the mangoes currently in the bowl. (The first person eats one-sixth, the second person eats one-fifth, and so on.) Ultimately, three mangoes are left in the bowl.

The problem presented learning opportunities for the class; it also helped bring out inherent misconceptions that students had with understanding the context of a problem, seeing a group as a unit, and working with complementary fractions. Below is a detailed examination of a second group’s work. Like the first table group, this one also consisted of four students. In her notebook, the organizer (O) sketched a representation of the problem’s description:

\[
\frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = 3
\]

Then the conversation started. Her symbolic notation captured the same “important” information as our “big bowl” group had but in a math sentence that is not “true.” Like many elementary school students, the organizer still viewed the equal sign as indicating and the answer is rather than as communicating equivalence. Her symbolic sentence did not represent a “math fact” to her but rather was a symbolic way to illustrate or keep track of the action in the problem.

O: I did three times two since three plus three equals six. Then I did one-third of six, which is two, and then added that to six to get eight. Then I did one-fourth of eight, which is two, and added it to eight to get eight plus two equals ten. Then I took one-fifth of ten, which is two, and added to get ten plus two equals twelve; and I did one-sixth of twelve and added it to twelve to get twelve plus two equals fourteen.

As is often the case, this table group had one student, the skeptic (S), who seemed to have done the work but never shared (until later). The doer (D) seemed to be the leader of the group; he tried to bring closure to everything that they had to offer as a group (whether it was right or wrong). The follower (F) was dissatisfied with working on paper; he wanted to use the snap cubes that were at the table to solve the problem. The conversation continued:

F: Like, you take six of them [picking up six snap cubes]. There was the three left [putting down three snap cubes on the table]. And that is half of them. So you need three more [adding three more snap cubes], and that was before the last guy ate. Then he ate one-third, so that means he had to eat three, which means we add three. But then I didn’t have enough time.

D: So, what is your answer?
F: I didn't get an answer. I didn't have enough time.

D: You are only adding two each time [a misconception].

F: Doesn't really make sense.

D: Yes, it does. Look.

The doer grabbed snap cubes to disprove the follower and to convince the girls at the table that they should really be multiplying each time because of the 1/2, 1/3, 1/4, 1/5, 1/6 (another misconception). The doer seemed to have convinced the organizer to take an alternate route, which the teacher noticed. She joined the next sequence of conversation:

What are we doing here?

O: Now I am figuring out another way. First I did three times two because he did one-half. So you get six. Then we have to do six times three for one-third. Then you get eighteen. We do eighteen times four for one-fourth, which is seventy-two. Then you have to do seventy-two times five, which is ... [getting the full approval of the doer (the boss at the table), they keep multiplying, finishing this problem with the answer of 2160 mangoes (the same “big bowl” misconception)].

F: [remaining unconvinced] How many [looking skeptical] did the king eat?

O and D: He ate one-sixth of that. He ate 360.

F: [deciding it is time to convince everyone at the table with his grand finale and putting down three snap cubes to represent the remaining three mangoes in the bowl] Prince 3 ate three [putting down three more cubes to make a total of six cubes on the table]. So this is how it was before Prince 3 came along. Before Prince 3, there were six mangoes.

I have a question. I thought that Prince 3 ate less. So how could it be fourths?

F: Before Prince 3 ate, this was how many were left [grabbing all six cubes], and this is how many he ate [dropping 3 cubes], which is three. That means there were three left.

OK [seeming convinced].

F: “Prince 2,” it says, [reading the problem] “ate one-third of what was then left before he ate.” So there were nine mangoes before he ate, and since there is six after, it means he ate three [dropping three more cubes].

O: Is that a better way to throw the mangoes back into the bowl?

F: I am working back in time [using the classic response of working backward].

D: So, basically, you did three times the last one [still unwilling to let go of her misconception of multiplying by the fraction’s inverse each time].

F: [Completely ignoring the doer’s observation because he is busy convincing the observer and the teacher, the follower grabs the nine cubes on the table.] This is before Prince 2 ate. And then Prince 1 came; he ate one-fourth of what was before him, which means nine is left after he ate [pointing to the nine cubes on the table]. So that means there were twelve.

This idea of focusing on what is left after a portion is eaten is difficult for some students to grasp. To most students, working backward means simply “undoing” the given operation. This is evident in the two groups’ continued belief that multiplying by the fraction’s inverse—which resulted in the big bowl of mangoes—will be part of the solution. The students worked hard to rationalize their answer because they were so confident of their strategy.

As the follower indicated, “working backward in time” in this problem requires that one continually readjust the “whole” amount at each step of the story. (For example, nine is the whole that Prince 2 saw, but nine is three-fourths of what Prince 1 saw.) The ability to visualize one-fourth and three-fourths as a complementary relationship is not trivial.

This shift in perspective of solving through complementary relationships rather than through inverse operations resulted in a once-reticent student finally finding voice and affirmation. The skeptic, who had never spoken, suddenly jumped up and said she “got it” but claimed she had not said so because she was unsure that she was correct until the
follower described it with snap cubes.

S: I was right! [The four students continue to build backward, focusing on what is left in the bowl.]

F: Before the king came, there were eighteen; and then the king ate a sixth, and there were fifteen. [At this point, the follower seems to have convinced everyone.]

D: Actually, I think [the follower] is right! [providing closure by giving the follower credit for the demonstration of the solution using snap cubes and for the convincing verbal explanation. Then the quiet skeptic jumps in again.]

S: Finally, credit.

D: Wow, [the skeptic] had it right all this time. Wow!

Yes, this was a “wow” moment indeed. The NCTM Illuminations link shows other solution strategies. However, one must be aware that what may seem obvious to us can be conceptually difficult for a child. After all, perhaps the king truly does have a really big bowl.