
$\frac{1}{5}$
$\frac{2}{5}$
$\frac{1}{2}$
$\frac{3}{5}$
$\frac{4}{5}$
0
1
half of
$\frac{1}{5} = \frac{1}{10}$

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Moving beyond Brownies and Pizza

A carefully planned sequence of contextual problems in a multiweek unit and a strong emphasis on student discourse helped fourth graders use number lines to develop a rich understanding of comparing fractions.

By Daniel W. Freeman and Theresa A. Jorgensen

A lack of fractional understanding is a well-documented obstacle to student achievement in upper elementary and middle school math (National Center for Educational Statistics {NCES] 1999; Lamon 1999; National Research Council [NRC] 2001). Lamon (1999) notes that one major conceptual hurdle that students must overcome is the idea that fractions are numbers in and of themselves, not a composition of two, distinct, whole numbers. Further, it is likely that students fail to recognize fractions as discrete numbers because much of school mathematics focuses on understanding fractions as parts of wholes or parts of sets. To a great extent, children's earliest experiences with fractions are situated in part-whole contexts where both the part and the whole are whole numbers. Many of these experiences involve partitioning items like brownies or pizzas into equal regions. These are worthwhile endeavors for young children, but at some point, students must transition to thinking about fractions in ways that are more sophisticated.

The use of a number line when teaching fractions helps students better develop a sense of the magnitude and relationships of the numbers.


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Fractions as measures

The Common Core State Standards for Mathematics (CCSSM) expect students to understand arithmetic operations on fractions by the time they leave fourth grade (CCSSI 2010). Inherent in this expectation is the notion that fractions are numbers to which students can extend previous understandings of operations on whole numbers. Students may not be well prepared to meet this expectation if they think about fractions exclusively in terms of brownies, pizzas, or other area models (Charalambous and Pitta-Pantazi 2007). Students need opportunities to develop and explore their understanding of fractions as measures, that is, understanding both the relative size of fractions (e.g., 3/4 is a bigger number than 1/2) and understanding how fractions measure specific intervals (e.g., an eraser is 3/4 inch wide). The number line model is a logical context within which these explorations can take place.

Three goals

Like other topics in elementary and middle school math, a rich student understanding of fractions may be short-circuited by the early introduction of standard algorithms (National Mathematics Advisory Panel [NMAP] 2008; NRC 2001; Van de Walle, Karp, and Bay-Williams 2010). When students employ traditional methods for ordering, comparing, and operating with fractions before they have a solid grasp of fraction concepts, they may end up following procedures with little understanding. They may know what to do while simultaneously being unable to explain why they are doing it (Skemp 1976). We need only think about the invert-and-multiply algorithm to realize how easy it is to operate with fractions apart from meaningful understanding.

When my fourth-grade class began to deeply explore fractions, I was interested in pursuing three concurrent goals:

1. Build students' understanding of fractions as numbers with a definite magnitude (e.g., 3/4 falls between 3/5 and 1 on the number line).
2. Increase students' understanding of measuring with fractions.
3. Develop fraction number sense by avoiding early introduction of traditional fraction algorithms.

I had taught many of these same students in third grade at an urban elementary school in which more than 90 percent of students qualify for free or reduced-price school meals. Our fraction work the previous year had focused on realistic contexts, logical reasoning, and exploring multiple representations. Students had represented fractions with student-generated drawings, fraction circles, Cuisenaire® Rods, grid paper, fraction strips and tiles, and sets of concrete objects.

The authors of *Adding It Up* recommend the use of the number line in the context of rational numbers because it "may help students develop a sense of the magnitudes and

relationships of those numbers that is less clear in other representations" (NRC 2001, p. 418). Our class had already used the number line extensively for adding and subtracting whole numbers in third grade. We had also used it earlier in fourth grade to explore factors, multiples, divisibility rules, and division with three-digit dividends. The children were familiar with the number line as a concept, a representation, and a tool; so, I reasoned, using the number line as our primary representation for comparing fractions would be natural.

As I prepared for these lessons, I first identified six ways that two fractions might be related to each other so that the relationship would suggest a comparison strategy. I then sequenced these relationships from less complex to more complex so that each strategy might help lead to the next (see **table 1**). Finally, I wrote six problems of each type. I situated each problem in a real-world linear context to support students' development of the measure understanding of fractions. (See the **more4u** box at the end of the article for how to access the problems from each category.)

A shift in thinking, at their own pace

For a period of five weeks, students encountered these groups of problems in the sequence indicated in **table 1**. Students worked in pairs to compare the fractional quantities in each problem set. At first, many students used part-whole concepts and diagrams to aid them in their work. However, a number of students noticed the linear context of each problem and began shifting to linear representations. I encouraged these efforts and asked students to share their work during our whole-group debriefing sessions. I continued to use the number line exclusively during whole-group discussions. I displayed student-drawn number lines on the walls and used these as reference points during lessons. Within a few weeks, nearly all the students began to demonstrate an understanding of fractions as actual quantities on the number line.

At the same time, I was careful not to reprimand or correct students who continued to rely on part-whole models. I wanted each student to develop understanding at his or her own pace. I also wanted each student to feel comfortable enough to share his or her own thinking—even if the thinking was less sophisticated or incorrect.

One day toward the end of these five weeks, I wrote the fractions two-thirds and three-fifths on the board and asked the class to compare the numbers. Students gathered on the floor around me with marker boards in front of them. After several minutes, I gave them their signal that it was time to share. Nearly every

TABLE 1

For more than a month, student pairs sequentially compared the fractional quantities in each type of problem set describing relationships between the fractions.

Sequence of fraction relationships in comparison problems	Example
1. Both fractions are one unit fraction more than a half (the number 1/2 on the number line).	5/8 and 7/12
2. Both fractions are one unit fraction less than a half (the number 1/2 on the number line).	4/10 and 5/12
3. Both fractions are one unit fraction less than one (the number 1 on the number line).	8/9 and 7/8
4. Both fractions can be used in the context of money (fractions of a dollar).	3/5 and 7/10
5. More of a bigger unit fraction or less of a smaller unit fraction	7/8 and 6/9
6. One fraction can be expressed in terms of another.	1/3 and 4/6

FIGURE 1

The number line was clearly central to students' thinking. Although both denominators were odd, and Aiden could draw how he decomposed fractions, he could not explain it thoroughly.

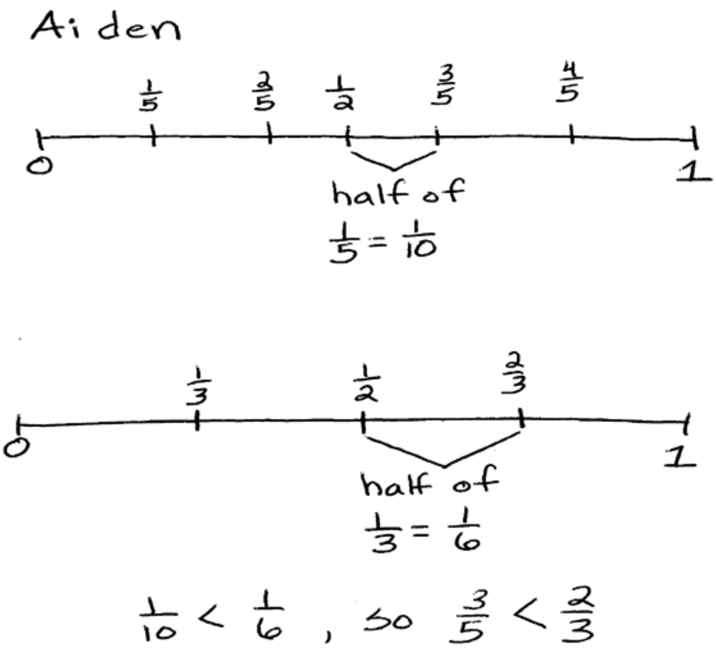
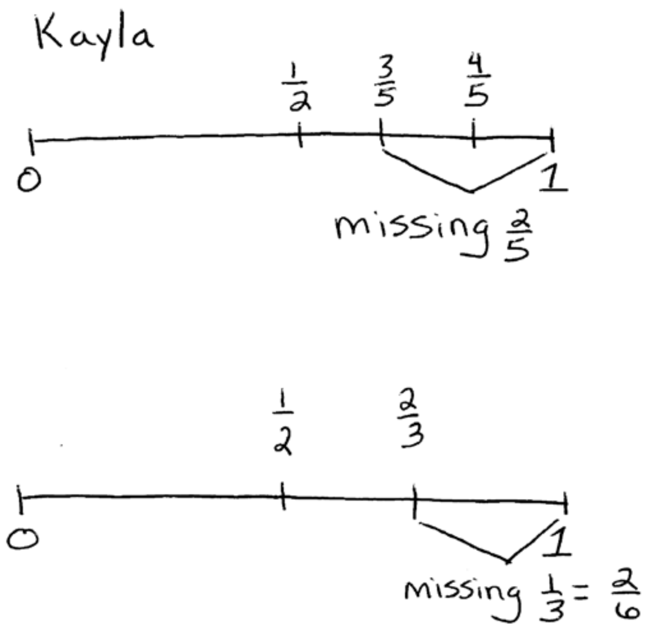


FIGURE 2

Like many of her peers, Kayla used more than one representation to demonstrate her ability to decompose fractions and to make an accurate comparison.



hand shot up. I made a digital recording of this conversation on a laptop; the following vignettes are transcribed from that recording. The vignettes typify the nature of the many conversations that had taken place in the previous five weeks during debriefing sessions. The diagrams in the figures are re-creations of students' original diagrams. Aiden and Kayla began the class conversation.

Aiden: I know that three-fifths is a half of a fifth bigger than one-half. And I know that two-thirds is half of a third bigger than one-half. Since half of one-fifth is the same as one-tenth, and half of one-third is one-sixth, that means three-fifths is one-tenth more than one-half. And two-thirds is one-sixth more than one-half. So two-thirds is bigger because the part bigger than one-half of two-thirds is more than the part bigger than one-half of three-fifths. [*See* **fig. 1**.]

Kayla: I agree with Aiden about the answer, but I didn't think about it that way. I noticed that three-fifths is missing two-fifths in order to be one, and two-thirds is missing one-third in order to be one. If you think about what's missing from a whole, you can think about one-third and two-fifths. [*See* **fig. 2**.] Now I am thinking about one-third. I know one-third is the same as adding one-sixth plus one-sixth. It takes two groups of one-sixth to make one-third. So what I am really comparing now is two-fifths and two-sixths. That's easy since the numerators are equal. Two-fifths is bigger. That means three-fifths is missing more than two-thirds. Three-fifths is smaller since it is farther from one.

Different perspectives, same number line

Although Aiden and Kayla reasoned about the relative size of the fractions from different perspectives, the number line was clearly central to their thinking. Aiden's explanation and drawing show that he could decompose these fractions, even though both denominators were odd. On the basis of previous conversations with Aiden, I was aware that he could determine halves when confronted with odd denominators. Although he did not explain it thoroughly, I imagined that he reasoned that two-and-a-half fifths and one-and-a-half thirds were both equal to one-half. Therefore, without

resorting to fraction algorithms, he concluded that 3/5 – 1/2 = (1/2)/5 = 1/10 and that 2/3 – 1/2 = (1/2)/3 = 1/6. This comparison task did not explicitly call for arithmetic manipulation, but Aiden was already operating with these fractions in an intuitive manner. Many children learn whole-number addition facts best by developing strong number sense first. I was encouraged to see Aiden's developing number sense with fractions, knowing that he would be well-positioned for the time when he formally learns fraction arithmetic operations.

Kayla also demonstrated the ability to decompose fractions and to use this knowledge to make an accurate comparison. Kayla was representative of many others in the class who were learning to decompose fractions so they could facilitate their fraction comparisons. These students were making connections between what they knew about whole numbers and what they were learning about fractions. Kayla might still have been thinking in terms of area models because she spoke about "three-fifths missing more," but she also referenced the number line representation when she determined which fraction is closer to one. This was an interesting window into Kayla's thinking process as she used more than one representation to make sense of fractions for herself. Kayla was moving along a continuum from less sophisticated to more sophisticated ways of understanding fractions.

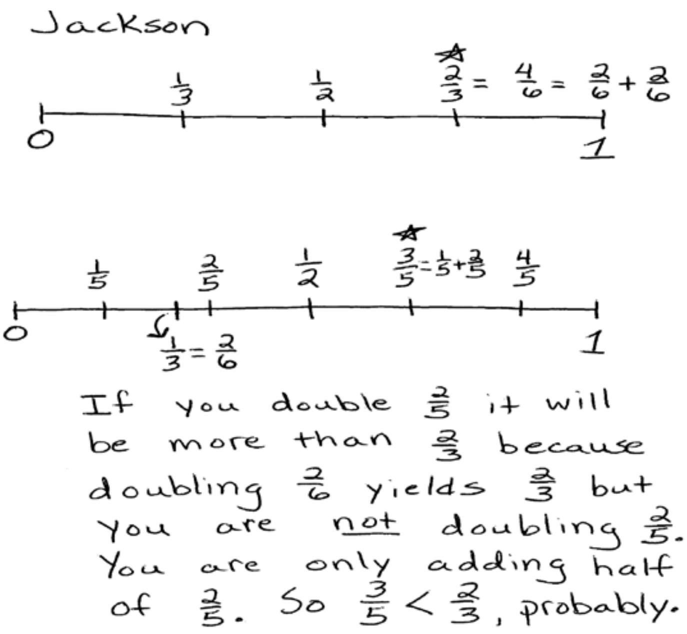
A third student, Samantha, then volunteered her ideas:

> I got a different answer. I am pretty sure it's wrong because everyone else has convinced me. I drew a picture on my marker board. I drew a bar to represent two-thirds and another to represent three-fifths. I got the bars exactly equal. But then when I split the bars into the thirds and fifths, I must not have done it very good. When I look at my picture, they look equal.

Samantha's response indicated that students were at various places in their understanding and their use of representations. Her comments also demonstrated what can occur when teachers foster an atmosphere where students expect to learn from one another and where they feel comfortable admitting their mistakes.

FIGURE 3

Jackson recognized the imprecise nature of his estimate. He demonstrated sophisticated reasoning, explaining that doubling the smaller of two fractions that are close together on the number line would probably yield a bigger fraction than adding half of the larger fraction to itself.



Specifically, Samantha may have benefited from realizing her own imprecision rather than having someone else point it out to her.

After Samantha finished, Jackson shared his strategy for how to compare the fractions:

> It's not precise, but I think it works. I was thinking that two-thirds is almost twice the size of two-fifths, because two-thirds is exactly two times as big as two-sixths, and two-fifths is a little bigger than two-sixths. That means two-thirds is not quite twice as big as it [*two-fifths*] is. And if you add another one-fifth to two-fifths, you are not even coming close to doubling it [*two-fifths*]. You would have to add two-fifths in order to double it [*two-fifths*]. You are, like, only adding half of what you need to double. So you would need to almost double two-fifths in order to make four-sixths, but you aren't doubling it by adding one-fifth. [*See* **fig. 3**]

I was unsure if any students followed Jackson's thinking. I invited Jackson to come to the front

to explain with the help of some diagrams. He drew two number lines and explained his thinking with reference to these number lines.

Jackson: I was thinking at first about two-thirds and two-fifths. Two-fifths is a lot smaller than two-thirds. It's not quite half the size, because half the size would be two-sixths, but two-fifths is a little more than two-sixths.

Teacher: Maybe you could clarify. It's not really clear to everyone.

Jackson: I mean two-thirds is almost twice as big as two-fifths because two-thirds is actually twice as big as two-sixths. Maybe I should say four-sixths is the same as two-thirds because two-sixths and two-sixths again is two-thirds. But two-fifths is a little bigger than two-sixths. So, that means that if you put two-fifths and two-fifths together, you are going to get something bigger than two-thirds. But we are only adding on one-fifth to that two-fifths. So my estimate is that two-thirds must be bigger, since you are only adding half of two-fifths, not a whole two-fifths.

Teacher: What you said the second time around helped clear things up for some people. Let me just summarize for those who are still mulling it over. Jackson reasons that if you double the smaller of two fractions that are really close together on the number line, the result of that doubling will probably be bigger than only adding half of the larger fraction to itself. It's just an estimate, but it is very reasonable since two-fifths and two-sixths are fairly close together on the number line. Think about two whole numbers: seven and eight. They are close together on the number line. Jackson's reasoning is like saying seven times two is more than eight plus four. Doubling seven is more than just adding half of eight to eight.

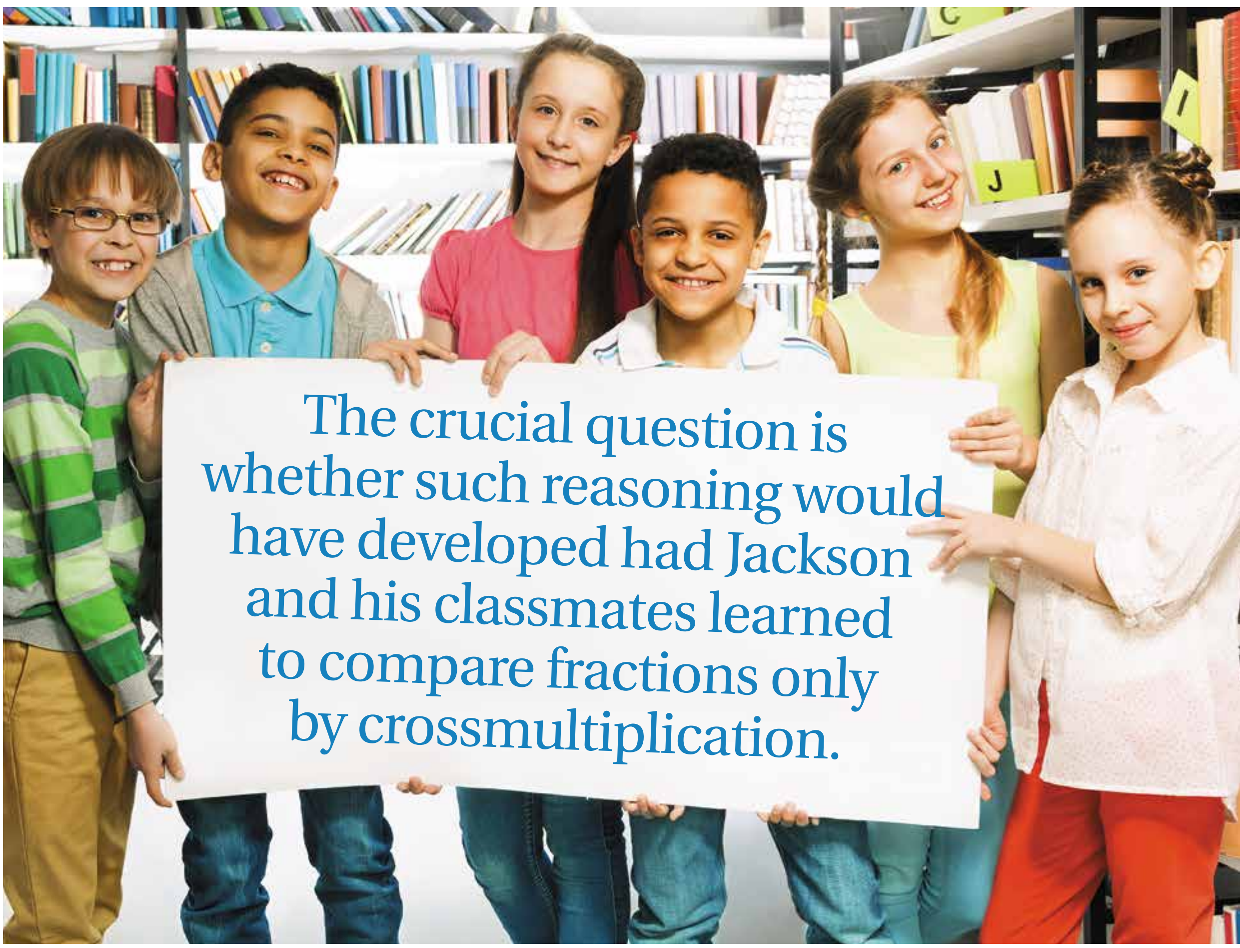


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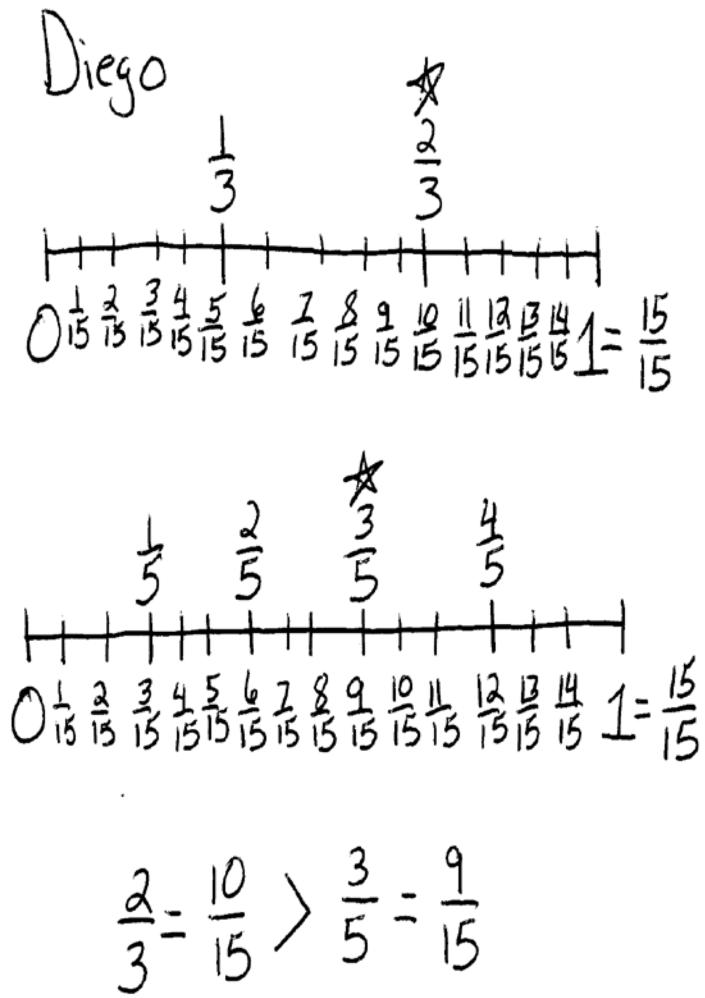
FIGURE 4

Diego drew two number lines and split them on the basis of the other fraction's denominator, resulting in fifteenths for both. He then counted the number of fifteenths up to the original fractions to determine the larger of the two.



Jackson's reasoning was sophisticated and demonstrative of someone who is developing a solid understanding of fractions. His use of the number line to illustrate and justify his thinking also indicates his growing conceptualization of the measure subconstruct of fractions. Like some of the other students in class, Jackson easily decomposed these fractions and used his understanding of whole-number operations to make a reasonable comparison. Some teachers may doubt their students could reason like this in fourth grade. However, the important pedagogical principle is not that every fourth-grade student should be able to reason like Jackson. Rather, the crucial question is whether such reasoning would have developed had Jackson and his classmates learned to compare fractions only by cross-multiplication.

What a discovery!

After waiting excitedly during Jackson's explanation, Diego was eager to share his thinking:

> We talked about how you can split one-third into two-sixths or one-fourth into two-eighths. So I drew two number lines, and I split the number lines based on the other fraction's denominator. I drew two-thirds on a number line and split each third into five parts. That gave me fifteenths. I drew three-fifths on a number line and split each fifth into three parts. That also gave me fifteenths. So, I counted the number of fifteenths up to the original fraction. The two-thirds number line had ten-fifteenths, and the three-fifths number line had nine-fifteenths. Ten-fifteenths is bigger than nine-fifteenths since you are comparing the same units. So two-thirds is bigger. [*See* **fig. 4**.]

Recall the instructional sequence listed earlier, and note that before having this conversation, the students and I had never discussed Diego's strategy. Students had previously discovered that they could express one fraction in terms of another, but we had not explored any contexts that might suggest comparing both fractions via a separate common denominator. Watching Diego forge this connection for himself was thrilling, as was witnessing his peers realizing what a *discovery* Diego had made! Not every student in every class will make this connection independently. However, I believe that all students ought to have sufficient opportunity to develop these connections before teachers impose the traditional approach. For the students in my class who did not initially understand Diego's method, the kind of conceptual work we had just undertaken set the stage for eventual acquisition of the standard algorithms and provided me with critical insight into their understanding of fractions. This allowed me to then plan for additional instruction to facilitate these students' development.

Magnitude, measurement, and relative size

All students in this class made progress in relation to the three goals I had set for them. The

specific students featured here were indicative of many left unmentioned. Like Jackson, students were building an understanding of fractions as numbers with a definite magnitude. As illustrated by students' diagrams, they were beginning to understand the measure subconstruct and to use the number line as a tool for representing fractions. In particular, students were using ideas connected to the measure subconstruct as a way to reason about the size of fractions in comparison to other fractions and to one whole. Finally, as Kayla's comments indicate, she and other students were developing fraction number sense as they determined the relative size of fractions without resorting to traditional methods of comparison. However, Diego's final comments show that we were on the threshold of a natural opportunity to explore the common-denominator approach.

These vignettes illustrate a pathway for developing the important concepts that relate to the measure subconstruct of fractions. Understanding these concepts moves children along the continuum toward the increasingly abstract uses of fractions, which they will encounter in middle and high school.

Common Core Connections

4.NF.B.3b
4.MD.A
4.NF.A.2

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