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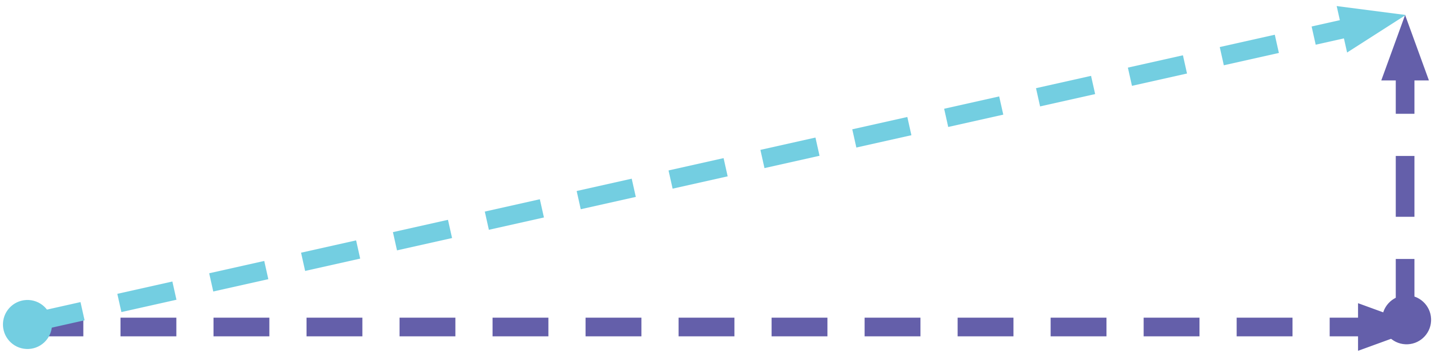
Encouraging optimization of energy



Designing strategies to optimize energy is a concern that we face in our daily lives. Optimizing energy is important because we need to conserve resources on our planet to ensure that resources are always available. Scientists in different fields often encounter situations in which they need to optimize. For instance, engineers frequently minimize the material used in construction, and biologists advise on spending the least amount of nonrenewable natural resources. Additionally, employees in the business world who travel might visit multiple locations and want to reduce travel time and expenses. Avoiding using the same road twice would minimize backtracking and costs associated with it (this becomes the Traveling Salesman problem). In our daily schedules, we find ourselves running such errands as visiting stores, the school, and the bank, and we often plan our routes to save gas and time. We could design a delivery map in our heads, then sketch it on paper, representing each location with a dot and representing roads between those locations with lines. People in a branch of mathematics called *graph theory* commonly call a diagram consisting of dots and lines a *graph* (see **fig. 1**).

The activity presented here corresponds to a variety of mathematics concepts. The Problem Solving, Reasoning and Proof, Communication, Representation, and Connections Standards of the National Council of Teachers of Mathematics are fully supported through the development of this activity (NCTM 2000, 2014). In addition, the Common Core's Standards for Mathematical Practice (SMPs) of *Reason abstractly and quantitatively*, *Construct viable arguments and critique the reasoning of others*, *Model with mathematics*, and *Use appropriate tools strategically* (CCSSI 2010) are embedded in the activity. In modern times, a valid procedure to resolve a problem for the planet, the community, or in daily life includes using minimal resources from our environment, analyzing a complex problem, and discovering optimization patterns as well as the supporting arguments for findings.

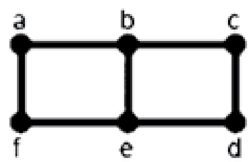
Exploring optimization through graphing

This activity portrays cities as dots (*vertices*), and the roads between the cities are connected with some lines (*edges*). The goal of the activity is to find a path that visits all the cities and all

FIGURE 1

The optimal path visits all the cities and uses all the roads without overlap.

Graph 1



Efficient path: *b, c, d, e, b, a, f, e*

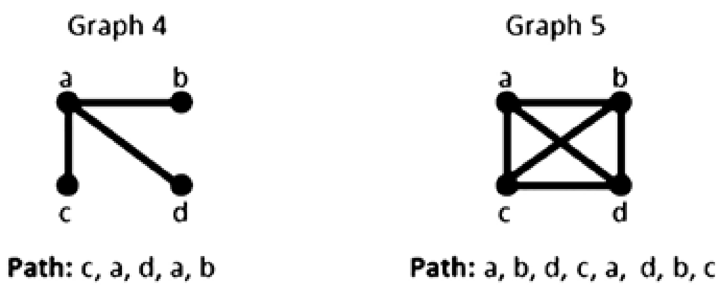
the roads. The game is won by the player whose path does not use the same road twice. When designing a path, labeling the dots, or vertices (using letters), on this graph is helpful; after this is done, the path will be simply a sequence of letters. If we consider that we are using energy (gasoline) each time we visit a city, we are really looking into the optimization of energy. In **figure 1**, the path *a, b, c, d, e, f, a, b, e* visits all the cities and all the roads, but the road that connects cities *a* and *b* is used twice, so it is not an efficient path. The optimal path for this graph is is *b, c, d, e, b, a, f, e* (see **fig. 1**).

FIGURE 2

Two simple graphs and paths (not necessarily optimal) visit all edges.

FIGURE 3

These two simple graphs have paths that visit all the edges, although they are not necessarily optimal.



Goal and history of the activity

Students are expected to draw several graphs and design the most efficient path for each graph. A second goal is to discover patterns regarding the graphs that have efficient paths. In mathematics, an efficient path in the previous graph (see **fig. 1**) is called the Euler Path. Euler was a Swiss mathematician who was walking through various points in a city called Königsberg (now Kaliningrad, Russia). Seven bridges connected the points, and Euler wondered if he could visit all those points in the city using all seven bridges without using a bridge twice (Newman 1956). Euler proved the impossibility of such a walk and found a formula based on how many lines connect to each dot on the graph (this is called the *degree of a vertex*).

Appropriate grade levels for the activity

This task can be implemented in grades 3–6. For the lower grades, the purpose is to encourage students' optimization skills via game-based activities. For older children, the main focus is to develop strategies with the specific graphs and establish a direct connection with real-life situations.

1. Providing sample graphs

In the first step of the implementation, students were expected to optimize the paths by finding an efficient path without overlap. **Figures 2** and **3** show some graphs that assisted students in understanding the goals. Each graph has a path that visits all the edges even if it visits some edges twice. Students were to find the most efficient path possible.

Notice that finding an efficient path is the same as drawing the graph without removing the pencil from the paper and without overlapping a line. Hence, an easier way to pose this problem is this: *Could you go through all the lines without going through the same line twice?* Students could practice on a separate piece of paper to find efficient paths. Here are some interesting questions to ask students while they participate in the activity. For your convenience in implementation, an answer to each of the following questions is provided for verification purposes.

- Were you able to find an efficient path in graphs 2, 3, 4, and 5? (*No, only for graphs 2 and 3.*)
- Describe a pattern you notice in the graphs that had an efficient winning path. (*Answers will vary.*)

FIGURE 4

An efficient path is impossible for graph 6.

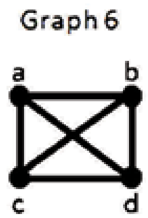


FIGURE 5

Students used task 1 to explore graph 5.

Task 1: In the graph below, label the vertices. Then compute the degree of each vertex and write it inside of a circle next to its label and, finally, find a winning path for the graph.

2. Making decisions

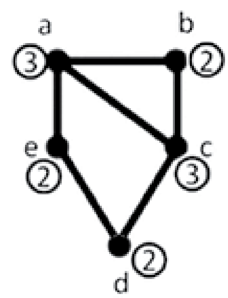
In this part of the activity, students explore more examples and provide multiple path solutions when possible. In graph 6 (see **fig. 4**), an efficient path is impossible. Students were encouraged to spend time first trying to discover an efficient path; then after they had exhausted all possibilities, a class discussion ensued. Students discussed the impossibility of the existence of an efficient path on a given example. Exhausting all possibilities can support the conclusion that a solution is impossible. In this step, the teacher could ask students to list all possibilities and try to recognize features that determine an efficient path. As opposed to graph 6 in **figure 4**, where an efficient path is impossible, we have graph 1 (see **fig. 1**), where an efficient path is possible if we start at a convenient vertex. Some of these questions supported a classroom discussion:

- Is there an efficient path that starts at vertex *a*? (*No.*)

FIGURE 6

The results of task 1 are illustrated below. Students noticed that vertices *b, d*, and *e* have degree 2 (even), and vertices *a* and *c* have degree 3 (odd). The teacher pointed out that, as a consequence, vertices *b, d*, and *e*, have "even degree" and vertices *a* and *c* have "odd degree." The teacher asked key questions.

- Exclude the beginning vertex and the ending vertex in the winning path *a,b,c,d,e,a,c* for a moment. Did you notice, by following a winning path, that each time you arrive at a vertex, you have to leave that vertex?
- What does this mean in terms of the edges connected to that vertex? (*Answer: It means that through a winning path, the number of edges that arrive matches the number of edges that leave the vertex.*)
- What does this tell you about the degree of that vertex? (*Answer: The degree is even.*)
- What can you conclude about the degrees of all the vertices when there is such a winning path? (*Answer: The degree is even for all the vertices, except perhaps for the ones where the winning path starts or ends.*) The following task was performed; this is the task that allowed students to discover patterns of winning paths.



Winning Path: a,b,c,d,e,a,c

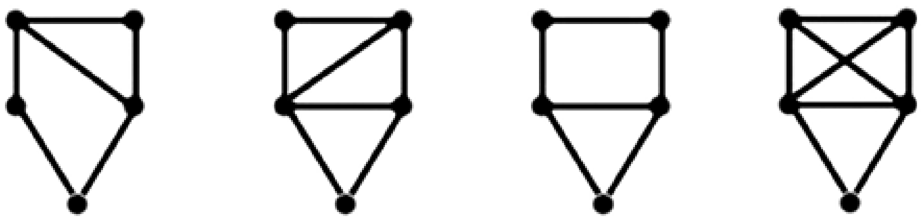
- Is there an efficient path that starts at vertex *c*? (*No.*) At vertex *f*? (*No.*) Why is this response the same as the previous one? (*Because of symmetry, vertex* a *and vertex* c *are the same if we flip the graph.*)

FIGURE 7

Students can write their own ideas, patterns, or examples to help them visualize their findings and reason through the process.

Task 2: In each graph below, label the vertices. Then compute the degree of each vertex and write it inside a circle next to its label. Finally, find a winning path for the graph.

- Did you find any pattern between the graphs that have winning paths and the degrees of the vertices? If not, try the graphs in the next two figures.
- After you find your pattern, test it with a graph that you design yourself or with the graphs provided in **figure 8**.



- Is there an efficient path that starts at *e*? (*Yes:* e, d, c, b, a, f, e, b.) Is this path unique? (*No, here is another one*: e, f, a, b, c, d, e, b.)
- How many paths start at vertex *e*? (*Six:*
  1. e, d, c, b, a, f, e, b
  2. e, d, c, b, e, f, a, b
  3. e, f, a, b, c, e, d, b
  4. e, f, a, b, e, d, c, b
  5. e, b, c, d, e, f, a, b
  6. e, b, a, f, e, d, c, b)

A final graph is interesting to analyze in this part of the activity. Students answered some of these questions regarding the graph in **figure 4**:

- Can you find an efficient path? (*No.*)
- Why is it enough to explore only the paths that start at vertex *a*? (*Because of symmetry; by flipping the graph in one way or another, each vertex looks like* a.)
- Write a path that starts at *a* and continues visiting the edges in graph 5. If you cannot finish writing an efficient path, stop and write a reason why you could not continue. (*This is just an example:* a, b, c, d, a, c, *stopped. I could not finish an efficient path because I would need to use one of the three edges connected to* c, *which I already used.*)

3. Formulating patterns

Encourage students to recognize patterns of the graphs that have efficient solution paths (see **fig. 5**). Recall that the degree of a vertex is the number of lines that are connected to it. By computing the degree of the angle at each vertex and writing efficient paths, students can generalize if a graph has an efficient path (see **fig. 6**). Before the final step of the activity, students use task 2 to write their own patterns and examples (see **fig. 7**).

4. Presenting results

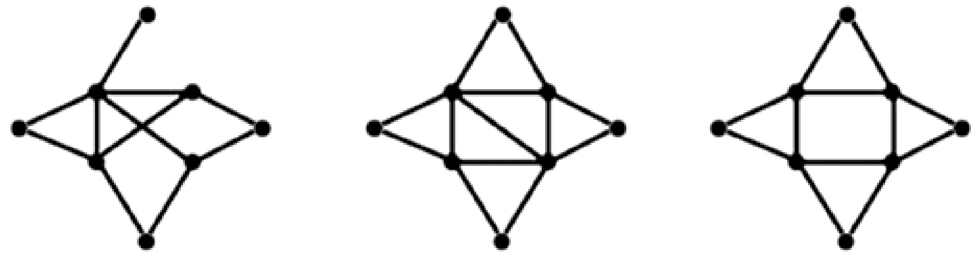
In the final step of the activity, students presented their findings. They also tested the patterns that other students discovered. Here are some questions to explore with the entire class:

- When does a graph have an efficient path? (*When there are exactly two or no vertices of odd degree*) What pattern did you find?
- If a graph has an efficient path and we remove an edge, does the resulting graph have an efficient path? (*It depends; if after removing the edge, the number of vertices of odd degree is either 0 or 2, then it has an efficient graph.*)
- If a graph has an efficient path and we add an extra edge, does the resulting graph have an efficient path? (*It depends; if after adding the edge, the number of vertices of odd degree is either 0 or 2, then it has an efficient graph.*)
- If I have an efficient path but I lose my graph, can I re-create it from the information on the efficient path? (*Yes, we can just start drawing*

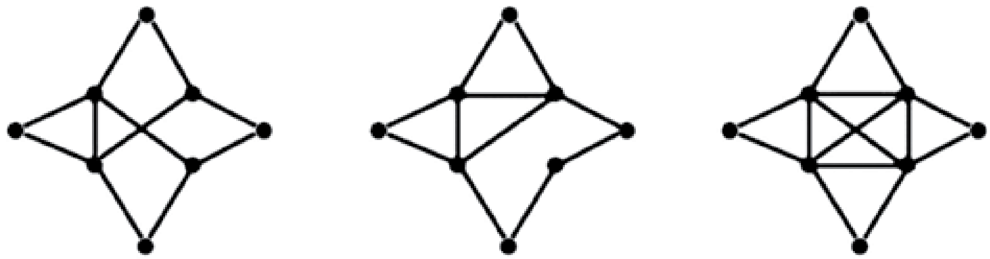
FIGURE 8

After students found a pattern in task 2, they could test it with the graphs in these figures.

(a)



(b)



the vertices one at a time and follow the path to draw the lines.)

- What are some possible applications to real life of determining the most efficient paths? (*Minimizing energy, delivering pizza, using the least amount of resources*)

Pursuing optimization

The activity presented here encourages students to practice optimizing natural resources in daily-life situations. It also helps them with abstraction of a physical model into a mathematical one. The process of traveling through edges is an optimization skill that contributes to the resolution of complex problems in the STEM fields. All these topics (exploration, analysis, patterns, justification, and discussion) are truly exemplified when teaching mathematics in the early years.

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