

Unwrapping **Students'**

A stack of several triangular sandwiches, likely club sandwiches, are shown. They are made with white bread and filled with layers of sliced meat (possibly ham or turkey), melted yellow cheese, and green vegetables like lettuce and onions. A single wooden toothpick is used to hold the sandwiches together, visible at the bottom of the stack. The background is a solid light orange color.

Sandwiching
formative assessment items
and instruction can yield
insight into why students use
particular strategies
or notation.

Ideas about Fractions

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Supporting students to develop an understanding of the meaning of fractions is an important goal of elementary school mathematics (NCTM 2000; CCSSI 2010) (see the sidebar on p. 168). This involves developing partitioning strategies, creating representations, naming fractional quantities, and using symbolic notation. This article describes how teachers can use a formative assessment problem to elicit and understand students' ideas about the meaning of fractions.

Mr. Lin, a fourth-grade teacher, knew that his students had to understand the meaning of fractions so that they could understand fraction equivalence and operations in their upcoming fractions unit. He was curious about students' fractions understanding, so he asked his class to solve an equal-sharing problem:

6 children are sharing 8 small sandwiches. They are sharing so that each child gets the same amount.

How many sandwiches will 1 child get?

Mr. Lin circulated around the classroom as students worked and asked each student about his or her thinking. In the vignette that follows, Mr. Lin talks with a student to learn more about his understanding of fractions.

Mr. Lin: Tell me about what you did here.

Abdi: Well, it says there's six children and eight sandwiches. That's enough for each kid to get a sandwich. [See fig. 1.]

Mr. Lin: How did you show that in your drawing?

Abdi: That's these [pointing to the six undivided sandwiches].

Mr. Lin: So you gave one of these to each person? [He touches each of the undivided sandwiches.]

Abdi: Yeah. Then there were two sandwiches leftover, so I figured out how much to give each person. I tried half, but that's only four-halves, and I need six. So I did each sandwich into threes [pointing to the two sandwiches sliced into thirds].

Mr. Lin: Ah, I see how you did that. If I'm one of the kids, can you point to what I get to eat? [Abdi points to the top-left whole sandwich and the left-most one-third.] Got it. How much sandwich is that altogether? What would you call that?

Abdi: One and one-third [pointing to where he wrote "1/3"]. Is that how you write it?

Mr. Lin: What do you think? [writing himself a note—"one and one-third"—next to where Abdi had written "1/3."]

Abdi: I'm pretty sure that's it.

Mr. Lin: Thanks for sharing your thinking with me.

In this brief conversation about one problem, Mr. Lin learned about how Abdi thinks about

TABLE 1

Examples of equal-sharing problems give a broad range of children access to familiar sharing experiences and give their teachers starting points for instruction.

Grade 1	Grade 2	Grades 3–5
2 children are sharing 3 crackers. They are sharing so that each child gets the same amount. How much cracker will 1 child get?	3 children are sharing 10 bricks of clay. They are sharing so each child gets the same amount. How many bricks of clay will 1 child get?	6 children are sharing 8 small sandwiches. They are sharing so each child gets the same amount. How many sandwiches will 1 child get?
2 children are sharing 5 large cookies. They are sharing so each child gets the same amount. How much cookie will 1 child get?	3 children are sharing 8 muffins. They are sharing so each child gets the same amount. How many muffins will 1 child get?	6 children are sharing 10 burritos. They are sharing so each child gets the same amount. How much burrito will 1 child get?

Note: We often provide students with a model of the objects. In figure 1, for example (see p. 163), the problem included the eight square "sandwiches." As you can see, we have selected other contexts, such as familiar foods that are often shared. We select quantities so that the number of items (sandwiches) is greater than the number of people sharing so we can see if students distribute whole items before partitioning as well as how students deal with the remaining wholes. Finally, we select the number of people so that each person's share involves fractions other than halves (e.g., thirds, sixths, etc.); we avoid such numbers as five or seven sharers because partitioning shapes into these fractional pieces is quite difficult, even for adults.

partitioning, uses diagrams to represent equivalent parts, identifies and names fractional quantities, and uses symbolic notation for fractions. Imagine if Mr. Lin had the same information about all students in his class. What could he learn? Imagine further if his grade-level colleagues had the same information about their students. What could the team learn from examining their students' thinking and then planning together for upcoming instruction?

Data-driven decision making is common practice in schools. At Mr. Lin's school, we found that a single formative assessment problem offered insight into student thinking, helping us understand how students made sense of problems, what strategies they used, and what errors contributed to incorrect responses. It fueled teachers' discussions about fraction instruction and enabled them to track important changes in student understandings across the school as the staff engaged in collective work to improve learning and instruction.

The fraction problem described in this article was designed using research on children's thinking in elementary mathematics (Carpenter et al. 2014; Empson and Levi 2011), and it enables teachers to uncover students' understanding of—

1. partitioning strategies;
2. representations; and
3. language and notation

—all of which are essential aspects of understanding the meaning of fractions.

Using an equal-sharing problem

Involving “a total number of items to be distributed to a given number of groups, usually people” (Empson and Levi 2011, p. 8), equal-sharing problems allow children to draw on familiar sharing experiences. They also provide access for a broad range of children and enable us to understand their starting points for instruction (see **table 1**).

To learn about students' ideas, we ask them to explain their thinking aloud. Examining students' work without their explanations is

TABLE 2

We have used the questions below to help us understand students' thinking. We do not ask all of them of each student but instead select those that make sense in the moment.

Understanding students' thinking about fractions	
General questions	<p>Tell me about what you did.</p> <p>What do you think?</p> <p>I noticed that you revised. Tell me about that.</p> <p>What's happening in this story?</p> <p>What feels hard/tricky in this problem? (This question is helpful when students are stuck or frustrated. It is unnecessary that students complete the problem for a teacher to learn about their thinking.)</p>
Questions about sharing strategy	<p>How did you share the sandwiches?</p> <p>If I'm one of the kids, can you point to what I get to eat?</p> <p>Does everyone get the same amount? (How do you know?)</p>
Questions about drawing	<p>Are these pieces [<i>pointing to two or more pieces in the drawing</i>] the same size?</p> <p>How much is this? What do you call this? [<i>pointing to one of the pieces in the drawing</i>]</p> <p>How did you show that in your drawing?</p> <p>If I'm one of the kids, can you point to what I get to eat?</p>
Questions about language	<p>How much sandwich is that altogether?</p> <p>What do you call this [<i>pointing to a piece in the drawing</i>] amount?</p> <p>(When the student has already written a fraction)</p> <p>How do you say this?</p> <p>How many sandwiches will one child get?</p> <p>I heard you say “_____.” Where do you see that in your drawing?</p>
Questions about notation	<p>How would you write that?</p> <p>What does each of these numbers mean in your drawing? Or, what does this [<i>pointing to a portion of the written answer</i>] mean in your drawing?</p>

insufficient because their work may not indicate why they used particular strategies or notation. Therefore, after reading the problem aloud, we ask questions as each student works or finishes. The questions are intended to reveal students' understanding and use of strategies; it is important to take the stance of uncovering students' thinking and not to provide guidance or instruction. In the opening vignette, Mr. Lin

did not correct Abdi's written notation. One may wonder why he did not use this opportunity to teach the written notation. Mr. Lin's goal at that moment was to learn about Abdi's thinking—and he knew that the upcoming fractions unit would provide many opportunities to work on notation. Therefore, he made the decision to simply ask, "What do you think?" in response to Abdi's question, "Is that how you write it?"

TABLE 3

These children's strategies for solving equal-sharing problems are adapted from Empson and Levi 2011, p. 25.

Problem: 6 children are sharing 8 small sandwiches. They are sharing so that each child gets the same amount. How many sandwiches will 1 child get?

Strategy name	Strategy description
Nonanticipatory sharing	Child does not think in advance of both number of sharers and amount to be shared. For example, child splits each sandwich into halves, because halves are easy to make, and gives each person one-half. Child may or may not decide to split the last two sandwiches into sixths. Each person gets one-half sandwich and "two little pieces," if the last two sandwiches are split. Example: See Muna's work in figure 2a .
Additive coordination: sharing one item at a time	Child represents each sandwich, splitting the first sandwich into sixths because that is the number of sharers. Each person gets one-sixth piece. Child repeats the process until all eight sandwiches are shared. Each person gets eight-sixths sandwiches altogether. Example: See Sahra's work in figure 2b .
Additive coordination: sharing groups of items	Child represents each sandwich and realizes that six pieces can be created by splitting two sandwiches each into thirds. Each person gets one-third. Child moves on to another group of items and continues similarly until all the sandwiches are used up. Each person gets four-thirds sandwiches. OR Child represents each sandwich. Realizing that there are more sandwiches than people, child gives each person a whole sandwich. Child moves on to remaining two sandwiches and divides them into sixths or thirds. Each person gets one and two-sixths or one and one-third sandwiches. Example: See Abdi's work in figure 1 .
Ratio (repeated halving, factors)	Child may or may not represent all the sandwiches and people. Uses knowledge of repeated halving or multiplication factors to transform the problem into a simpler problem: three children sharing four sandwiches. Solves the simpler problem. Each child gets four-thirds sandwiches.
Multiplicative coordination	Child does not need to represent each sandwich. Child understands that a thing shared by b people is a/b , so eight sandwiches shared by six people means each person gets eight-sixths sandwiches.

to see if Abdi might revise his thinking on his own. Although Abdi did not do so, students often revise their thinking when explaining aloud because putting thoughts into words can push students to clarify their ideas (Chapin, O'Connor, and Anderson 2009). **Table 2** shows the types of questions we use to interview students.

Students' strategies for sharing sandwiches

The multiplicative relationship between the numerator and denominator is an important idea for students to understand (Empson and Levi 2011). Equal-sharing problems provide an opportunity to learn how students are relating two quantities: number of items and number of people sharing. Research has shown that children's sharing strategies develop along predictable trajectories (Empson and Levi 2011) (see **table 3**). When using early strategies, children typically do not coordinate the two quantities in advance. Instead, they typically draw, cut, fold, or split—often in halves—and then reflect on the results. For example, another fourth grader, Muna, split sandwiches in half and gave them out until two sandwiches were left. She “threw away” the remaining sandwiches because “there wasn't enough for the six people to each get another (whole) sandwich” (see **fig. 2a**). Empson and Levi (2011) call this strategy *non-anticipatory sharing*.

Once children understand the multiplicative relationship in an equal-sharing problem, their strategies involve coordinating the two quantities. This can happen in several ways. For example, Sahra partitioned each sandwich into sixths and distributed a sixth from each sandwich to each person so that each person got a total of eight-sixths sandwiches (see **fig. 2b**). This strategy is called *sharing one item at a time*.

Abdi used a sharing-groups-of-items strategy (see **fig. 1**); he first shared six whole sandwiches (by giving each person a whole sandwich), and then he shared the remaining two sandwiches (by giving each person one-third of a sandwich).

The most sophisticated strategies (called *ratio: repeated halving or factors* and *multiplicative coordination*) are more typical among middle school students and involve using ratios, knowledge of division, and the idea that fractions can be quotients.

FIGURE 1

Using a sharing-groups-of-items strategy, Abdi first gave each of six people one whole sandwich and then cut the remaining two sandwiches to give each person one-third of a sandwich.

Name Abdi

Problem: 6 children are sharing 8 small sandwiches. They are sharing so that each child gets the same amount.

How many sandwiches will 1 child get? $1\frac{1}{3}$

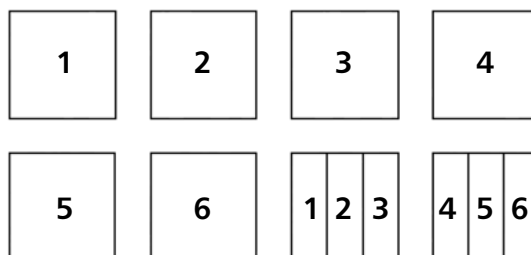


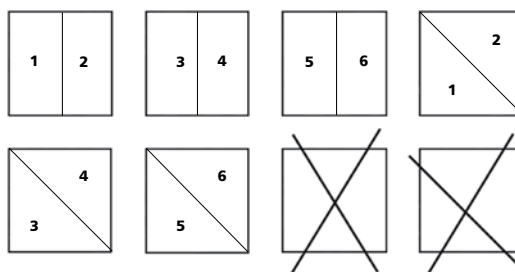
FIGURE 2

Researchers have found that children's sharing strategies develop along predictable trajectories.

(a) Muna's nonanticipatory-sharing strategy involves sharing only the first six sandwiches by partitioning them all into halves. She gives each child two halves but calls them two (whole) sandwiches.

Name Muna

How many sandwiches will 1 child get? 2

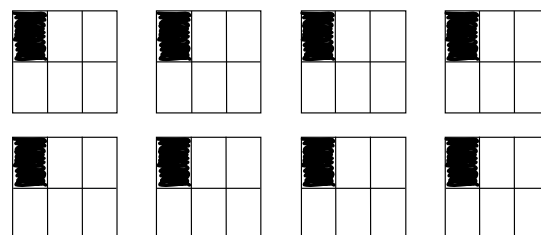


“There isn't enough for the six people to all get more.”

(b) Sahra's sharing-one-item-at-a-time strategy partitions each sandwich into sixths and distributes one-sixth from each sandwich to each person for a total of eight-sixths sandwiches.

Name Sahra

How many sandwiches will 1 child get? 8

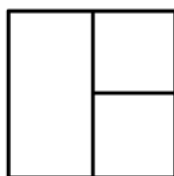
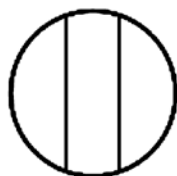
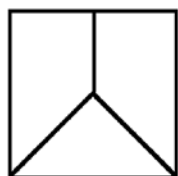


These are common ways that students attempt to draw thirds.

(a) Some students draw “thirds” inside the whole.



(b) Students’ inaccurate drawings of thirds are often due to underdeveloped spatial reasoning or fine motor skills.



Examining students’ representations

As children begin developing ideas about fractions, models that support dividing or partitioning into equal-size parts can help them learn how to create fractional quantities using part-whole relationships (Pothier and Sawada 1990). When we use this problem with students, we often include predrawn objects. For example, in **figure 1**, the problem included the eight square “sandwiches.”

In addition to revealing their sharing strategies, students’ drawings offer insight into their understanding of other fraction ideas. For instance, some students who are using a non-anticipatory-sharing strategy ignore the “whole” when they draw each portion separately (see **fig. 3a**).

Students’ drawings can also reveal what they understand about “equal parts.” In the opening vignette, Abdi split the last two sandwiches into thirds by drawing two vertical lines to create three equal-size pieces. Sometimes when students attempt to make thirds, they first make halves and then cut one of the halves in half again. Their drawings reveal a misunderstanding that “thirds” simply means three pieces, without regard for the size of those pieces (see **fig. 3b**). Other times, students explain that they need to partition an object into thirds (or sixths), but they are unable to make equal-size pieces with their drawings—often due to underdeveloped spatial reasoning or fine motor skills.

Figure 3b shows examples highlighting the importance of asking students about their drawings.

In **figure 2a**, Muna shows “half” of a sandwich in two different ways—by cutting vertically and by cutting diagonally, suggesting that Muna is beginning to understand that fractional pieces must be equivalent but do not have to be congruent—a key concept for understanding what fractions mean (Empson and Levi 2011).

Fraction language and notation

Initially, as fraction understanding develops in elementary grades, children often use informal language and notation. Empson and Levi (2011) explain that teachers should introduce the logic underlying fraction language and notation because it is not intuitive. Students have difficulties shifting focus from the language and written notation of whole numbers to the language and written notation of fractions (Hiebert 1988). Equal-sharing problems offer an opportunity to learn about students’ use of fraction terminology and notation.

After a student has had a chance to reason through the problem, we typically ask, “How many sandwiches will one child get?” and record the student’s response using words (e.g., one and two-sixths). By using words to record



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their responses, we do not assume that students also know the corresponding symbolic notation. We see this in Abdi's explanation (see the opening vignette) when he said that each child gets "one and one-third," even though he did not know how to write this with conventional symbolic notation.

We have found that verbal responses from children with little or no formal instruction about fraction terminology typically falls into one of three categories: expressed—

1. in terms of pieces;
2. in terms of whole numbers;
3. using inaccurate fraction language.

Sahra's answer was expressed in terms of pieces when she said that each child would get eight pieces (see **fig. 2b**). Muna, who said each child would get two sandwiches, shows what it sounds like to express an answer in terms of whole numbers (see **fig. 2a**). Finally, a common way that students use inaccurate fraction language is to overgeneralize the term *half*. For example, students create a drawing similar to Abdi's but instead refer to the third-size pieces as *halves*.

Because students have many informal experiences with fraction notation and because they



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Student drawings offer valuable insight into their understanding of mathematical ideas.

are sense makers, we saw a variety of written responses from students when we asked them to write an answer to *How many sandwiches will one child get?* We frequently saw whole-number responses from students who had few experiences with the symbolic form of fractions, although the unit attached to the whole number varied. For example, Muna's response of "two sandwiches" (see **fig. 2a**) is an example of using "whole sandwiches" as the unit. Alternatively, students used an informal unit that implied something smaller than a whole sandwich (such as "pieces" or "little sandwiches"), as shown in Sahra's response (see **fig. 2b**).

Learning about students' use of fraction terminology is important as it provides insight into whether students are using fraction terminology to describe parts of a whole as well as what terms



Formative assessment tasks offer insight into student thinking that helps teachers understand how children make sense of problems and strategize finding solutions.

TABLE 4

Mr. Lin, the teacher in the opening vignette, used a recording sheet to keep track of what he learned about his students' understanding of partitioning strategies, representations, language, and notation. The data collected helped inform upcoming instruction.

Student	Recording sheet					Drawing	Fraction language and notation	
	Nonanticipatory	One item at a time	Groups of items	Ratio (repeated halving, factors)	Multiplicative coordination		How does the student describe the quantity (as wholes, pieces, halves, a fraction, a mixed number)?	Symbolic notation
Abdi			X			Thirds are the same size.	"a whole and one-third"	$1\frac{1}{3}$
Muna	X					Has two ways of showing halves	wholes	2
Sahra		X				Cut all into 2×3 arrays	pieces	8
Imran			X			Cut six in half; two in sixths (3×2 arrays)	"one and two-sixths"	$1\frac{2}{6}$
Josiah			X			Cut six in half; cut two into "three unequal pieces" (one-half and two-fourths)	"one-half and three-thirds"	$1\frac{1}{2}$ and $\frac{3}{3}$
Munira			X			Gave out wholes; initially cut last two into fourths, revised to thirds	"one and one-third"	$1\frac{1}{3}$
Mylla			X			Gave out wholes; two in sixths (1×6 arrays)	"one and two-sixths"	$1\frac{2}{6}$
Alberto			X			Gave out wholes; initially cut last two into "peace sign" thirds by making two vertical cuts instead	"one whole and one-third"	$1\frac{1}{3}$
Victor	X					Gave out four wholes; cut remaining four in half	"one whole and one-half"	$1\frac{1}{2}$
Kiya		X				Cut all into 1×6 arrays	"eight sixes"	$\frac{8}{6}$
Miski			X			Cut all into thirds; lots of erasing and redrawing as he "tried to make them the same size"	"Four-thirds; that's the same as one and one-third."	$\frac{4}{3} = 1\frac{1}{3}$
Jonah	X					Cut all in half; gave some people two halves and some people three halves	"little sandwiches"	"two or three little sandwiches"

students may already know (such as *half*) and how they are using the terms. By asking students to record their response using symbolic notation, we uncover students' ideas about writing fractions and the conventions involved. We also see how students coordinate fraction language and symbols. As teachers plan for instruction, this information helps guide decisions about how and when to introduce and support students in learning to use fraction language and notation.

Using student data to inform instruction

Let's return to Mr. Lin's use of the formative assessment problem to learn about his students' understanding of the meaning of fractions. He gave the problem to the whole class at the same time, asking students to work individually. Similar to his conversation with Abdi, Mr. Lin asked each student how they solved the problem and used a recording sheet (see **table 4**) to efficiently keep track of students' ideas across the entire class. He then used this information to make instructional decisions about the upcoming fraction unit.

When looking at his class data (see **table 4**), Mr. Lin noticed that most students were using a groups-of-items sharing strategy. He also saw that roughly half his students were able to use accurate fraction language and notation. On the basis of these data, Mr. Lin planned a series of minilessons designed to support students who were using nonanticipatory and one-item-at-a-time sharing strategies. He also used these lessons to support the development of students' language and notation.

For example, Mr. Lin's first fractions lesson included an "open strategy share" (Kazemi and Hintz 2014) where the goal "is to bring out a range of possible ways to solve the same problem and build students' repertoire of strategies" (p. 18). Using students' work from the formative assessment, Mr. Lin invited Kiya and Mylla to share their strategies with the whole class. He facilitated the strategy share by asking *how* and *why* questions, such as "How did you think about sharing the sandwiches?" and "Why did you cut only some of the sandwiches in thirds?"

In the next lesson, Mr. Lin posed similar equal-sharing problems and invited students to share their strategies based on the fraction ideas



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that could emerge through classroom discussion. Through the use of talk moves (Chapin, O'Connor, and Anderson 2009) and sequencing (of the strategies) (Smith et al. 2009), Mr. Lin oriented students to particular sharing strategies, representations, and fraction language and notation. Based on the instructional recommendations of Empson and Levi (2011), Mr. Lin first focused on developing fraction language. He recorded students' verbal responses in word form (such as "eight-sixths" or "one whole and two-sixths"). Once fraction language was well established, he began to emphasize fraction notation during the strategy sharing.

In subsequent lessons, Mr. Lin used students' thinking about equal-sharing problems to introduce new fraction concepts. For example, when introducing the idea of equivalent fractions, Mr. Lin selected two strategies that resulted in equivalent, but different, responses. If Mr. Lin were using the student work from the recording sheet in **table 4**, he might use Imran's strategy (which resulted in $1 \frac{2}{6}$ sandwiches for each person) and Alberto's strategy (which resulted in $1 \frac{1}{3}$ sandwiches for each person). After both Imran and Alberto shared, Mr. Lin would ask the entire class to consider if $1 \frac{1}{3}$ is equal to $1 \frac{2}{6}$ and how they might defend their response. Empson and Levi (2011) offer more suggestions for how teachers might use students' current understandings and strategies to make instructional decisions.

Worth the time

Eliciting, responding to, and advancing children's mathematical thinking lie at the heart of great teaching. The time spent giving this

At the heart of all great teaching are eliciting, responding to, and advancing children's mathematical thinking.

Understanding the meaning of fractions

These are the NCTM Standards as well as standards from the Common Core that focus on teaching students the meaning of fractions.

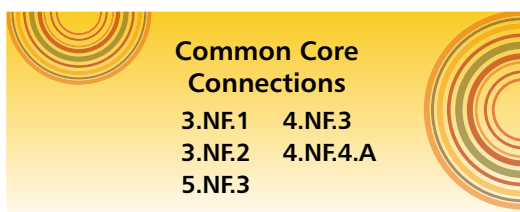
(a) From *Principles and Standards for School Mathematics* (NCTM 2000, p. 148):

In grades 3–5, all students should develop understanding of fractions as parts of whole units, as parts of a collection, as locations on number lines, and as divisions of whole numbers.

(b) From the *Common Core State Standards for Mathematics* (CCSSM) (CCSSI 2010):

- 3.NF.1: Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.
- 4.NF.3: Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.
- 3.NF.2: Understand a fraction as a number on the number line. . . .
- 4.NF.4A: Understand a fraction a/b as a multiple of $1/b$.
- 5.NF.3: Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). . . .

formative assessment problem to students and talking with them about their thinking is worth it, as it provides valuable information about student understanding that helps teachers like Mr. Lin plan and make decisions about fractions instruction.



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