Unpacking the Division Int

JUANMONINO/THINKSTOCK

Chew on why $13/7 = 13 \div 7$ as we explore two classroom lessons that develop conceptual understanding by building on children's knowledge of whole-number division.

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Between the Wholes: Promoting Fraction and Decimal Understanding

erpretation of a Fraction

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One of the challenges in learning fractions is understanding how and why a fraction can have multiple interpretations. As presented in one textbook, a fraction is "a symbol, such as 2/3, 5/1, or 8/5, used to name a part of a whole, a part of a set, a location on a number line, or a division of whole numbers" (Charles et al. 2012, p. 475). How can a fraction take so many forms? In particular, why is a fraction also a division of whole numbers (e.g., $13/7 = 13 \div 7$)?

We will present examples of classroom lessons that support children in developing conceptual understanding of the division interpretation of a fraction (i.e., $m/n = m \div n$ for whole numbers m and nonzero n) by building on children's knowledge of whole-number division. Children demonstrate conceptual understanding by (1) using the partitive interpretation of division to construct a definition for the division of any two whole numbers and (2) using established definitions and observations to show why the fraction m/n equals the division $m \div n$.


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Drawing on our experiences in classroom instruction and education research, we begin by offering a rationale for examining the division interpretation of a fraction and then describe two approaches to teaching this topic—one lesson from fourth grade, and the other from sixth grade. Next, we discuss the ways in which these two classroom examples exemplify *Principles and Standards for School Mathematics* (NCTM 2000) and integrate the Standards for Mathematical Practice and mathematical content from the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010). We conclude by considering how the instructional approaches outlined in this article can be extended to support children in developing conceptual understanding of other mathematical topics.

Why the division interpretation of a fraction matters

Behr and his colleagues (1992) proposed that the multiple interpretations of a fraction may be one reason why understanding fractions is a "formidable learning task." The researchers identified at least six ways that fractions can be interpreted:

1. A part-to-whole comparison
2. A decimal
3. A ratio
4. An indicated division (quotient)
5. An operator
6. A measure of continuous or discrete quantities (pp. 92–93)

One of the more challenging interpretations for children to learn is the division interpretation of a fraction. In the Secondary Mathematics Project (Kerslake 1986), twelve- and thirteen-year-old students received the following problem:

> Three bars of chocolate are to be shared equally between five children. How much should each child get?

Only 66 percent of twelve-year-olds and 63 percent of thirteen-year-olds in the study were able to identify 3/5 as the answer. Moreover, when presented with the computation 3 ÷ 5 without any context, even fewer students were able to identify the quotient 3 ÷ 5 as the number 3/5. These results suggest that the division interpretation of a fraction is a concept that needs to be learned with conceptual understanding.

We propose that the division interpretation of a fraction is a critical component of the elementary and middle school mathematics curriculum for the following reasons.

It extends children's knowledge of whole-number division

In the context of whole numbers (i.e., whole-number quotients), children learn two interpretations of whole-number division (CCSSI 2010; NCTM 2006): the partitive interpretation (e.g., 48 ÷ 6 is the number of objects in each share when forty-eight objects are partitioned equally into six shares) and the measurement interpretation (e.g., 48 ÷ 6 is the number of shares when forty-eight objects are partitioned into equal shares of six objects each). The division interpretation of a fraction is a direct extension of the partitive interpretation of division (e.g., 48 ÷ 7 is the fraction of objects in each share when forty-eight objects are partitioned equally into seven shares).

It provides an immediate quotient to the division of any two whole numbers

Division in the context of whole numbers requires recall of multiplication facts to produce a quotient and restricts division to cases where the dividend is a multiple of the divisor. In the context of fractions, however, the division interpretation of a fraction produces an

immediate quotient to the division of any two whole numbers.

It justifies the "retirement" of the division sign

The division sign is rarely used beyond middle school mathematics because the division interpretation of a fraction extends to rational numbers and real numbers (e.g., $-6\pi \div 2 = -6\pi/2$). Therefore, in the study of algebra and more-advanced mathematics, the fraction notation replaces the division sign, and the symbol m/n is interpreted as both the number m/n as well as the division of m by n.

Given the importance of the division interpretation of a fraction, how does a teacher help children develop a conceptual understanding of the topic? In the following sections, we provide two classroom examples.

A fourth-grade classroom lesson

After defining a fraction on the number line and connecting the definition of a fraction to counting "multiples" of the fraction $1/n$ (e.g., counting one copy of 1/5 is 1/5, counting two copies of 1/5 is 2/5, etc.) in previous lessons, the fourth-grade class revisits the following problem:

> Five people share ten granola bars. How many granola bars does each person get when they share equally?

The children recall the solution, $10 \div 5 = 2$, which represents the partitive interpretation of division. The teacher then presents the following problem:

> Five people share three granola bars. How much of a granola bar does each person get when they share equally?

The teacher begins the discussion by asking the class,

> If $10 \div 5$ was how much each person got when five people shared ten granola bars, how much would each person get when five people share three granola bars?

The children propose the answer $3 \div 5$ but immediately recognize that the division has no whole-number solution. As one child points out, "Only three people would get bars—one bar a person; the other two would get nothing."

Another child observes, "We need to break the bars into smaller parts, into fractions, so that everyone can get some."

In response, the teacher asks, "What is the fraction, the number that equals $3 \div 5$? How can we find it?"

The children begin the solution process by first considering how much each person would get if five people were sharing only one granola bar. Students draw one granola bar and partition it into five congruent sections, labeling each section 1/5 (see **fig. 1**). They conclude that, in the case of one granola bar, one person would get the first 1/5 of the granola bar, the second person would get the second 1/5 of the granola bar, and so forth.

The children replicate and extend the single granola bar model to the case of three granola bars (see **fig. 2**). The children then translate the

FIGURE 1

Drawing one granola bar and partitioning it into five congruent sections, each labeled 1/5, students concluded that if five people shared a granola bar, the first person would get the first 1/5 of the granola bar, the second person would get the second 1/5, and so on.

FIGURE 2

Students replicated and extended the single granola bar model to the case of three granola bars.

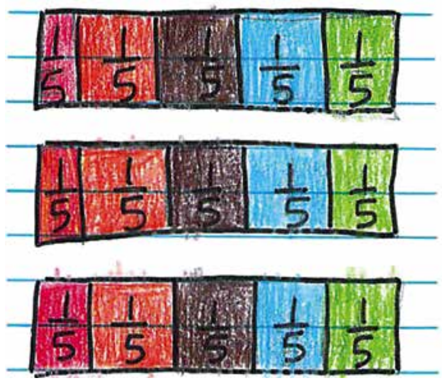


FIGURE 3

Students translated the pictorial representation to a number line and counted three copies of 1/5 for each person, concluding that the answer is 3/5 granola bar.

FIGURE 4

Using a set of animated slides, the class determined that separate copies of 1/5 could be linked to display the segment [0, 3] as five groups of congruent segments, each of length 3/5. Five groups of 3/5 on the number line represent five people, each given 3/5 of a granola bar, totaling three granola bars.

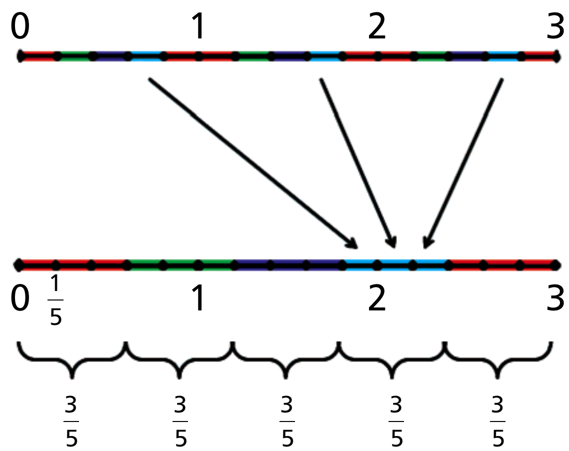
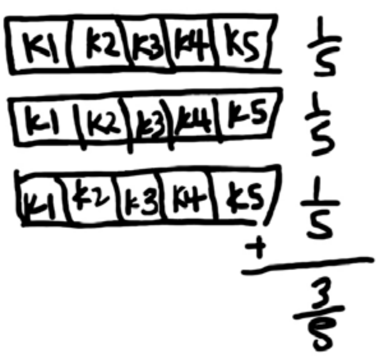


FIGURE 5

For the problem of five people sharing three sandwiches, the sixth-grade class used fraction addition to conclude that each person would get 3/5 of a sandwich.



pictorial representation to the number line (see **fig. 3**) and count three copies of 1/5 for each person to conclude the answer is 3/5 of a granola bar. The teacher supplements the children's explanation by using a set of animated slides (see **fig. 4**) to show that the separate copies of 1/5 can be concatenated to display the segment [0, 3] as five groups of congruent segments, each of length 3/5. In other words, five groups of 3/5 on the number line represent five people each given 3/5 of a granola bar, and the resulting total is three granola bars. On the basis of these observations, the teacher and children deduce that the division 3 ÷ 5 equals the fraction 3/5. After working through several more examples (e.g., 7 ÷ 4), the class concludes $m \div n = m/n$ for any whole number m and nonzero n.

A sixth-grade classroom lesson

After the children learn about equivalent fractions, adding fractions, and multiplying fractions in prior lessons, the sixth-grade teacher asks them to solve the following problem:

> On a trip, five children brought nothing to eat. Ms. Violet had only three sub sandwiches. If the five students shared the three sandwiches equally, how much did each one get?

Students begin by drawing three sandwiches. Recognizing that there are not enough sandwiches to give one to each person, the children decide to partition each sandwich into five congruent parts and assign the first part of each sandwich to "K1," the second part of each sandwich to "K2," and so forth. The children explain that because each sandwich is cut into five congruent parts, each part represents 1/5 of a sandwich. Using fraction addition, students conclude that each person gets 3/5 of a sandwich (see **fig. 5**).

Building on the children's solution, the teacher prompts them to draw on their knowledge of whole-number division to construct the division interpretation of a fraction. "We have found the answer is 3/5, but let's take a step back and think about what the problem is saying. What does it mean for five students to share three sandwiches equally?"

One child responds, "It means we are taking three sandwiches and want to make five equal groups."

"Where have we seen that before—taking a number of things and making equal groups?" the teacher asks.

The children immediately recall, "Division!" With this basic notion of division, the children formulate the division statement 3 ÷ 5 = 3/5.

When the teacher asks the children to connect the division statement 3 ÷ 5 = 3/5 to what they have previously learned about division and fractions, the children make three key observations. First, they observe that two whole numbers can be divided even when the quotient is not a whole number. Previously, in the context of whole numbers, the children had learned to define division only when the quotient is a whole number. Now in the context of fractions, the children observe that they are able to define 3 ÷ 5 and find the number that it equals, namely 3/5.

Building on this extended definition of whole-number division, students also observe that the division statement 3 ÷ 5 = 3/5 should correspond to one true multiplication statement. During the unit on whole-number division, they had learned that "division with whole numbers is another way of expressing multiplication" and that "every true division statement leads to one true multiplication statement." By extending this definition to the context of fractions, the children observe that 3 ÷ 5 = 3/5 means 3/5 × 5 must equal 3, which the children confirm by using fraction multiplication (see **fig. 6**).

Third, the children observe that a fraction now has a new representation: It can be expressed as division. Up to this point, students have learned to represent the fraction 3/5 as a point on the number line, a decimal, an equivalent fraction, a sum of fractions, and a product of fractions. Now they can represent the fraction 3/5 as the division 3 ÷ 5. The question is why?

"If we look at the number line," the teacher asks, "Why is the fraction 3/5 the same number as the division 3 ÷ 5?"

The children respond by first considering how the segment [0, 3] can be partitioned into five equal groups. They observe that the number 3 naturally partitions into three equal groups—segments [0, 1], [1, 2], and [2, 3]—but not five groups. If, however, the segments [0, 1], [1, 2], and [2, 3] are each partitioned into fifths, the children observe that, by equivalent fractions, the segment [0, 3] becomes fifteen copies of 1/5 (see **fig. 7**). Because fifteen can be divided into five groups, each with three (units), the children reason that fifteen copies of 1/5 can be partitioned into five groups, each with three copies of 1/5. Therefore, by writing the number 3 as the equivalent fraction 15/5, the segment [0, 3] can be thought of as fifteen copies of 1/5 that can be grouped into five groups of 3/5. By this reasoning, the children observe that 3 ÷ 5 = 3/5.

FIGURE 6

Sixth graders verified the corresponding multiplication statement for the division statement 3 ÷ 5 = 3/5.

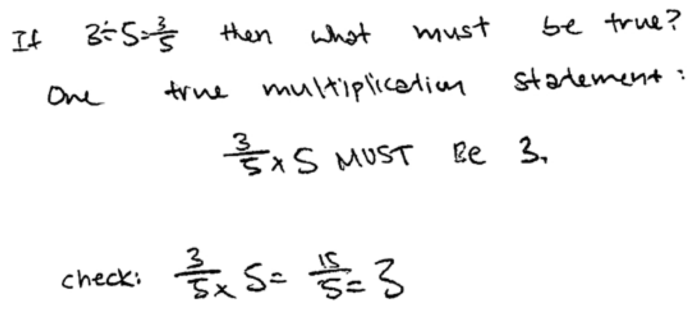
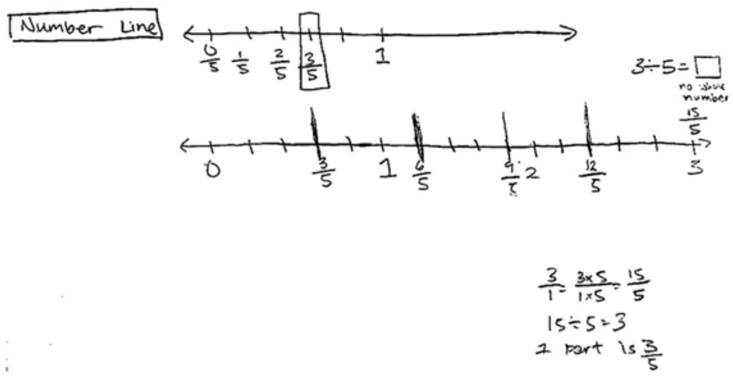


FIGURE 7

Students' number line represents their reasoning that 3 ÷ 5 = 3/5.



Analyses of the classroom lessons

Although the two lessons described above present two different approaches to teaching the division interpretation of a fraction and were taught prior to the implementation of CCSSM, both exemplify *Principles and Standards for School Mathematics* (NCTM 2000) and integrate the Common Core's Standards for Mathematical Practice and standards of mathematical content

Students build conceptual understanding by analyzing why the division interpretation of a fraction is true through multiple representations.

(CCSSI 2010) to support the development of conceptual understanding.

NCTM's Learning Principle and Reasoning and Proof Standard

The NCTM Learning Principle emphasizes learning mathematics "with understanding, actively building new knowledge from experience and prior knowledge" (2000, p. 20). Both classroom lessons apply this principle by actively engaging students in extending their prior knowledge of whole-number division to the division interpretation of a fraction. The understanding is developed by involving the children in mathematical reasoning and constructing arguments for specific cases that can be generalized, as highlighted in NCTM's Reasoning and Proof Standard (2000, p. 189). By analyzing why the division interpretation of a fraction is true through multiple representations, students build conceptual understanding of the topic rather than memorize the fact without understanding.

CCSSM's precision practice and number and operations content standards

Besides helping the children "make sense of problems"; "reason abstractly and quantitatively"; and "use stated assumptions, definitions, and previously established results in constructing arguments" (CCSSI 2010, p. 6), both teachers help their students "attend to precision" (p. 7) by (1) providing a precise definition of division of whole numbers and (2) giving precise mathematical meaning to the process of sharing three items equally among five people (3 ÷ 5), which provides clarity for identifying the unit as one item rather than as a set of three items.

Students engage in these Standards forMathematical Practice (SMP) as they "solve word problems involving division of whole numbers leading to answers in the form of fractions" (CCSSI 2010, p. 36). Moreover, the children doing the sixth-grade lesson are able to use their knowledge of fraction multiplication to "interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3," as recommended in CCSSM grade 5 standard 5.NF.B.3 (p. 36). Children in earlier grades can use their knowledge of counting "multiples" of the fraction $1/n$ to conclude that "when 3 wholes are shared equally among 4 people, each person has a share of size 3/4" because counting 3 copies of 1/4 is 3/4. Therefore, the two classroom examples presented in this article provide two ways for children to access the reasoning behind the division interpretation of a fraction. Each approach helps children develop conceptual understanding by building on their prior knowledge of fractions.

Beyond division interpretation of a fraction

In this article, we have unpacked the importance of the division interpretation of a fraction and explored two classroom lessons that support children in developing conceptual understanding by extending their knowledge of division from the context of whole numbers. Exploratory evidence indicates that mathematics instruction of this form can have a positive impact on student learning (Poon 2014). Through the sample lessons presented in this article, we hope readers will gain a better understanding of how to develop lessons and approaches that help children learn the division interpretation, as well as other fraction and decimal topics, with conceptual understanding.

Common Core Connections
5.NF.B.3
SMP 3
SMP 6

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