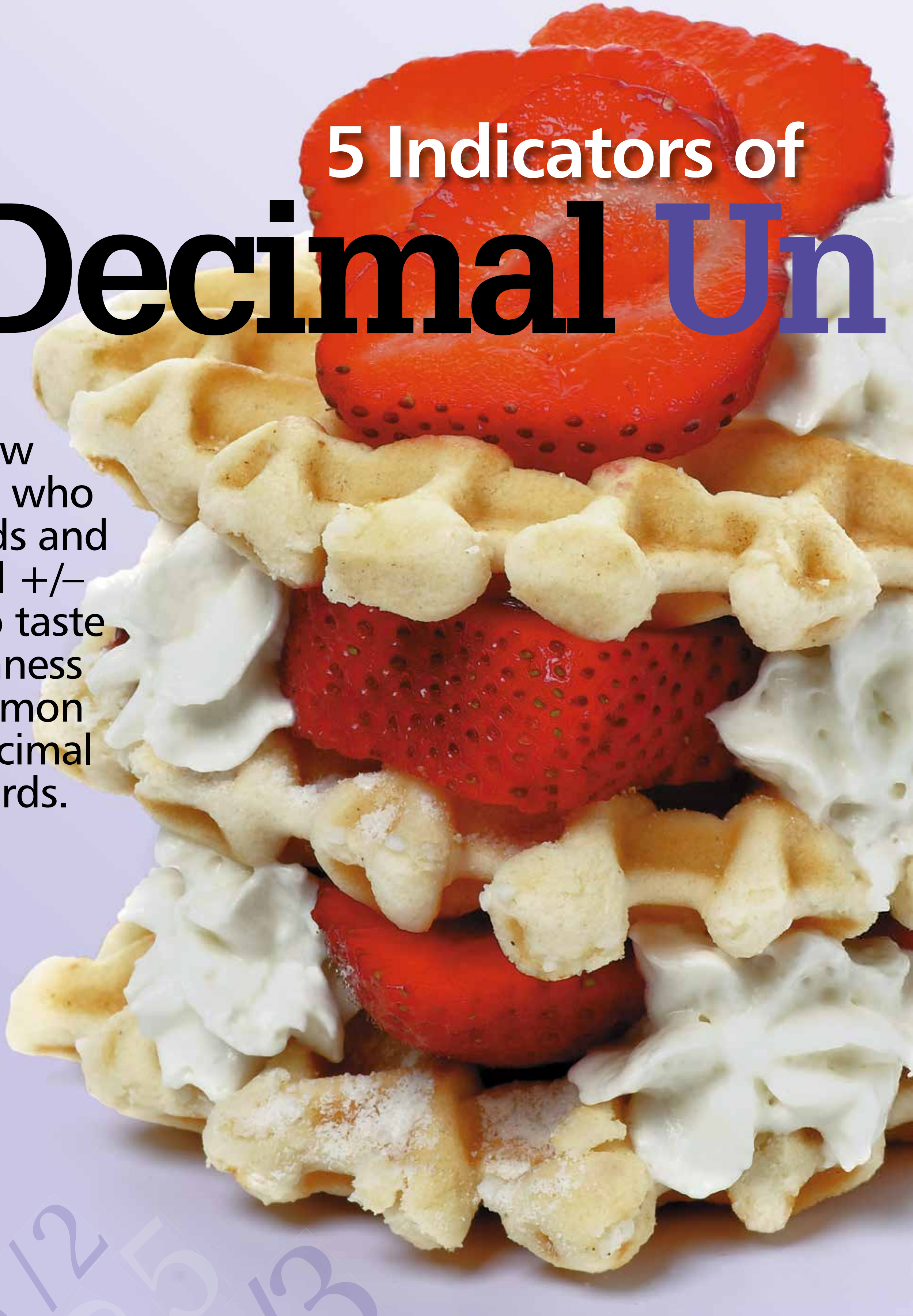
5 Indicators of

Decimal Un



Follow children who used grids and decimal +/– charts to taste the richness of Common Core decimal standards.

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Between the Wholes:
Promoting Fraction and Decimal Understanding

derstandings

ZHEKOS/THINKSTOCK

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Violet is a fourth-grade student in an urban district in the upper Midwest who, when asked to name and compare 0.38 to 0.4, responded in a way that is familiar to many fourth-grade teachers:

Violet: Zero point four.

Interviewer: OK, and how about this one?

Violet: Zero point thirty-eight.

Interviewer: And, which one is larger, or are they equal?

Violet: Thirty-eight.

Interviewer: Thirty-eight is larger, and why is that?

Violet: Because thirty-eight is bigger than four.

Too often students like Violet use unreliable procedures for comparing and operating with decimals on the basis of their misunderstanding of decimals as numbers distinct from whole numbers (Hiebert and Wearne 1986; Roche and Clarke 2004). How do you move students from this type of thinking to a rich conceptual understanding of decimals? What might a rich conceptual understanding look like?

The authors of this article collaborated with fourth-grade teachers from two schools to support implementation of a research-based fraction and decimal curriculum (Rational Number Project: Fraction Operations and Initial Decimal Ideas). Through this study, we identified five indicators of rich conceptual understanding of decimals, which we describe. Next we show how these indicators comprise a framework for monitoring student learning and guiding instruction.

The lessons

The content for the curriculum addressed the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) for decimal learning (see **table 1**). The lessons reflect our understanding of the common errors that students face when first encountering decimals (Hiebert and Wearne 1986; Grossman 1983; Yujng and Zhou 2005; Cramer et al. 2009). This research on decimal learning suggests that students have difficulty connecting decimal symbols with pictorial representations. Often students will apply whole-number ideas to decimals, disregard the placement of the decimal point, and invent inconsistent algorithms based on their interpretation of the situations.

TABLE 1

Research on decimal learning suggests that students have difficulty connecting decimal symbols with pictorial representations. The lessons addressed these Common Core State Standards for Mathematics.

Common Core State Standards

Grade 4	Grade 5
Express a fraction with a denominator of 10 as an equivalent fraction with a denominator of 100. Use this technique to add two fractions with respective denominators of 10 and 100. (4.NF.C.5)	Read, write, and compare decimals to the thousandths place. (5.NBT.A.3.A) Compare two decimals to the thousandths place based on meanings of the digits in each place. (5.NBT.A.3.B)
Use decimal notation for fractions with denominators of 10 and 100. (4.NF.C.6)	Add and subtract decimals to the hundredths place, using concrete models or drawings and strategies based on place value. Relate the strategy to a written method and explain the reasoning used. (5.NBT.B.7)
Compare two decimals in hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record results of comparisons with symbols >, <, =, and justify conclusions by using a visual fraction model. (4.NF.C.7)	

Helping students develop sound conceptual knowledge of decimals is a complicated mission. According to *Principles to Actions: Ensuring Mathematical Success for All*, one aspect of effective teaching involves "engaging students in making connections among different mathematical representations to deepen understanding of mathematical concepts and procedures and as tools for problem solving" (NCTM 2014, p. 24). The decimal lessons used in this study reflect this well-researched premise that building connections within and between multiple representations is key to addressing common misunderstandings (NRC 2001; Cramer 2003).

Students in this study began exploring decimal ideas by using pictorial models and connecting those representations to written and verbal symbols, then progressing to reasoning strategies that relied on mental images of the models. An image of a 10 × 10 decimal grid was used as a visual model to help illuminate the underlying structure of the decimal number system for students. Students' understanding of the mathematical structure of the decimal number system was reinforced as they made translations to other representations (see **fig. 1**). Students also used the visuals to explore order and equivalence ideas.

Using a 10 × 10 grid

Students began to develop a conceptual understanding of addition and subtraction procedures with a decimal +/– board (see **fig. 2**), an adaptation of the single 10 × 10 grid to one that allows students to show the decimals being added (or subtracted) separately on the 10 × 10 grids at the top and the solution on the bottom grid (Cramer et al. 2009).

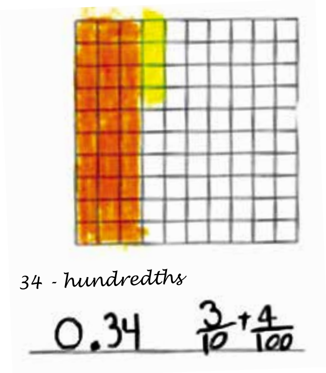
Identifying the five indicators of emerging decimal understanding

Student classwork and postinstructional interviews conducted with sixteen students ranging in ability were examined in an effort to create an understanding of what a rich conceptual understanding of decimal ideas looks like. The decimal ordering tasks were used with permission from the *Eliciting Mathematics Misconceptions Project* that was funded by the

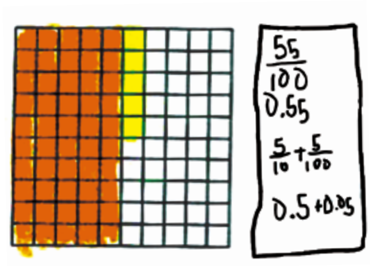
FIGURE 1

Sample activities with the decimal models show how students build meaning for decimals as numbers.

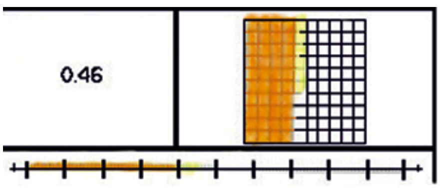
(a) Students first used fraction language and symbols to name pictorial representations for decimals.



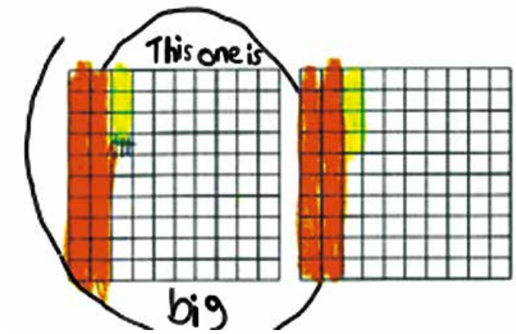
(b) Then they translated to decimal language and symbols (e.g., naming an image of 0.55 with the words *five-tenths* and *five-hundredths*, with the written symbols 50/100 or 5/10 + 5/100).



(c) From a decimal drawn on a 10 × 10 grid, students translated to a number line.



(d) A hundreds-grid representation of 0.23 and 0.24 was used to compare magnitude.



US Department of Education Institute of Education Sciences (award no. R305A110306-12). As expected, many student responses closely mirrored the instructional methods that leaned heavily on multiple representations using language as a bridge between them.

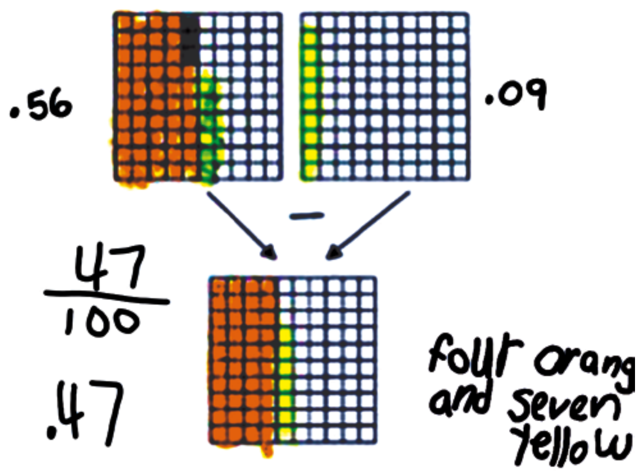
The indicators

The team identified indicators common among students who demonstrated understanding of decimal ideas:

1. Using precise mathematical language when working with decimals
2. Accurately using models to represent decimals
3. Decomposing and composing decimals based on mental images of the models and/or place-value understandings to order decimals

FIGURE 2

This example of the decimal +/– board shows a subtraction solution.



4. Using an understanding of the relative size of decimals to guide their estimation for operations with decimals
5. Using a model and their ability to compose and decompose decimals to interpret addition and subtraction operations and build meaning for work with symbols.

Examples of students who showed understanding

Like other students who showed a good understanding of initial decimal ideas, Nadia was able to read decimals using *accurate decimal language* (indicator 1) and was comfortable *representing* decimals on the 10 × 10 grid (indicator 2). She was able to *decompose* decimals and *use mental images* of the 10 × 10 grid model to guide her thinking on order tasks (indicator 3). When asked to order 0.4 and 0.38, she responded as follows:

> The one that is bigger is this one [*pointing to* 0.4]; four of the bars are filled in, and this one is only three bars with eight yellows. It is only two away from being the same as this one.

Notice how Nadia relied on color in her explanation. (On the grids, students used orange crayons to color tenths and yellow to color hundredths).

In this next example, Nadia names each decimal and mentally breaks the decimal 0.175 into its place-value components to prove that 0.2 is greater. Nadia's class coined the phrase *itty-bitty yellows* to represent thousandths.

Although examining thinking among successful students developed the key indicators, we gained further insight into what a rich, conceptual understanding of decimals might look like by examining misunderstandings.

> One hundred seventy-five thousandths; two-tenths. Two-tenths is bigger because it, like, it has two-tenths, and this one has one-tenth and seven yellows, and the little yellows, itty-bitty yellows.

When asked to order four decimals (0.245, 0.025, 0.249, 0.3), Nadia handled the complexity by tracking the size of each decimal as she built mental representations while decomposing each number into its place-value components:

Nadia: This one [*pointing to* 0.025] is probably smallest because it is only two yellows and five itty-bitty yellows. Because that one [*pointing to* 0.245] has two bars filled in, four yellows and five itty-bitty yellows; and this one [*pointing to* 0.049] has two bars filled in, four little yellows and then nine itty-bitty yellows.

Interviewer: What about this last one [*pointing to* 0.3]?

Nadia: This is three whole bars; orange.

Nadia's reasoning at this point is closely tied to the mental images of the decimal 10 × 10 grid and the colors used to represent decimals. In contrast, other successful students transitioned from the model used in class to ordering strategies that involved *decomposing decimals into their place-value* components with less dependence on the mental images of the grid (indicator 3).

When Alex compared 0.4 and 0.38, he decomposed the numbers without referring to the class model and was able to order them based on his understanding of place value:

Alex: This one [*pointing to* 0.04], zero and four-tenths, is larger.

Interviewer: And how do you know that?

Alex: Because this one [*pointing to* 0.038] is only zero and three-tenths, and then eight-hundredths still doesn't make it bigger.

Alex also used this method when ordering four numbers (0.245, 0.249, 0.3, and 0.025) but also recombined (*composed*) the components to name the numbers in thousandths:

> This one [*pointing to* 0.025] is the smallest because it doesn't have any tenths, and

then this one [*pointing to* 0.3] is the biggest because it has three-tenths; and both of these have two-tenths. Then two and forty-nine–hundredths is bigger—wait—is bigger than two and forty-five–hundredths. I mean zero and 249 thousandths is bigger than zero and 245 thousandths, I meant to say.

Note that neither Nadia nor Alex used a rule that involved adding a zero to the decimals so they could compare the decimals as whole numbers; they were able to reason through this comparison by decomposing the decimals to compare their values. Nadia relied more heavily on her mental images of the model; Alex moved beyond the mental images and used place-value language.

Students who ordered decimals conceptually, as described above, and had experience with the decimal +/– board to model fraction addition and subtraction problems could make reasonable estimates for addition and subtraction tasks (indicator 4). Estimation was guided by mental representations of the visuals used in class to model decimals. Kayla provided an example of how students were able to do this when she was asked to—

> picture fifty-seven–hundredths on the decimal +/– board. If you took away nine-thousandths, would the amount left shaded be more than one-half or less than one-half? Explain without finding the exact answer.

Kayla responded,

> More than one-half, because these nine-thousandths would only take up one of the hundredths because you just split one of those into tenths, and it would be nine; and so it would be five-hun—yeah, 561 thousandths.

When given the same estimation task, Alex estimated by relying on decomposing decimals, his mental images of the 10 × 10 grid, and then reconfiguring (composing) the decimal parts into an estimate of the final amount:

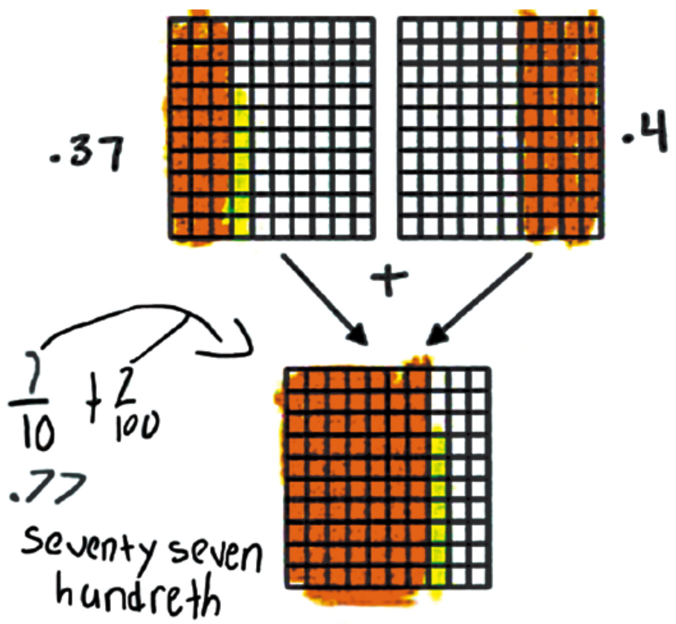
> Bigger, because nine-thousandths is, like, like nine itty-bitty yellows, and that's only taking away like one (hundredth) away from that (57 hundredths). Probably not even one, but then you would, it would still be fifty-six–hundredths and something.

According to indicator 5, students who have a good understanding of initial decimal ideas should be able to interpret addition and subtraction with a model, compose and decompose decimals, and use a model to build meaning for work with symbols. Kayla's classwork (see **fig. 3**) provided an example of how the decimal +/– board can be used to conceptually add two decimals (0.37 and 0.4). She shaded thirty-seven–hundredths on the left 10 × 10 grid as three orange bars (0.3) and seven yellows (0.07). She modeled four-tenths on the right grid as four orange bars. She then combined the two amounts on the bottom grid, combining three orange bars and four orange bars and added on the seven yellow squares to complete the sum. Notice how she was able to record her work using symbols. She saw the final sum of seven-tenths and seven-hundredths as fractions and translated that amount to decimal symbols and words.

Although examining thinking among successful students developed the key indicators, we gained further insight into what a rich,

FIGURE 3

Kayla solved 0.37 + 0.4 on the decimal +/– board.



→ reflect and discuss

5 Indicators of Decimal Understandings

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether it works. By collecting information about what goes on in our classrooms, and then analyzing and evaluating this information, we identify and explore our own practices and underlying beliefs. The following questions related to "5 Indicators of Decimal Understandings" by Kathleen Cramer, Debra Monson, Sue Ahrendt, Karen Colum, Bethann Wiley, and Terry Wyberg are suggested prompts to aid you in reflecting on the article and on how the authors' idea might benefit your own classroom practice. You are encouraged to reflect on the article independently as well as discuss it with your colleagues.

- How might you translate the five indicators of decimal understanding into formative assessments to use while you are teaching your decimal unit?
- According to *NCTM's Principles to Actions: Ensuring Mathematical Success for All*, "Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (2014, p. 10).
    - ♦ To help students see the similarities and differences in the solutions, how would you connect Nadia and Alex's strategies for ordering 0.38 and 0.4?
    - ♦ What questions might you ask students so they see how both children composed and decomposed the decimals to order them?
- The last section shares some observations on students' comparative use of the 10 × 10 grid and the number line as models for decimals and assertions about how to best to use these two models.
    - ♦ What is your experience with the number line model for fractions and decimals as compared with other models?
    - ♦ Have you noticed any particular challenges that students have with the number line model as compared with other models?

We invite you to tell us how you used Reflect and Discuss as part of your professional development. The TCM *Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be emailed to* Teaching Children Mathematics *at* ***tcm@nctm.org***. *Please include Readers Exchange in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgment. Letters are also edited for style and content.*

conceptual understanding of decimals might look like by examining misunderstandings.

Examples of student misunderstandings

As many teachers know, not all students will construct the same decimal understandings as depicted by the students above. Differences in time on task, prior knowledge, and language issues all influence students' access to new ideas. The list of indicators suggested here provides a framework to assess students' understanding. Because the indicators align closely with the way decimals are taught, instruction can be matched once misunderstandings are identified.

Struggles with language

When examining the interviews of students who continued to struggle with decimals, we saw that they did not consistently use correct decimal language and/or they did not correctly use the model to represent the decimals. Often, the mental images they had for decimals were not strong enough to overcome whole-number thinking. For example, when Ty was asked to name and then order 0.103 and 0.13, he answered, "Zero and a hundred and three, zero and thirteen." Because he had not adapted his understanding of number to include decimals, it was reasonable for him to mistakenly conclude that 0.103 > 0.13.

Struggles with representations

Students who failed to construct accurate representations for decimals were at a disadvantage as they attempted to order decimals or build meaning for operating with decimals. Although Rodrigo could name decimals to thousandths correctly, he was unable to represent decimals using the 10 × 10 grid. He misunderstood how color was used to represent tenths and hundredths. When asked to represent 0.5 on the grid, he colored five small squares orange (rather than five bars) and six small squares yellow for 0.06, explaining that tenths were orange and hundredths were yellow. Without an accurate model for decimals, Rodrigo was unable to factor in the size difference between tenths and hundredths, and he predictably resorted to whole-number thinking when ordering decimals and operating with them. He actually altered the model to fit his whole-number ideas.

Struggling without models to interpret addition and subtraction

Lily could model decimals on the grid and order decimals by decomposing numbers into their place-value components. However, when she was given the more complex tasks of adding and subtracting decimals, she initially resorted

to whole-number thinking. When asked to add 0.37 + 0.5 on the decimal +/– board, Lily first said the answer was 0.42. However, when asked to act out the addition on the board, she identified her error. She said, "I would do the five orange and then the three orange and then the seven. It would give me eighty-seven–hundredths."

Responding to struggles

By being aware of the indicators for decimal understanding, teachers can support students like Ty, Rodrigo, and Lily. Teachers can monitor their students' ability to use decimal language. They can, themselves, appropriately use decimal language and require students to do the same. As teachers monitor students' work with models, they should identify students who struggle with representations and present more opportunities for these students to work with the models. Teachers can provide opportunities for students to talk through ordering ideas, helping to highlight how successful students use language, mental representations of the class models, and skill in decomposing and composing decimals to order them.

As students move toward adding and subtracting decimals, teachers can provide opportunities for successful students to verbalize their strategies for composing and decomposing decimals. As teachers and students verbalize strategies using accurate language, they will support the students who are still struggling to overcome whole-number thinking and build a conceptual understanding of the operations.

Using mental images

Our picture of understanding in terms of the five indicators closely aligns with a pedagogy that emphasizes instruction using multiple representations and connections within and between them. Students who displayed solid understandings of decimal concepts first developed appropriate language and strong mental images of decimals using the 10×10 grid.

Those who were able to show flexibility in naming decimals like 0.43 in multiple ways, such as “forty-three–hundredths” and “four-tenths and three-hundredths,” demonstrated how language and mental images can be used to compose and decompose numbers. Students who were successful with composing and decomposing numbers used this flexibility to compare decimals and estimate sums and differences.

The grid and not the number line gave students strong mental images that supported their reasoning with decimals. This was consistent throughout the interviews and in our previous research (Wyberg et al. 2011; Cramer, Wyberg, and Leavitt 2008). We conjecture that the number line is a more abstract model, and although it does not support students’ construction of mental images for decimals, it may offer a representation in which students reconceptualize their understanding of unit, partitioning, and place value. Students’ understanding of decimals deepens when they make connections between these two models.

With appropriate instruction using multiple models, we believe all students can incorporate decimals into their understanding of number. Because some children may need more time than others to construct meaning for decimals as described here, students should have sufficient opportunities to engage with models and make connections to ensure that they all meet the CCSSM decimal standards.

Common Core Connections

4.NF.C.6 **5.NBT.A.3.B**
5.NBT.A.3.A **5.NBT.A.4**
4.NF.C.7 **5.NBT.B.7**

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