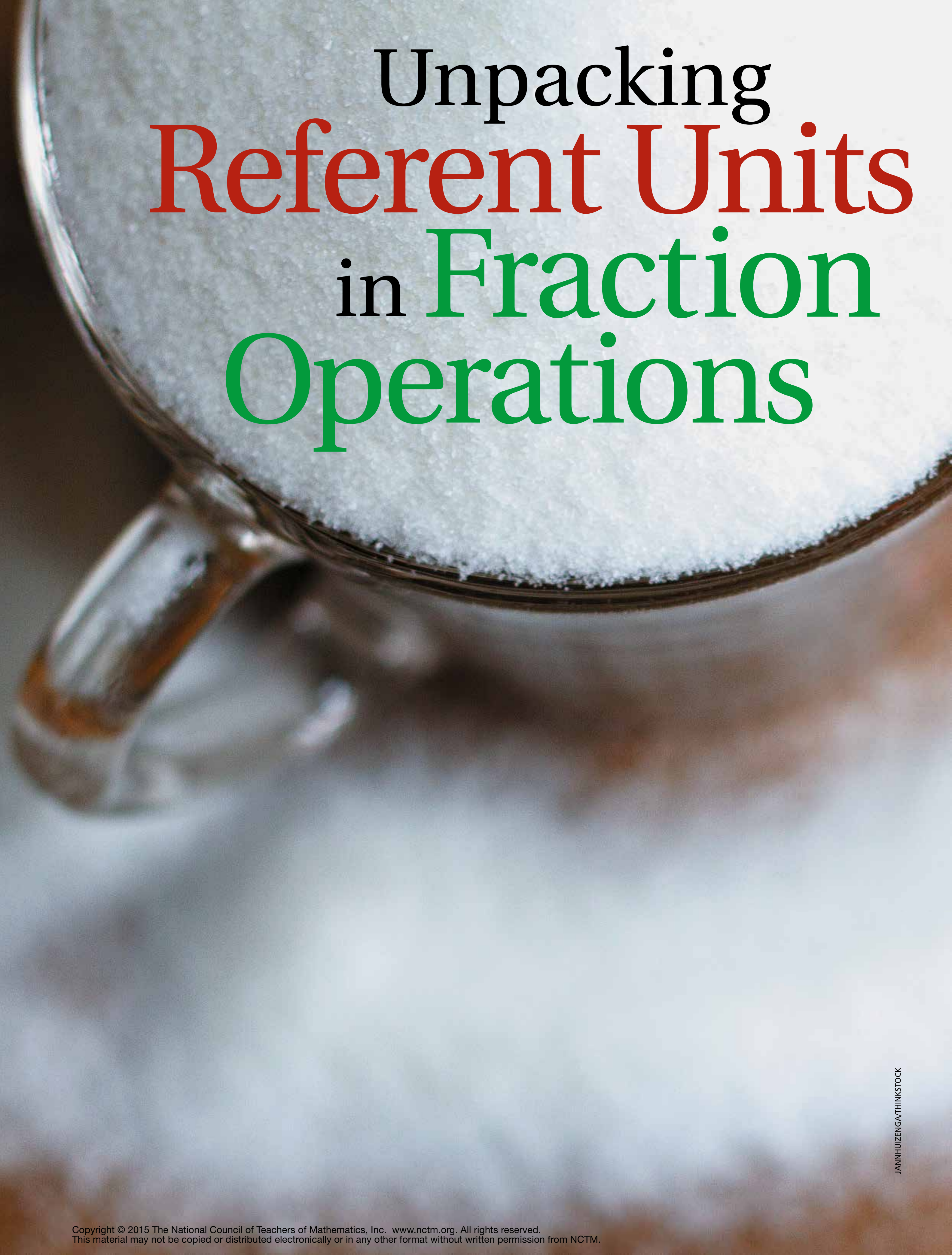
Unpacking Referent Units in Fraction Operations



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Using cups of sugar, this sequence of division tasks for K–grade 12 and adult learners highlights how seeing "wholes" results in fewer "holes" in reasoning.

Although fraction operations are procedurally straightforward, they are complex, because they require learners to conceptualize different units and view quantities in multiple ways. Why, when dividing fractions, do we invert and multiply? Prospective secondary school teachers sometimes provide an algebraic explanation. This proof may leverage the elegance of algebraic symbolic manipulation (see **fig. 1**), but it does not address the underlying relationships that are so important for learners striving to understand fractions. Furthermore, we know that secondary and elementary school preservice teachers have difficulty providing a conceptual explanation for fraction division (Ball 1990; Borko et al. 1992), and we have found that most experienced teachers have not had opportunities to work through, in a meaning-making way, the principles associated with making sense of fraction division.

In this article, we walk the reader through a lesson that has been used with prospective elementary and secondary school teachers for the purpose of highlighting key fraction concepts and principles necessary for teachers to understand if they are to support students in becoming mathematically proficient (NRC 2001) in the manner reflected in the Common Core State Standards in Mathematics (CCSSM) (CCSSI 2010). We do not suggest that K–grade 12 students would learn these ideas in one lesson; and, in fact, we believe that the ideas embedded in this lesson require sustained work for children. (For a more encompassing discussion of students' understanding of the domain of rational numbers with implications for instruction, we refer readers to a book by Empson and Levi [2011].) As such, we highlight key concepts for adult learners, hereafter referred to as *learners*.

FIGURE 1 Prospective secondary school teachers sometimes provide this algebraic explanation for why we invert and multiply when dividing fractions.

$$\frac{a}{b}\div\frac{c}{d}=\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{\frac{a}{b}}{\frac{c}{d}}\times 1=\frac{\frac{a}{b}}{\frac{c}{d}}\times\frac{\frac{b}{c}}{\frac{d}{d}}=\frac{\frac{a}{b}\times\frac{d}{c}}{1}=\frac{a}{b}\times\frac{d}{c}$$

Fraction understanding is often facilitated by drawing on real-life contexts (Clement 2004; Lesh, Post, and Behr 1987), such as cooking, and then coordinating the contextual quantities with other representations. Consider the following cooking context:

> A recipe calls for 1/3 cup of sugar, and you have 2 1/3 cups of sugar in the pantry. If you have unlimited supplies of the other ingredients, how many recipes could you bake?

Many learners can make sense of this problem by noting that there are three-thirds in one whole cup, six-thirds in two whole cups, and seven-thirds, or seven recipes, in two and one-third cups. When asked to bake the same recipe, but this time with one and one-half cups of sugar, learners see how making a seemingly insignificant change to a number can result in a substantially more difficult problem to conceptualize. This problem, although identical in structure to the previous problem, raises the issue of remainders. Some learners respond that they could make four recipes and would have one-sixth cup of sugar left. This answer is mathematically correct but does not address the crucial idea that units play in fractions. Because the remainder is expressed in terms of sugar, the learner has failed to grapple with the important conceptual idea that one-sixth *cup of sugar is, simultaneously,* one-half *the sugar for a recipe.* An example dialogue between a teacher and students about this issue follows:

Student 1: With one and one-half cups of sugar, I could make four recipes with one-sixth leftover.

Teacher: Can you talk a little about how you thought about that? How did you know that you could make four recipes?

Student 1: Because each recipe requires one-third of a cup of sugar; so one cup would make three recipes, and another third cup would make the fourth recipe.

Teacher: So, the four recipes would require how much sugar?

Student 1: One and one-third cups of sugar.

Teacher: Can someone else tell us what [student 1] is thinking?

Student 2: Each recipe takes one-third cup of sugar. So you can make three recipes from one cup, and with one-half cup leftover, you can make one more recipe.

Teacher: Is there any sugar left?

Student 2: Yes, there is one-sixth left.

Teacher: One-sixth what?

Student 1: One-sixth recipe.

Student 2: Yeah. I agree.

Teacher: Can someone explain where the one-sixth came from?

Student 3: You had one and one-half cups of sugar, and you used one and one-third. One-half minus one-third is one-sixth. So you have one-sixth left.

Teacher: One-sixth what? One-sixth *recipe*, or one-sixth *cup of sugar*? [*Students seem confused.*] Maybe a diagram will help [*drawing* **fig. 2**]. Can someone use this diagram to explain how many recipes can be made?

Student 4: Yes. Each one-third cup of sugar is one recipe. So the one-sixth leftover is one-sixth of a cup of sugar, which is one-half of a recipe.

Teacher: [*to students 1, 2, and 3*] Do you see how [student 4] is reasoning? [*Students seem confused.*] This is a good confusion. What I mean is that there is a complicated mathematical concept here, and if you work through your confusion, you will have developed a deeper understanding of an important mathematical idea. This amount [*pointing to the one-sixth remaining cup of sugar, in red, on the far right of* **fig. 2**] can be thought about as one-sixth of something [*gesturing over the three rectangles on the left side of* **fig. 2**, *representing one cup of sugar or three recipes*], and, at the same time, as one-half of something [*gesturing over one green rectangle, referring to one 1/3 cup of sugar,*

or a single recipe*]. Can you make sense of the difference?

Student 1: Oh, I think so. That is one-sixth of a cup of sugar, which is the same as one-half of a recipe.

Teacher: That's right.

Student 1: So, is the answer one-sixth or one-half?

Teacher: Great question! They both make sense, so long as you keep track of the unit. The answer is either one-sixth *of a cup of sugar*, which is expressed as a remainder, or one-half *of a recipe*, which is expressed as part of the quotient; but since the original question asked for recipes, we would say *four-and-one-half recipes*. And understanding multiplication or division of fractions requires us to be able to make sense of both of these ways of reasoning. So, this is a good confusion to work through! Actually, you [*to the class*] have all seen this before.

Students: We have?

Teacher: Yes. Consider that you have seven marbles or seven chocolate chip cookies, and you want to share them fairly among three people. How might you do that?

Student 2: Each gets two, with one leftover. Or, we could split the last cookie.

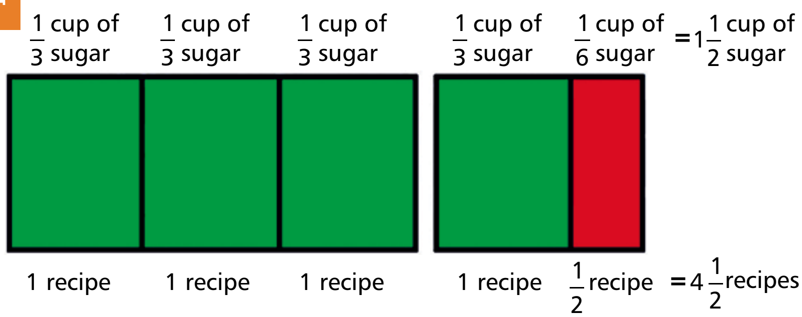
Teacher: Yes, and we might represent your reasoning symbolically using division:

$$\begin{array}{r} 2\ \text{R}\ 1\text{, or } 2\frac{1}{3} \\ 3\,\overline{)\,7} \\ \underline{-6} \\ 1 \end{array}$$

[*The teacher continues.*] Notice the two ways to write the answer. The first way, 2 R 1, represents the case whereby each of the three people gets two marbles or two cookies, and one is left that has not been shared. In this case, the 2 is the quotient and the 1 is the remainder. In the second case, the 1/3 refers to the result of sharing the remainder with the three people and expressing it as part of the quotient. We do not usually cut up marbles, but if the cookies are delicious, the three people might share the last one. The issue here is the same as the issue with the remainder 1/6.

FIGURE 2

The teacher asked students to use the diagram below to explain how many recipes could be made from the problem in the cooking context.



This vignette highlights a common difficulty experienced by students. The notion that a quantity might, simultaneously, be viewed as one-sixth (of one whole cup of sugar) and one-half (of a recipe, or of one-third cup of sugar) (see **fig. 2**) is a fundamental mathematical concept required for the understanding of multiplication and division. Furthermore, note how multiple representations—spoken language, real-life contexts, a diagram, gestures, and written symbols—were all invoked to help students see connections among these ideas. (For an example of a pensive student making sense of the meaning of division but incorrectly conceptualizing the remainder as a quotient, see http://www.sci.sdsu.edu/CRMSE/IMAP/vid_frac_div.html.)

Learning fractions conceptually is particularly challenging, especially when we consider that even when learners or K–grade 12 students do not try to conceptualize any meaning for the units, they may still be able to efficiently apply procedures. However, to make sense of fraction operations conceptually, one must attend explicitly to the units and to the fact that *multiplication and division require one to conceptualize multiple units and view a quantity in two ways*, whereas addition and subtraction require one to conceptualize only a single unit. The previous vignette highlights how learners are challenged to conceptualize two quantities with multiplication and division, and below we highlight how students may also sometimes be challenged to conceptualize only one unit with subtraction.

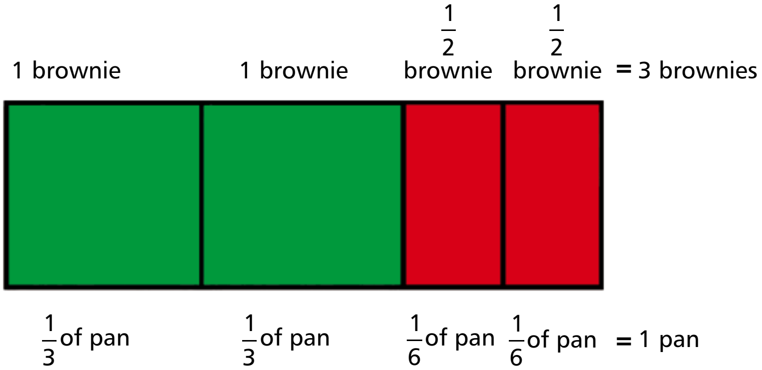
To demonstrate the significance of the idea that multiplication and division require one to conceptualize multiple units and therefore view a quantity in two ways, we compare these operations to the operations of addition and subtraction. Implicit when adding or subtracting

fractions, or any other type of number, is that each number refers to the same whole or unit. For example, when one applies the standard procedure for adding or subtracting 1/2 + 1/3, one is assuming that the one-half and the one-third refer to the same-size whole or unit, an assumption that is critical but also hidden from a learner operating only procedurally. But in multiplying fractions, such as 1/2 × 1/3 (which might be thought of as *what is one-half of one-third of a whole)*, the one-half and one-third do not refer to the same-size unit. Instead, the one-third might be thought of as referring to a whole of the measured quantity, whereas the one-half refers to one-third of the whole of the measured quantity. In a colloquial sense, multiplication is *gossip math,* whereas addition and subtraction are not, because in 4 × 3, the four is talking about the three—that is, the four refers to the number of threes (or the three refers to the number of fours)—whereas in 4 + 3, neither number is referring to the other, but instead both refer to parts of the same inferred whole.

To help see the multiple units involved in multiplication, again it helps to draw on multiple representations. Applying a context to 1/2 × 1/3 and using a similar double labeled diagram (see **fig. 3**) as used with the previous division problem, the multiple referents become more discernible. For example, imagine a pan of brownies cut into thirds, so each brownie is one-third of the pan. Then cut one brownie in half and take this piece. Your final piece may be thought of, simultaneously, as one-half of a brownie and one-sixth of the pan.

FIGURE 3

Using double labels helps learners simultaneously think of referents in two different ways.



Another complicating aspect of learning fraction multiplication is the fact that the commutative property of multiplication is procedurally simple—one can easily compute to determine that $a \times b$ and $b \times a$ result in the same product—but conceptually complex and unintuitive. Consider drawing a diagram for the context of 1/3 × (2 1/2 pies) versus 2 1/2 × (1/3 of a pie). In the first case, the unit is two-and-a-half pies; in the second case, the unit is one-third of a pie. Because these two expressions refer to different units, the conceptualization of these two products differ considerably and are, thus, challenging for one trying to make sense.

Finally, we note that whereas learners often struggle to see two units when multiplying and dividing, they also sometimes incorrectly see two units when subtracting. For example, consider finishing writing a story problem for 1/2 – 1/3 that starts like this:

> Pat has one-half of a brownie leftover in the refrigerator. For lunch, Pat eats . . .

A common (incorrect) story problem provided by learners is—

> Pat has one-half of a brownie leftover in the refrigerator. For lunch, Pat eats one-third of the (remaining) brownie. How much brownie did Pat have left?

This is incorrect because in this problem, the one-third is not referring to the original whole, but instead is referring to the one-half remaining brownie. The student's incorrect story problem is actually correctly representing 1/2 – (1/3 × 1/2). For the story problem to correctly represent 1/2 – 1/3, each term must refer to the same whole:

> Pat has one-half of a leftover brownie in the refrigerator. On his diet, Pat may eat one-third of a brownie for lunch. If Pat eats an amount equivalent to one-third of a brownie, how much brownie does Pat have left after lunch?

Again, notice how the multiple representations—in this case, the symbols, the oral language, and the context—may support students

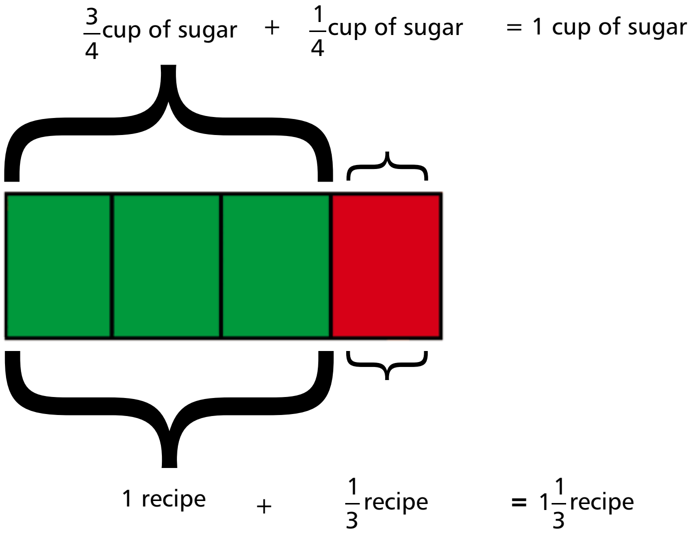
as they grapple with the referent for the unit. To further highlight the principle that in multiplication or division, one must conceptualize multiple units and view a quantity in two ways, consider another problem.

> Cynthia would like to bake cookies. She has one cup of sugar in the pantry. How many batches of cookies can she make using the entire cup if each batch requires—
>
> A) 1/2 cup? B) 1/3 cup?
> C) 3/4 cup? D) 4/5 cup?

Learners tend to have no problem with the first two situations, which have no remainder, but they have to think harder about the last two. Regarding how many recipes could be baked if each required three-fourths cup of sugar, consider the diagram in **figure 4**. The learners who struggle with this have trouble simultaneously viewing one segment as one-fourth cup of sugar and one-third of a recipe; that is, they struggle to view the given quantity in two ways.

FIGURE 4

Students often have difficulty simultaneously viewing a given quantity in two ways.

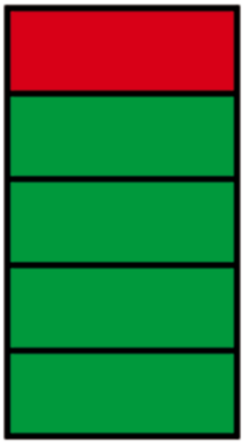


When students work problem D, they again have to consider the fact that four-fifths cup of sugar is equivalent to one recipe, and they

FIGURE 5

Again, when students work problem D, they must equate four-fifths cup of sugar to one recipe, and they must observe that the leftover part is simultaneously one-fifth of one unit (cups of sugar) and one-fourth of the other unit (recipes).

Use the picture to the right to explain why $1 \div \frac{4}{5} = \frac{5}{4}$.



must conceptualize two units—*cups of sugar* and *recipes*—observing that the part left is simultaneously one-fifth of one unit (cups of sugar) and one-fourth of the other unit (recipes) (see **fig. 5**).

To represent Cynthia's baking situations using symbolic notation, learners must recognize these as division problems: 1 ÷ 3/4 and 1 ÷ 4/5. For instance, the latter example might be represented by the question, "If we had one cup of sugar and each recipe called for four-fifths cup of sugar, how many recipes could we bake?"

One might think, "One whole has one four-fifths with one-fifth of a whole left. But that one-fifth of a whole cup of sugar is one-fourth of a four-fifths–cup recipe. Therefore, one whole has 1 1/4 four-fifths."

Again, note the principle that learners must simultaneously see the same quantity as one-fifth (of a whole) and one-fourth (of four-fifths.)

In the problems we have solved, we may notice a pattern:

1 ÷ 1/2 = 2/1
1 ÷ 1/3 = 3/1
1 ÷ 3/4 = 4/3
1 ÷ 4/5 = 5/4

The answer to "How many *a/b* are in one?" is always *b/a*. In conceptualizing additional tasks, like 1 ÷ 5/6 and 1 ÷ 3/7, one may note that the reciprocal of a number tells how many of each original number are in one.

The final step in the derivation for the fractions-division rule is to consider dividing something like 3 ÷ 4/5, which can be interpreted as "How many four-fifths are in three?" We know how many four-fifths are in one, and three times as many four-fifths are in three as are in one. When faced with 3 ÷ 4/5, we might think of approaching this in two steps: First, note that there are 5/4 four-fifths in one. Second, there are three times as many four-fifths in three as in one, or 3 × 5/4.

Attending to the referent unit

Students struggle with fractions, and we believe that one of the major reasons is that to understand fractions requires attending to the referent unit. When one eats one-fourth of a package containing two cupcakes, one has eaten one-half of a cupcake, and to understand this requires one to recognize that a given amount can, simultaneously, be one-fourth of one referent unit (a package) and one-half of another referent unit (a cupcake.) One instructional implication that follows is to make the unit, or whole, explicit whenever talking about fractions. When working with operations that require changing referent units, such as in multiplication and division, this instructional implication may seem necessary, but we believe that making the unit explicit ought to be started as soon as we begin teaching fractions, even in kindergarten! For example, instead of saying *one-half*, say *one-half of one cupcake*. If we can help students see *wholes* in their reasoning, beginning in kindergarten, we suspect that they will have fewer *holes* in their reasoning!

Common Core Connections
6.NS.1
6.NS.2

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