Inspection-Worthy Mist

Carefully select and leverage student errors for

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Jana recently attended a ten-day professional development workshop during which she learned about the importance of viewing students' mistakes as learning opportunities (Boaler 2016). Throughout the workshop, Jana and her colleagues celebrated their mathematical mistakes and not only corrected the mistakes but also learned from them. In this setting, she was introduced to the motto, "Mistakes are expected, inspected, and respected" (Seeley 2016, p. 26), which she planned to use during the upcoming school year.

When the school year started, Jana introduced her students to the motto and began reinforcing a positive view of mistakes as learning opportunities. In enacting the motto, though, she faced a new dilemma: Which mistakes should the class inspect? Should the class inspect *all* mistakes? Or are some mistakes more "inspection worthy" than others?

Jana's dilemma is not uncommon. As we become aware of the role of mistakes in learning, we desire to feature students' mistakes in class discussions. Often, discussing a student's mistake provides an opportunity to *critique the reasoning of others*, which is part of the Common Core Standards for Mathematical Practice (SMP 3, CCSSI 2010, p. 6–7), particularly when mistakes are not limited to computational errors. Further, inspecting mistakes opens a space for enhancing students' conceptual understandings (Boaler 2015; Borasi 1996).

As Jana previously asked, though, which mistakes are most appropriate for class inspection? With this question in mind, the purpose of this article is to support the reader in selecting mistakes that can be

akes: Which? And Why?

whole-class discussions to benefit the learning of all.

leveraged to benefit the learning of all students. Specifically, we focus on *which* and *why*: *which* mistakes to inspect and *why* these mistakes are inspection worthy. In the next section, we introduce types of mistakes along with ideas to consider when deciding whether a mistake is worthy of class inspection. Then we apply these ideas to a scenario taken from Jana's classroom.

Which mistakes and why

In considering *which* mistakes to inspect and *why*, we looked at student work from different lessons across different grade levels, focusing on the mistakes that were made and whether they were featured in whole-class discussions. From this process, we identified three types of errors along with ideas to consider when deciding whether inspecting an error will benefit all learners. Before describing the types of mistakes, though, we share two key ideas that arose during this process: The first involved what constitutes a mistake. From our viewpoint, a mistake is not limited to a computational error. Rather, mistakes include mathematical thinking, answers, and strategies that are either incorrect or unjustifiable. The second idea involved the mathematical goals, which serve as a lens through which to view all errors. Specifically, throughout our discussion, whether or not we explicitly state so, the mathematical goals of the lesson and/or learning trajectory should be in the foreground of selecting mistakes for inspection. With these ideas in mind, in the following sections, we describe *which* mistakes and *why*.

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Guiding questions to determine which procedural errors are inspection worthy

1. Is the error pervasive or fairly common throughout the class? For example, consider a classroom of third-grade students who are practicing subtraction with three-digit numbers involving regrouping. As the teacher circulates around the room, she notices several students making the procedural error of subtracting the minuend from the subtrahend in the ones place (see **fig. 1a**). Given that several students have made this procedural error, it is worthy of class inspection.
2. Is the error in line with the lesson's goals? Suppose, for example, a fifth-grade class has been studying multiplication with decimals when the teacher notices that a student has incorrectly placed the decimal in the product (see **fig. 1b**) by "lining up the decimals" as if it were an addition problem. Although the error is not necessarily pervasive, correct placement of the decimal in the product represents a fundamental component of the lesson's goals and is, therefore, worthy of inspection.

FIGURE 1

Because several students made the same procedural error, the teacher considered (a) a pervasive procedural error worthy of inspection. When a procedural error represents a fundamental component of the lesson goal, as in (b), we can consider it inspection worthy.

(a)

$$\begin{array}{r} 384 \\ -\ 276 \\ \hline 112 \end{array}$$

(b)

$$\begin{array}{r} \overset{2}{2}.4 \\ \times\quad .7 \\ \hline 16.8 \end{array}$$

Procedural errors

Procedural errors include mistakes in algorithms or other routine procedures. Sometimes procedural errors can be insignificant. For example, a student may write, "3 × 4 = 11." Although correcting his or her mistake is important for the student in this scenario, discussing what seems to be a trivial mistake is unlikely to enhance the mathematical development of all learners. Other procedural errors, though, can potentially enhance all students' mathematical development and are, therefore, worthy of class inspection. To aid in identifying which procedural errors are inspection worthy, we offer two guiding questions along with examples (see the **sidebar** above).

Why?

Inspecting procedural errors that are pervasive and/or aligned with lesson goals offers the class the opportunity to not only identify and correct the error but also justify the reasoning behind correct procedures. By making connections between procedures and their underlying mathematical reasons, students have a chance to "focus attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas" (Stein et al. 2000, p. 16). In doing so, those students who made the error receive support while all students' understanding is enriched through the sharing of mathematical justifications.

Inappropriate solution processes

Our second type of mistake involves the solution processes for word problems. Often in these instances, an inspection of the computations alone may not reveal a mistake. That is, the computations may be correct. However, considering the computations in relation to the problem context reveals the error, in that the processes represented by the computations are not appropriate for the problem and represent faulty reasoning or a misunderstanding regarding the problem context. Here, we present two examples to illustrate this type of mistake.

The Sharing Chocolate problem

The Sharing Chocolate problem (Enns 2014, p. 139) reads, in part, as follows:

> Two groups of friends are sharing chocolate bars. Each group wants to share the chocolate bars fairly so that every person gets the same amount and no chocolate remains. In the first group of friends, four students receive three chocolate bars. How much chocolate did each person get in the first group?

Without consideration of the problem, the student's work (see **fig. 2a**) is computationally correct. The mistake is recognized, though, when one considers the problem. That is, the student's mistake is with his solution process, which does not align with the problem context.

The Peach Tarts problem

Consider the work in our second example (see **fig. 2b**). The student has correctly multiplied ten by two-thirds and represented her answer with a model. Now contemplate this work in light of the problem the student was solving:

> Ms. Stangle wants to make peach tarts for her friends. She needs two-thirds of a peach for each tart, and she has 10 peaches. What is the greatest number of tarts that she can make with 10 peaches? (Chapin, O'Connor, and Anderson 2003, p. 31)

Is 10 × 2/3 the appropriate process to use when solving this problem? Actually, the Peach Tarts problem is a division problem: The goal is to determine how many two-thirds are in ten. Therefore, the correct solution process for this problem involves either computing 10 ÷ 2/3 or developing a representation (e.g., a picture or concrete manipulatives) that embodies ten divided by two-thirds (see **fig. 3a**). As a result, although the work in **figure 2b** is computationally correct, it does not align with the problem and is, therefore, a mistake representing an inappropriate solution process.

Not all inappropriate solution processes represent inspection-worthy mistakes, though. Consider the work in **figure 3b**, where the student has performed a variety of computations in an attempt to "do something" with the numbers. In talking with the student privately, the teacher found that the reasoning behind the computations was unrelated to the problem context. As a result, inspection of this mistake would likely focus on trying to understand why the student performed the various computations and the errors in them rather than the mathematics represented within the problem context. Therefore, this discussion would not deepen the class' understanding of the problem context and is not inspection worthy.

FIGURE 2

A student's work (a) on the Sharing Chocolate problem (Enns 2014, p. 139) is computationally correct, but his solution process does not align with the problem context; (b) correct computations—in this case, on the Peach Tarts problem—can still represent a mistake in terms of the solution process.

(a) Sharing Chocolate **(b) Peach Tarts**

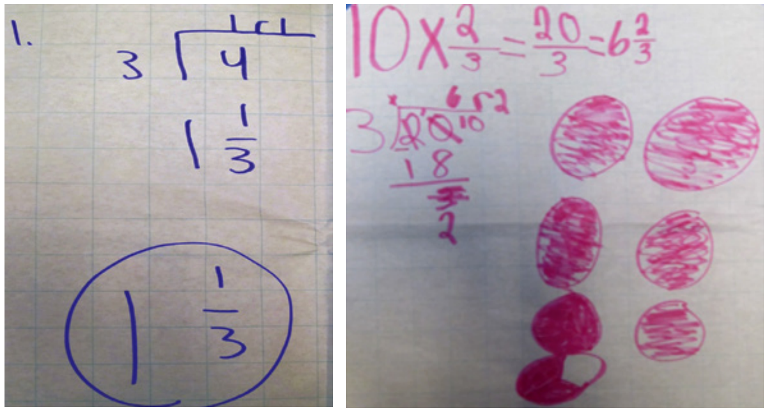

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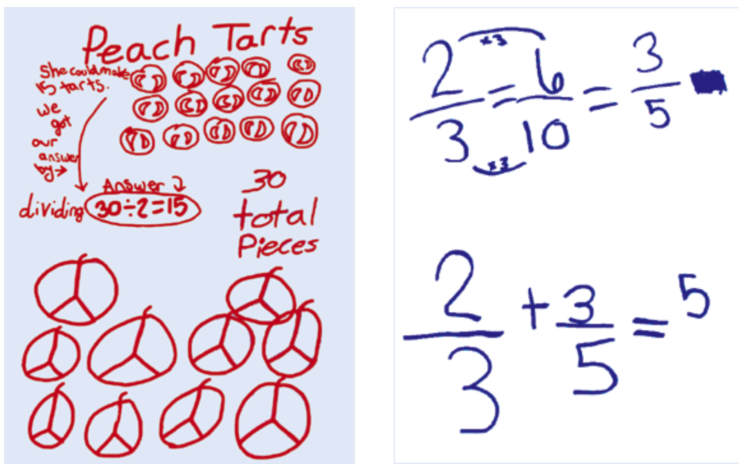
FIGURE 3

A student had (a) a correct solution to the Peach Tarts problem. Not all (b) inappropriate solution processes represent inspection-worthy mistakes.

(a) A correct solution **(b) An inappropriate solution**



Why?

In **figures 2a** and **b**, the mistakes represent opportunities to engage students in reasoning about key mathematical ideas represented within problem contexts. In the Peach Tarts problem, all students would likely benefit from discussing the problem aspects that indicate that it is, in fact, a division problem rather than a multiplication problem. Similarly, a discussion of the mistake in **figure 2a** would provide all students with a meaningful opportunity to assess the reasonableness of this answer. In general, "understanding occurs as a by-product of solving problems and reflecting on the thinking that went into those problem solutions" (Lambdin 2003, p. 11). Reflecting on the mistakes contained within these problem solutions serves to enhance students' understandings (Boaler 2015). As a result, errors that represent inappropriate solution processes are inspection worthy.

Misconceptions

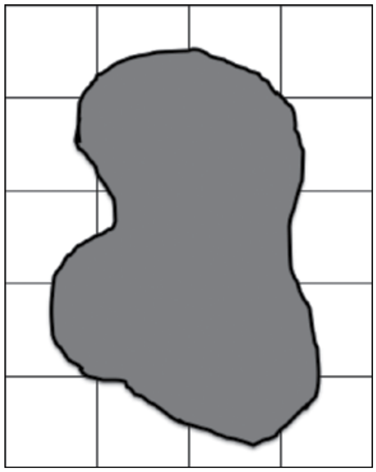
Mistakes that reveal students' mathematical misconceptions are also worthy of class

FIGURE 4

Some problems typically reveal a view or opinion that students have based on their previous misunderstandings or wrong thinking.

The Spilled-Juice Problem

On his homework, Jesse needed to find the perimeter and area of a rectangle. Unfortunately, he spilled his juice on the rectangle (see image below). Help Jesse find the area and perimeter of the rectangle.



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inspection. We define a misconception as a view or opinion that students mistakenly hold that is based on their previous misunderstandings or wrong thinking. For instance, consider the Spilled Juice problem (see **fig. 4**). Third-grade students normally cover the rectangle with square tiles, (see **fig. 5a**) and count the tiles to find the perimeter and area. To aid in this discussion of mistakes, we provide **figure 5b** as well, which shows a color-coded arrangement of the tiles. Once covered, two common mistakes representing mathematical misconceptions arise as students count the tiles:

1. To find perimeter, students often mistakenly count the "border tiles" (in red in **fig. 5b**). Whether students report the perimeter to be fourteen (a literal count of the border tiles) or eighteen (double-counting the corners), counting the squares represents a misconception regarding perimeter. That is, the students do not recognize perimeter as a measurement of length that should be found by counting the units of length (i.e., the sides of the squares) that make up the perimeter of the rectangle.

2. In determining area, students will state that the area is represented by the "inside squares" and then limit the area to the enclosed blue squares (see **fig. 5b**). This misconception regarding what constitutes the area of a figure is likely related to viewing the border tiles as representing the perimeter.

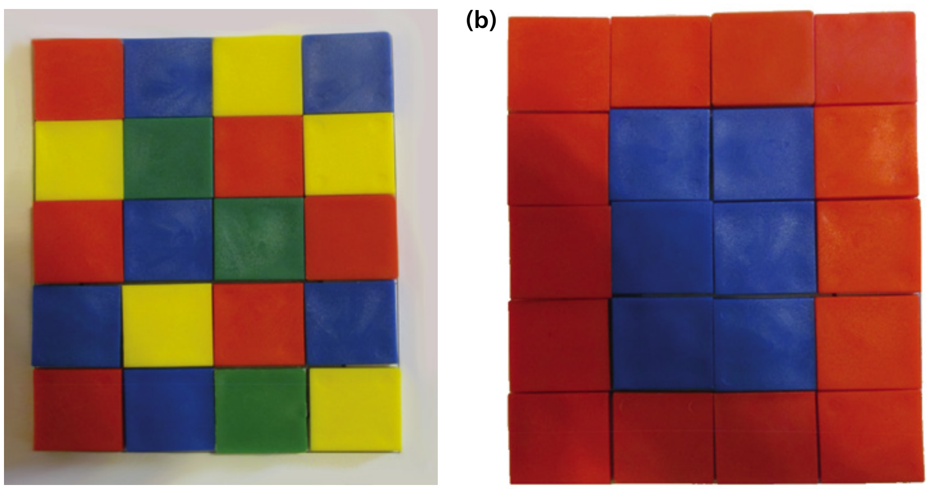
Why?

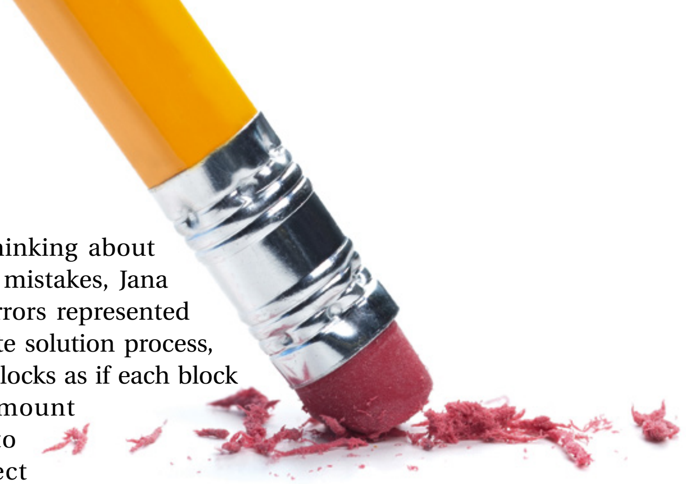
The two featured mistakes represent fundamental misconceptions regarding the lesson's mathematical goals (i.e., perimeter and area) and are, therefore, worthy of class inspection. By discussing these mistakes, students have

FIGURE 5

Two common misconceptions emerge when representing the problem with tiles.

(a) (b)





the opportunity to grapple with the underlying concepts; in this case, what are perimeter and area, and how are they measured?

> Explicit confrontation of preconceptions or misconceptions creates cognitive dissonance in which students begin to question and rethink their preconceptions, and further instruction and reflection can now help students understand the new concept. (Tobey and Fagan 2013, p. 181)

Inspecting mistakes that represent misconceptions can support all students in either *correcting* or *refining* their understandings of the concepts. Therefore, mistakes that involve fundamental misconceptions related to the lesson's mathematical goals are worthy of inspection.

Revisiting Jana's dilemma

With a focus on *which* and *why*, we now consider Jana's dilemma from the opening scenario. In an introductory lesson on area, Jana posed the Quilt task (see **fig. 6**) to her students. Her goal was to support students in seeing area as the amount of space covered by a figure. Jana gave students copies of the task and a set of pattern blocks. As students worked, Jana circulated around the classroom, inquiring about students' strategies. Jana's dilemma of which mistakes were inspection worthy arose as she noted three mistakes (see **table 1**).

In considering these errors, Jana recognized Emily's mistake as a procedural error because she had miscounted. Because this error was not pervasive, Jana chose not to have the class inspect it. In thinking about Caroline's and Ryan's mistakes, Jana noted that the two errors represented the same inappropriate solution process, that is, counting the blocks as if each block covered the same amount of area. She chose to have the class inspect Caroline's mistake (see **fig. 7**) and used the Imagine the Alternative strategy (Rathouz 2011). The following discussion ensued.

Teacher: So, without thinking about whether the answer is right, let's look at this covering of the quilts. Think for a moment, how might a student use this arrangement of the blocks to argue that quilt B is larger than quilt A? Trey?

Trey: Maybe it's 'cause quilt B has more blocks.

Teacher: Would someone like to talk some about this idea? Lizzy?

Lizzy: B has twelve blocks, but A has only seven blocks.

Manuel: I don't think you can just count the blocks.

Teacher: Why do you say that, Manuel?

Manuel: 'Cause, like, two green blocks is not bigger than one yellow block.

Teacher: Hmmmm, that's an interesting point. Let's all think about that for a moment.

As students discussed this mistake, fundamental concepts related to the lesson's focus of area began to arise, providing the cognitive dissonance needed to disrupt students' current

TABLE 1

Walking around the classroom and asking students about their strategies brought Jana's dilemma to the forefront.

Summary of mistakes for the Quilt task

Student	Description of strategy	Additional information
Emily	Covered the quilts with triangles and then miscounted; concluded that Quilt B with 24 triangles was larger than Quilt A with 22 triangles.	Emily's mistake of miscounting the blocks was unique to Emily.
Caroline	Covered the quilts with an assortment of blocks and then counted; concluded that Quilt B with 12 blocks was larger than Quilt A with 7 blocks.	Caroline's mistake of counting blocks without consideration for their different sizes was a common mistake throughout the class.
Ryan	Covered the quilts with an assortment of blocks and then counted; concluded the quilts covered the same area because both had 15 blocks.	Ryan's mistake was basically the same as Caroline's, with the exception that his led to the correct answer (i.e., the quilts covered the same area).

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FIGURE 6

In posing the Quilt task, Jana's goal was to support students in seeing area as the amount of space covered by a figure.

The Quilt Task

Adrianna is trying to decide which quilt she should buy for her bed. Each quilt was created using pattern blocks. Adrianna wants to choose a quilt that covers the greatest area. Should she choose Quilt A or Quilt B?

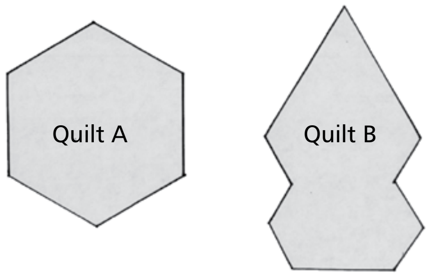

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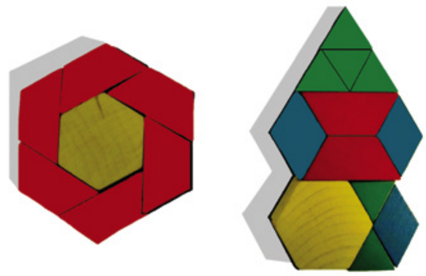
FIGURE 7

The class inspected Caroline's mistake and used the Imagine the Alternative strategy.



In our sister journals

The April 2018 issues of *Mathematics Teacher* (*MT*), *Mathematics Teaching in the Middle School* (*MTMS*), and *Teaching Children Mathematics* (*TCM*) contain a trio of articles discussing how to leverage mistakes. Each article is its respective journal's Twitter chat offering as well as the free online preview, available to NCTM members only.

MT contains "Making Room for Inspecting Mistakes," by Alyson E. Lischka, Natasha E. Gerstenschlager, D. Christopher Stephens, Angela T. Barlow, and Jeremy F. Strayer (pp. 332–39), which describes how selecting errors to discuss in class and trying three alternative lesson ideas can help move students toward deeper understanding. *MT*'s Twitter chat occurs on the fourth Wednesday of each month.

In *MTMS*, "Examining Mistakes to Shift Student Thinking," by James C. Willingham, Jeremy F. Strayer, Angela T. Barlow, and Alyson E. Lischka (pp. 324–32), delves into four research-based criteria on using mistakes during a problem-solving lesson on ratio reasoning. The Twitter chat for *MTMS* occurs on the third Wednesday of each month.

In *TCM*, "Inspection-Worthy Mistakes: Which? And Why?" by Angela T. Barlow, Lucy A. Watson, Amdeberhan A. Tessema, Alyson E. Lischka, and Jeremy F. Strayer, explores how carefully selecting and leveraging student errors for whole-class discussions can benefit the learning of all. **TCM's Twitter chat** is always on the second Wednesday of each month. This month, it is **April 11 at 9:00 p.m. EDT**.

thinking so that the focus might shift toward more productive ways of comparing area.

Conclusion

The inspection of mistakes can play a powerful role in an individual's learning process (Boaler 2015). In our own lessons, we have noted that students who might be hesitant to share their own ideas are often willing to inspect other's mistakes. Through discussions of inspection-worthy mistakes, errors can be leveraged to benefit the learning of *each and every* student. By focusing on *which* and *why*, our students benefit from expecting, inspecting, and respecting mistakes.

RELATED NCTM RESOURCE

For more on assessment, see *Using Classroom Assessment to Improve Student Learning: Math Problems Aligned with NCTM and Common Core State Standards* (NCTM 2011).

Common Core Connections

2.NBT.7	5.NBT.7
3.MD.5	5.NF.3
3.MD.6	5.NF.7
3.MD.8	

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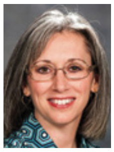



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Recommendations

We recognize the importance of considering *how* to engage students in inspecting mistakes. We recommend beginning with the Unknown Student strategy (Barlow, Gerstenschlager, and Harmon 2016), before moving to the Get the Goof strategy (Pace and Ortiz 2016) and then the Imagine the Alternative strategy (Rathouz 2011). Also, Bray (2013) provides a framework for designing a lesson that elicits and examines key mistakes.

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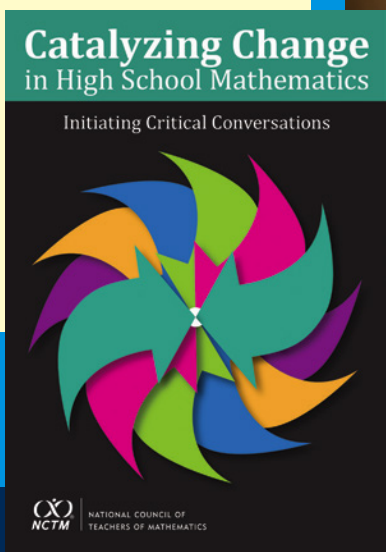

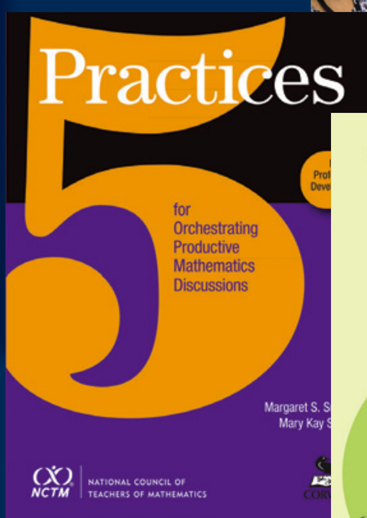

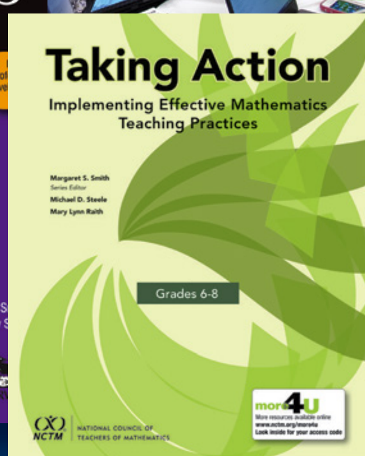

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