Journal for Research in Mathematics Education
2015, Vol. 46, No. 1, 39–87

The Development of Children's Algebraic Thinking: The Impact of a Comprehensive Early Algebra Intervention in Third Grade

Maria Blanton
TERC

Ana Stephens
University of Wisconsin Madison

Eric Knuth
University of Wisconsin Madison

Angela Murphy Gardiner
TERC

Isil Isler
University of Wisconsin Madison

Jee-Seon Kim
University of Wisconsin Madison

This article reports results from a study investigating the impact of a sustained, comprehensive early algebra intervention in third grade. Participants included 106 students; 39 received the early algebra intervention, and 67 received their district's regularly planned mathematics instruction. We share and discuss students' responses to a written pre- and post-assessment that addressed their understanding of several big ideas in the area of early algebra, including mathematical equivalence and equations, generalizing arithmetic, and functional thinking. We found that the intervention group significantly outperformed the nonintervention group and was more apt by posttest to use algebraic strategies to solve problems. Given the multitude of studies among adolescents documenting students' difficulties with algebra and the serious consequences of these difficulties, an important contribution of this research is the finding that—provided the appropriate instruction—children are capable of engaging successfully with a broad and diverse set of big algebraic ideas.

Key words: Algebra; Algebraic thinking; Early algebra; Elementary grades

The research reported here was supported in part by the National Science Foundation under DRK–12 Award No. 1207945. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

The teaching and learning of algebra in the United States has undergone a critical transformation over the last several decades. Prompted largely by the failure of historical paths to prepare students for success in formal school algebra (Hiebert et al., 2005; Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999; U.S. Department of Education [ED], Office of Educational Research and Improvement [OERI], National Center for Education Statistics [NCES], 1996, 1997, 1998) and in conjunction with the role of algebra as a gatekeeper (Schoenfeld, 1995), scholars have worked to examine traditional algebra education and formulate new recommendations for teaching and learning algebra (e.g., Kaput, 1998, 1999; Katz, 2007; Kilpatrick, Swafford, & Findell, 2001; RAND Mathematics Study Panel, 2003).

A central argument in reconceptualizing algebra has been that the "arithmetic-then-algebra" approach, where an arithmetic curriculum in the elementary grades is followed by a formal treatment of algebra in the secondary grades, has not allowed the time and space necessary for developing depth in students' algebraic thinking and, instead, has led to widespread school failure in mathematics and subsequent limited career and economic opportunities, particularly in STEM-related fields (Kaput, 1998, 1999, 2008; Moses & Cobb, 2001; Schoenfeld, 1995). As a result, it is now widely accepted that algebra should be treated as a longitudinal, Kindergarten through Grade 12 content strand whereby students have long-term, sustained algebra experiences in school mathematics, beginning in the elementary grades, that build their natural, informal intuitions about patterns and relationships into formalized ways of mathematical thinking (e.g., National Council of Teachers of Mathematics [NCTM], 2000, 2006; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010).

As a result of this shift, an emerging body of research on algebraic thinking in the elementary grades (hereinafter, *early algebra*) has provided important evidence regarding how children think algebraically (for more comprehensive overviews of this research, see Cai & Knuth, 2011; Carraher & Schliemann, 2007; Kaput, Carraher, & Blanton, 2008; Lins & Kaput, 2004). In particular, research has documented children's abilities to develop a relational understanding of the equal sign (Carpenter, Franke, & Levi, 2003; Carpenter, Levi, Berman, & Pligge, 2005; Falkner, Levi, & Carpenter, 1999); generalize mathematical structure by noticing regularity in arithmetic situations (Bastable & Schifter, 2008; Schifter, 1999; Schifter, Monk, Russell, & Bastable, 2008); use sophisticated tools to explore, generalize, and symbolize functional relationships (e.g., Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, in press; Carraher, Schliemann, Brizuela, & Earnest, 2006; Cooper & Warren, 2011; Moss, Beatty, Shillolo, & Barkin, 2008); build mathematical arguments that reflect more generalized forms than the empirical, case-based reasoning often used (Carpenter et al., 2003; Schifter, 2009); and reason about abstract quantities (e.g., length, volume) to represent algebraic relationships (Dougherty, 2003, 2008).

However, the majority of this work might not be viewed as sufficiently comprehensive in its approach because it has often focused on specific areas of early

algebra research to the exclusion of others (e.g., functional thinking vs. generalized arithmetic). This is understandable given the need for early algebra research, as a relatively new body of work, to narrow its focus initially in order to map out conceptual terrains in children's thinking. In addition, there is little research that systematically addresses how the development of children's algebraic thinking influences their understanding of important algebraic concepts in comparison to children who experience more traditional arithmetic-based instruction.

A fundamental assumption of early algebra education is that it will increase children's understanding of algebraic concepts that will serve them well in their largely arithmetic-based elementary school mathematics classrooms. In turn, this will increase their likelihood of success in the study of more advanced mathematics, particularly algebra, in secondary grades. Yet, with the exception of a few studies (e.g., Britt & Irwin, 2008; Schliemann, Carraher, & Brizuela, in press), this premise has been largely untested.

Recent initiatives in the United States have made understanding the effect of a sustained, comprehensive approach to early algebra education on the development of children's algebraic thinking a critical objective in algebra research. The Common Core State Standards for Mathematics (NGA & CCSSO, 2010), which reiterates the significance of early algebra in children's mathematics education beginning in kindergarten, and the National Mathematics Advisory Panel report (ED, 2008), which emphasizes the importance of student success in algebra and advocates for longitudinal research "to identify early predictors of success or failure in algebra" (p. 33), underscore the critical need to examine the effect of a comprehensive early algebra education. Moreover, through adoption of the Common Core State Standards for Mathematics, many states have elevated the role of algebra, leaving students potentially vulnerable to failure in the absence of algebra instruction that can address new learning goals.

Research Focus

Our focus in this study was to address the open question of the effect of sustained, comprehensive early algebra instruction on children's understanding of core algebraic practices and concepts (Kaput, 2008). Kaput (2008) proposed that the domain of algebra consists of both particular thinking practices and content strands. Specifically, he argued that algebraic thinking involves (a) making and expressing generalizations in increasingly formal and conventional symbol systems and (b) reasoning with symbolic forms. He further argued that these practices take place across three content strands:

> 1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and quantitative reasoning.
> 2. Algebra as the study of functions, relations, and joint variation.
> 3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics. (Kaput, 2008, p. 11)

By *sustained* early algebra instruction, we mean instruction that spans a significant amount of time (i.e., 1 year or more). By *comprehensive* early algebra instruction, we refer to an approach that intentionally integrates early algebraic practices into the elementary school curriculum across different conceptual domains that are recognized as important entry points into algebraic thinking (e.g., Carraher & Schliemann, 2007) so that these concepts and practices are broadly addressed in ways that build mathematical connections and are accessible to students at multiple levels of thinking. For example, the core algebraic thinking practice of *generalizing* (Kaput, 2008) can occur in different mathematical domains and can involve experiences as diverse as generalizing relationships between two covarying quantities in the study of functional relationships or generalizing arithmetic relationships regarding properties of number and operation. The core practice of *representing* generalizations (Kaput, 2008) offers another example. Although there are diverse ways in which to represent generalizations, variable notation as a representational tool is arguably the most recognizable cultural artifact of algebra (Kline, 1972). Yet, children in elementary grades are often introduced to variable—if at all—as a fixed, unknown quantity associated with missing-value problems (Blanton, Levi, Crites, & Dougherty, 2011; Lloyd, Herbel-Eisenmann, & Star, 2011). A comprehensive approach would highlight the additional roles of variable, including variable as varying quantity (e.g., in algebraic expressions or function rules), as generalized number (e.g., when representing fundamental properties), and even as parameter (e.g., when examining classes of functions).

Thus, the research focus of the study reported here is part of a systematic effort to explore the influence of early algebra instruction on student understanding. In this article, we share findings concerning (a) whether third-grade students are capable of learning a breadth of algebraic concepts with comprehensive early algebra instruction and (b) changes in their understanding of these concepts as a result of experiencing sustained instruction in early algebra over the course of 1 school year. We additionally examine the early algebra learning, or lack thereof, of students who experienced more typical (arithmetic-based) third-grade instruction to better understand the affordances of early algebra instruction.

The Basis for the Early Algebra Intervention

The work described here is situated in a broader, ongoing project that has as its goal the development of an early algebra learning progression (EALP) for Grades 3 through 7.[1] The early algebra intervention implemented in the study reported here was based on this EALP (a more detailed account of the development of the EALP is beyond the scope of this article; see Blanton, Stephens, Knuth, Gardiner, & Isler, 2014 for an account).

We based our construction of the EALP on Kaput's (2008) analysis of algebra

[1] Grade 3 was chosen as the starting point because it represents an important transition point for students into upper elementary grades. Grade 7 was chosen as the endpoint based on the expectation that students would be ready to begin a formal study of algebra in Grade 8.

in terms of thinking practices and content strands. In particular, we conducted an extensive analysis of several bodies of work in order to flesh out research findings about how children might engage in the core algebraic thinking practices within the content strands identified in Kaput's (2008) framework. This included an analysis of (a) research on teaching and learning algebra in Kindergarten through Grade 9, (b) state and national curricular standards and frameworks, (c) elementary and middle grades mathematics curricula, and (d) advanced (i.e., high school and postsecondary) mathematical perspectives on the sequencing of algebra content as evidenced in high school and college textbooks.

From our analysis and drawing from the language of learning progressions research (e.g., Shin, Stevens, Short, & Krajcik, 2009), we identified *five big ideas* that are represented in Kaput's (2008) content strands and around which much of early algebra research has matured (Blanton et al., 2011; Carraher & Schliemann, 2007; Kaput et al., 2008). These big ideas, which offer significant opportunities for engaging in the core algebraic thinking practices of generalizing, representing, justifying, and reasoning with mathematical relationships (Blanton et al., 2011; Kaput, 2008), include (a) equivalence, expressions, equations, and inequalities; (b) generalized arithmetic; (c) functional thinking; (d) variable; and (e) proportional reasoning.

The big idea of *equivalence, expressions, equations, and inequalities* includes developing a relational understanding of the equal sign, representing and reasoning with expressions and equations in their symbolic form, and describing relationships between and among generalized quantities that may or may not be equivalent. We take *generalized arithmetic* to involve generalizing arithmetic relationships, including fundamental properties of number and operation (e.g., the Commutative Property of Addition), and reasoning about the structure of arithmetic expressions rather than their computational value. *Functional thinking* involves generalizing relationships between covarying quantities and representing and reasoning with those relationships through natural language, algebraic (symbolic) notation, tables, and graphs. *Variable* refers to symbolic notation as a linguistic tool for representing mathematical ideas in succinct ways and includes the different roles variable plays in different mathematical contexts (Blanton et al., 2011). Finally, in the context of our work, *proportional reasoning* refers to opportunities for reasoning algebraically about two generalized quantities that are related in such a way that the ratio of one quantity to the other is invariant.

We do not claim that these five big ideas represent the only way to parse algebraic content. For example, one might use core algebraic thinking practices (Kaput, 2008) as another way to organize algebraic ideas. Moreover, the big ideas should not be taken as mutually exclusive. For example, we intentionally integrate variable throughout the other big ideas in our intervention because it reflects a more organic approach to instruction (i.e., students ideally learn about variable as a varying, unknown quantity in the study of functional relationships and as a generalized number when examining the fundamental properties). However, because of the seminal role of variable in algebra, we believed it was important to identify it as a

distinct big idea in the study of students' algebraic thinking (see also Blanton et al., 2011). Although our framework is not the only way to organize algebraic content, we view these five big ideas as fundamental to understanding algebra because they provide rich contexts in which algebraic thinking can occur (e.g., through generalizing arithmetic) or they represent central components of algebra as a discipline (e.g., variable).

In designing our third-grade, early algebra intervention, we used the aforementioned analysis of empirical research on children's algebraic thinking conducted in developing our EALP to hypothesize what algebraic concepts might reasonably be taught at the third-grade level. Table 1 provides the claims regarding what we would reasonably expect children in third grade to be able to do as related to the big ideas around which we organized our EALP. Such claims then formed the basis for the intervention, which is detailed further in the methods section.

Methods

In this study, a 1-year early algebra intervention was implemented in two third-grade classrooms, and pre- and post-intervention written assessments were administered to students. The assessments were also administered to third-grade students in classrooms in which the intervention was not implemented. Although the study design was not experimental (i.e., classrooms were not randomly assigned to treatment or control conditions), the intervention and nonintervention groups (described below) were demographically and academically comparable. It is therefore reasonable to examine the algebra learning that took place in each of these two groups as a way to better understand the affordances of the early algebra intervention reported here.

Participants

The third-grade students who participated in the study were all from one school district in the Northeastern United States. The school district had four elementary schools that were comparable academically and demographically, having a total enrollment of about 1,800 elementary students. Ten percent of the students were categorized as minority population, 20% as low socioeconomic status, and 5% as English Language Learners. Our analysis of the district's mathematics curriculum, *Growing with Mathematics* (Irons, 2003), indicated that it was primarily arithmetic focused and included little, if any, treatment of algebraic concepts. Teachers were expected to follow a district-mandated pacing guide that detailed the scope and sequence of the content they were to teach.

There were 106 third-grade students from two of the district's elementary schools who participated in the study. The intervention group (who received the early algebra intervention) consisted of 39 students from two intact classrooms in one of the schools, and the nonintervention group (who did not receive the intervention) consisted of 67 students from four intact classrooms—two at the school with the intervention group and two at a second school in the district. Intact groups were a necessary part of the study design because of the extensive nature (1 year) of the intervention. Given the reality of school structures, we could not randomly select students from different classrooms to form an intervention group.

Table 1
Summary of Student Expectations for Third Grade

Big ideas	Grade 3: Claims
Equivalence, Expressions, Equations, & Inequalities (EEEI)	• Interpret equations written in different formats (e.g., other than $a + b = c$) and evaluate as true or false • Solve open number sentences (e.g., 8 + 5 = ___ + 4), including by reasoning from the structural relationship in the equation • Use variable expressions to model linear problem situations • Identify the meaning of a variable used to represent an unknown quantity • Interpret an algebraic expression in the context of a problem • Model problem situations to produce linear equations of form $x + a = b$ • Analyze an equation to determine the value of a variable
Generalized Arithmetic (GA)	• Analyze information to conjecture an arithmetic relationship • Express the conjecture in words and/or variables • Identify values or domains of values for which a conjectured generalization is true • Describe the meaning of a repeated variable or different variables in the same equation • Identify a generalization in use (e.g., in computational work) • Justify an arithmetic generalization using either empirical arguments or representation-based arguments; examine limitations of empirical arguments
Functional Thinking (FT)	• Generate linear data and organize in a function table • Identify the meaning of a variable used to represent a varying quantity • Identify a recursive pattern and describe in words; use to predict near data • Identify a covariational relationship and describe in words • Identify a function rule and describe in words and variables • Use a function rule to predict far function values • Given a value of the dependent variable, determine the value of the independent variable (reversibility) • Construct a coordinate graph
Variable (Var)	• Use variables to represent arithmetic generalizations • Examine the meaning of a repeated variable or different variables in an equation or rule • Use variables to represent an unknown quantity (fixed or varying) • Understand that a variable represents the measure or amount of an object rather than the object itself • Interpret the meaning of a variable within a problem context • Use variables to represent linear problem situations • Describe a function rule using variables

Note. The big idea of Proportional Reasoning was not explicitly addressed at the third-grade level. Concepts associated with Variable were integrated in instruction throughout the other big ideas as appropriate.

Intervention classrooms were self-selected in that teachers volunteered their classrooms as intervention sites. It should be noted, however, that the intervention was taught by a member of our research team. Students from the two groups were comparable academically in that there was no statistically significant difference in either their performance at pretest on our algebra assessment or their performance on the state's standardized mathematics assessment, which students completed at the end of third grade after the conclusion of our intervention.[2]

Implementing the Early Algebra Intervention

Nineteen 1-hour early algebra lessons were taught throughout the school year in the two intervention classrooms. No more than one lesson was taught during a week. The lessons constituted about 10% of overall yearly instructional time in mathematics and took place during students' regularly scheduled mathematics instruction. However, because of early algebra's deep connections to arithmetic, topics in the regularly planned curriculum did not need to be eliminated to accommodate the intervention. That is, our early algebra intervention provided opportunities to develop not only students' understanding of algebraic concepts and practices, but also their understanding of important arithmetic concepts and skills, albeit in an "algebrafied" way that highlighted the underlying structure of the arithmetic. Students' comparable performance on the state's standardized mathematics assessment completed after the conclusion of the intervention supports the notion that in the intervention group, students' general mathematical knowledge was not negatively affected by this shift in focus. In summary, all participating students, regardless of whether they received the intervention, continued to receive the same overall amount of mathematics instructional time and approximately the same amount of time for topics addressed in the official curriculum.

Table 2 provides an overview of the intervention lessons, including the primary goals for each lesson, the order in which they occurred, the associated big algebraic ideas, and the number of distinct lessons. It is important to note that concepts were often revisited in subsequent lessons; thus, one should interpret the number of lessons per concept with caution. For example, as students engaged in lessons focused on generalizing arithmetic, they were encouraged to represent fundamental properties of number and operation (e.g., the Additive Identity) in different ways (e.g., as $a + 0 = a$ or $a = a + 0$), thus revisiting the concept of the equal sign as a relational symbol.

Each early algebra lesson began with small-group discussions on previously taught concepts. New concepts were then introduced through small-group problem solving and whole-class discussion. The tasks that provided the problem-solving

[2] In the district in which this study occurred, the administration of state standardized assessments begins at the end of third grade. Therefore, standardized assessment data were not available for participants in our study prior to the intervention.

focus were designed to engage students in the core algebraic practices: generalizing, representing, justifying, and reasoning with mathematical relationships. All lessons were taught by one member of the research team, a former elementary school teacher, and observed by at least one other member of the research team. The research team met twice weekly to plan lessons, discuss preliminary findings, and make appropriate revisions for upcoming lessons. See Appendix A for a sample lesson.

Data Collection

Data reported in this study consist of students' responses to a written assessment. Students were given 1 hour to complete the assessment. The assessment was administered to both the early algebra intervention group and the nonintervention group as a pretest at the beginning of the school year (prior to the instructional intervention), and as a posttest at the end of the school year (at the conclusion of the intervention for the early algebra group). The sample size for the pretest was $n = 104$ (for Items 1–6) and $n = 105$ (for Items 7–11); the sample size for the posttest was $n = 101$.[3]

The written assessment was based on the EALP and was designed to assess students' understanding of core algebraic concepts and practices within the five big ideas. Nine of the 11 assessment items were open response, one was multiple-choice, and one used a mixed format. To ensure that the items possessed good psychometric properties, we used items similar to those that had performed well psychometrically in previous research (e.g., Blanton, 2008; Carpenter et al., 2003; Carraher, Schliemann, & Schwartz, 2008; Kenney, Lindquist, & Heffernan, 2002; Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008; Knuth, Stephens, McNeil, & Alibali, 2006). All items were piloted and revised as necessary prior to administering the assessment (e.g., the wording of items was revised to improve readability for the participants). The same form was used as both pre- and post-test because the length of time between administrations (approximately 8 months) minimized our concern with test familiarity. We were concerned, however, that different items addressing the same big idea might lead to unexpected differences in responses for reasons such as problem context, number choice, or wording. Thus, the consistency of the items allowed us to more clearly attribute differences from pretest to posttest to growth in students' algebraic understanding.[4]

Appendix B provides the assessment items as well as an overview of the concepts and practices addressed by each of the assessment items. The content addressed in the assessment represents a subset of the overall lesson goals

[3] Differences in the pretest to posttest sample sizes are due to student absences when the assessment was given. The discrepancy in pretest sample size is due to administering the pretest to one additional third-grade student after the intervention began. Only assessment items that had not been addressed in the intervention by that point (Items 7–11) were analyzed for this student.

[4] We calculated Cronbach's alpha as an estimate of the reliability, which demonstrates a strong internal consistency for the assessment ($\alpha = 0.904$ for the posttest score).

Table 2
Summary of Instructional Sequence for the Third-Grade Intervention

Lesson focus (big ideas)	Number of lessons	Lesson goals (claims)
Relational understanding of the equal sign (EEEI)	2	• Identify meaning of "=" as expressing a relationship between quantities • Interpret equations written in various formats (e.g., other than $a + b = c$) to correctly assess an equivalence relationship (true/false number sentences) • Solve missing value problems by reasoning from the structural relationship in the equation (open number sentences)
Generalizing the Additive Identity and Additive Inverse properties Generalizing the Commutative Property of Addition (GA, Var)	1 1	• Analyze information to develop a conjecture about the arithmetic relationship • Express a conjecture in words • Develop a justification or argument to support the conjecture's truth; introduce representation-based arguments (Schifter, 2009) and examine the limitations of empirical arguments • Identify values for which a conjecture is true • Express a generalization (property) using variables • Examine the meaning of a repeated variable or different variables in the same equation • Examine the characteristic that a generalization (property) is true for all values of a variable in a given number domain • Identify a generalization (property) in use
Modeling problem situations with (linear) algebraic expressions (EEEI, Var)	2	• Identify a variable to represent an unknown quantity • Informally examine the role of variable as an unknown, varying quantity • Represent a quantity as an algebraic expression using variables • Interpret an algebraic expression in context • Identify different ways to write an expression
Modeling problem situations with (linear) equations in one variable (EEEI, Var)	2	• Identify a variable to represent an unknown quantity • Informally examine the role of variable as an unknown (fixed) quantity • Write an algebraic expression using variables • Express the relationship of two equivalent expressions as an equation. Identify different ways to write the equation.

Table 2
Summary of Instructional Sequence for the Third-Grade Intervention (*continued*)

Solving a problem situation involving one-step, single variable linear equation (additive) Solving a problem situation involving one-step, single variable linear equation (multiplicative) (EEEI, Var)	2 1	• Model a problem situation to produce a linear equation of the form $x + a = b$ or $ax = b$ • Identify different ways to write an equation • Analyze the structure of the equation to determine the value of the variable • Check the solution to an equation or determine if the solution is reasonable given the context of the problem • Informally examine the role of variable as an unknown, fixed quantity
Generalizing the Multiplicative Identity Property and the property that $a \times 0 = 0$ for all a. (GA, Var)	1	(see Generalized Arithmetic/Variable goals above, this table)
Solving a problem situation involving linear functions with one operation ($y = x + b$) Solving a problem situation involving linear functions with two operations ($y = mx + b$) (FT, Var)	3 4	• Generate data and organize in a function table • Identify variables (as number of objects or magnitude of a quantity, not as an object) and their roles as varying quantities • Identify a recursive pattern and describe in words; use the pattern to predict near data • Identify a covariational relationship and describe in words • Identify a function rule and describe in words and variables • Use a function rule to predict far function values • Examine the meaning of different variables in a function • Develop a justification for why a function rule works by reasoning from the problem context or the function table • Recognize that corresponding values in a function table must satisfy the function rule. That is, when function variables are substituted with corresponding values from the table, the result must be a true equation. • Use multiplicative relationships to reason proportionally about data (e.g., if 2 candies cost 10 cents, determine how much 4 candies would cost) • Construct a coordinate graph and attend to how discrete data are represented and issues of scale • Reversibility: given a value of the dependent variable and the function rule for a one-operation function, determine the value of the independent variable

identified for the third-grade intervention (see Table 2). Because of the time constraints of a 1-hour assessment, we selected a subset of goals that we viewed as particularly salient and from which we could collect reasonable data on written responses. Overall, the assessment was designed so that each of the big ideas was given relatively equal representation in terms of the amount of time it might take students to respond to items. The exception to this is proportional reasoning, which was not explicitly addressed in the third-grade intervention. With an eye toward a future longitudinal study, we included one proportional reasoning item as part of our assessment because of its connection to the big idea of functional thinking and its significance in later grades.

Data Analysis

Responses to the assessment items were coded for correctness as well as for the strategies students employed. Determining correctness was often straightforward (e.g., missing-value items, true–false items, and equation-solving items have only one correct response). At other times, what qualified as a correct response needed to be discussed among and agreed upon by the coders. Sometimes, a particular strategy code aligned with correctness (e.g., see Item 7's *letter* and *letter-related* codes in Figure 7 or Item 4's *structural* code in Figure 13). Sometimes, it was deemed inappropriate to code a particular item for correctness, and only the strategy used was coded (in particular, Items 3a, 3b, 4a, and 8b). Criteria for evaluating correctness are discussed with the results of student performance on individual assessment items.

Developing the coding scheme for strategy use. The development of the scheme for coding the strategies students used in response to the assessment items was an iterative process that began with identifying strategies from existing research regarding children's algebraic thinking. Student strategies were assigned external codes in cases where those strategies were well documented in the literature. For example, research shows that in order to find the missing number in a task such as 7 + 3 = ___ + 4 (Item 1a), students might add the numbers to the left of the equal sign, add all given numbers, or reason about quantities on both sides of the equal sign either computationally or relationally (Carpenter et al., 2003). If students add the numbers to the left of the equal sign (for a response of 10) or add all numbers (for a response of 14), they are said to have an *operational understanding* of the equal sign. Thus, we used an external code of *operational* to describe these strategies. External codes served as a starting point for the coding scheme, and any external codes that did not occur in student responses were eliminated.

With an initial (external) coding scheme in place, items were divided among three members of the project team[5] to continue the development of the coding scheme and coding of data. Project team members developed new (internal) codes to describe student strategies that emerged from patterns of responses in the data that were not captured by the initial (external) codes. When a strategy occurred frequently

[5] We wish to acknowledge Timothy Marum, a member of the project team, for his contributions in the analysis of the data reported here.

(i.e., in more than 5% of the responses), it became part of the coding scheme. When new codes were introduced, their meanings were negotiated among the coders. The assignment of internal codes then led to an updating of the coding scheme and a recoding of the data with these new codes in mind. After a stable set of codes (internal and external) was determined, the entire project team discussed the codes and their meanings, looked for redundancies, and agreed on a final set of codes.

There were times when a student's written work showed two strategies in response to one item. In these cases, the most sophisticated strategy was recorded. For example, in response to Item 2b (see Figure 1), students occasionally computed the sum on both sides of the equal sign but also provided a structural explanation. In such a case, the response was assigned the *structural* code.

Item 1a

Fill in the blank with the value that makes the following number sentence true.
How did you get your answer?

7 + 3 = ____ + 4 Why?

Item 2b

Circle True or False and explain your choice.

57 + 22 = 58 + 21 True False How do you know?

Strategy Code	**Description**	**Example**
Structural	Student notices structure in the equation and solves or determines equivalence without computing.	*Item 1a: 7 + 3 = 6 + 4, because if you take one away from the 7 and add it to the 3 you have 6 left.* *Item 2b: True, because you add one to 57 and minus one from 22.*
Computational	Student computes to find the missing value or to determine if the two sides of the equation have the same value.	*Item 1a: 7 + 3 = 6 + 4, because 7 + 3 = 10 and 6 + 4 = 10.* *Item 2b: True, because 57 + 22 = 79 and 58 + 21 = 79.*
Operational	Student adds the numbers to the left of the equal sign or adds all of the numbers given in the equation to get the solution.	*Item 1a: 7 + 3 = 10 + 4 since 7 + 3 = 10* *Item 1a: 7 + 3 = 14 + 4 since 7 + 3 + 4 = 14* *Item 2b: False, because 57 + 22 = 79, not 58* *Item 2b: False, because 57 + 22 + 58 + 21 = 158*

Figure 1. Coding scheme for Items 1a and 2b.

Finally, items that students left blank, or for which they provided responses such as "I don't know" or "?," were grouped into a *no response* category. These responses

were coded as incorrect in our statistical analyses of student performance. Responses that were not sufficiently frequent to constitute their own codes were placed into an *other* category, and responses that included no work or explanation were placed into an *answer only* category. The only exception to this latter case occurred for Item 1a: If students answered 10 or 14 but showed no work, research on children's responses to similar items (e.g., Falkner et al., 1999; Knuth et al., 2006) suggests that they likely used an operational strategy, and their responses were assigned the *operational* code. Because the *other* and *answer only* codes did not provide information about the types of strategies students used, we do not include these codes in our subsequent discussion of students' strategy use except when relevant given the nature of the item and students' responses.

Determining interrater reliability. As we noted earlier, coders discussed the meanings of each strategy code as it emerged in order to ensure consistency in coding. As part of this process, they developed a document that included a list of codes for each assessment item, a description of each code, and examples of student responses assigned to each particular code. Interrater reliability was established by having a project team member code a randomly selected 20% sample of responses initially coded by another member of the team and calculating the percent agreement on whether the response was correct or incorrect and what strategy the student used (i.e., the code assigned to the student's response). Reliability checks resulted in at least 80% agreement among coders on all items. Any discrepancies were discussed until full agreement was reached. The coding scheme was updated to take into account any changes resulting from interrater reliability assessments, and the data were recoded in light of these changes.

Analyzing students' responses. After responses were coded according to correctness (as a measure of student performance) and strategy use, they were analyzed quantitatively. This analysis provided insight into (a) the early algebra learning of students in the two groups (the intervention group and the nonintervention group) and (b) the difference in gains in pretest to posttest performance between the two groups. All students' responses were analyzed by item as well as by total percentage correct. Specifically, 2 × 2 frequency tables of group by item scores were created for each pre- and post-test item, and nonrandom associations between the group and item were statistically examined using Fisher's exact test. Total percentages of correct responses for each student on the pre- and post-tests were also calculated. Both the pretest and posttest distributions were unimodal and positively skewed, although the degree of skewness was reduced in the posttest (skewness = 0.47) compared to the pretest (skewness = 0.79). The peakedness of the pretest distribution (kurtosis = 3.40) was slightly sharper than a normal distribution (kurtosis = 3.00), whereas the posttest distribution (kurtosis = 1.93) was flatter than a normal distribution. The differences between the pre- and post-test scores were statistically tested using an F-test, and corresponding effect sizes were reported. Finally, students' strategy use was analyzed using descriptive statistics.

Results

In this section, we share the results of students' performance on the overall assessment, their performance on individual items, and the types of strategies they used on selected items.

Students' Overall Performance

We calculated students' percentages of correct responses on the pretest and the posttest and used these measures to compare gains demonstrated by the intervention group and the nonintervention group. Although there were no significant differences between the two groups at pretest (M = 18.22 and SD = 12.36 for the intervention group vs. M = 14.99 and SD = 10.58 for the nonintervention group, $F = 2.01$, $p = 0.16$, $d = 0.28$), the difference in gains between these groups from pretest to posttest was significant (M = 65.51 and SD = 21.01 for the intervention group vs. M = 21.97 and SD = 15.37 for the nonintervention group at posttest, $F = 143.6$, $p < 0.001$, $d = 2.37$). When the difference in pretest scores between the two groups was statistically controlled in an analysis of covariance ($F = 110.1$, $p < 0.001$, $d = 2.13$), the finding was consistent with the analysis of variance on the posttest alone, and the adjusted means (63.76 vs. 22.63) were similar to the unadjusted means of the posttest. In summary, with respect to the percentage of correct responses, students in the intervention group demonstrated a significant gain in their pretest to posttest scores (65.5 − 18.2 = 47.3%), substantially greater than that demonstrated by students in the nonintervention group (22.0 − 15.0 = 7%).

These results suggest that the comprehensive early algebra approach used here was appropriate academically for the third-grade students in the intervention group. That is, students in the intervention group seemed capable of learning the breadth of algebraic concepts and practices addressed in the intervention. Not only did they demonstrate a significant performance gain (47.3%), but they also achieved a relatively high posttest average (65.5%). The results also suggest that the arithmetic-focused curricular approach experienced by students in the nonintervention group did little to develop students' understanding of early algebra concepts and practices (at least those addressed in the assessment administered in this study) given their small performance gain (7%) and low posttest average (22%).

Students' Performance and Strategy Use on Individual Items

Results of our analysis of gains in students' performance on individual items indicate that, although there were no significant differences between the two groups' performances on each item at pretest, the intervention group showed statistically significant gains from pretest to posttest on every item except Item 10e and Item 11.[6] The nonintervention group did not show significant gains from pretest to posttest on any of the items. It is worth noting that Item 11 was a proportional reasoning item, and proportional reasoning was not explicitly addressed in

our third-grade intervention. Results for Item 10 are discussed further below. In what follows, we look more closely at a selected set of the individual assessment items. Because of the volume of data, we do not discuss results for every assessment item; however, the items selected for discussion represent the spectrum of content addressed in the intervention. Appendix C includes results of our analysis of all items coded for correctness.

Equivalence, expressions, equations, and inequalities. We discuss items in this section that address students' understanding of the equal sign, expressions, and equations.

Students' understanding of the equal sign. Items 1a and 2b were adapted from research-based tasks designed to elicit students' understanding of the equal sign (Carpenter et al., 2003; Knuth et al., 2006). Figure 1 includes these items, a description of the codes used to categorize student responses, and an example from students' written work of a response that warranted a given code.

Figure 2 illustrates students' pretest to posttest performances, in terms of percentage correct, for Items 1a and 2b. Results suggest that misconceptions about the equal sign persist if not addressed instructionally (as in the case of students in the nonintervention group) but can be successfully addressed through early algebra-based interventions (as in the case of the intervention group).

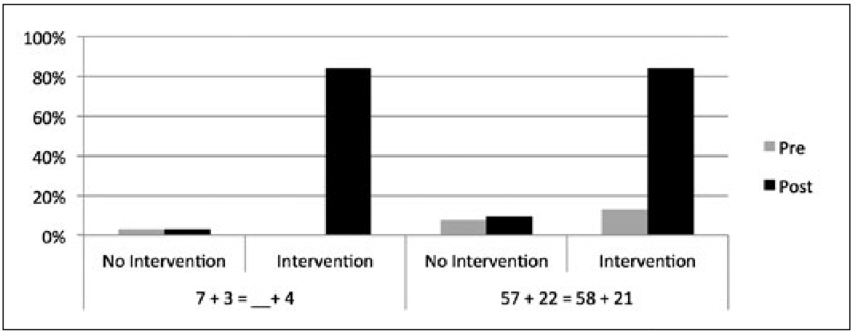


Figure 2. Percentage of students who provided correct responses on Items 1a and 2b.

However, we are interested not only in students' performance within each group but also in the types of strategies used and whether they reflected more algebraic approaches from pretest to posttest. Figure 3 shows the types of strategies students used when responding to Items 1a and 2b. All students used predominantly an (incorrect) operational strategy for both Items 1a and 2b at pretest.[7] For example, 95% of students in the intervention group and 83% of those in the nonintervention group used an operational strategy for Item 1a. Furthermore, no student used a

[6] As described earlier, this excludes those items that were not coded for correctness (i.e., Items 3a, 3b, 4a, and 8b).

(correct) structural strategy at pretest for either of Items 1a or 2b. Similarly, no student who received the intervention and only 2% of those who did not used a (correct) computational strategy at pretest for Item 1a. Only 3% of either group of students used this strategy at pretest for Item 2b. Thus, at pretest, all students overwhelmingly exhibited an operational understanding of the equal sign, almost no use of arithmetic (computational) approaches, and no use of algebraic (structural) approaches for reasoning about the two tasks.

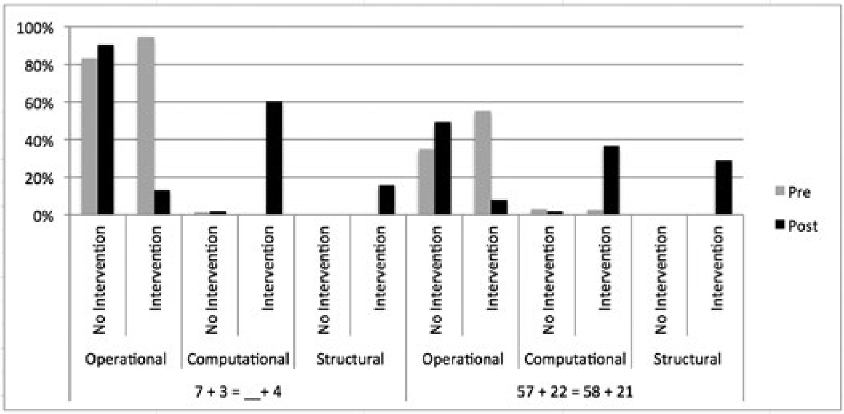


Figure 3. Percentage of students who used the indicated strategy on Items 1a and 2b. Frequency of no response for Item 1a was 0% at pretest and posttest for the intervention group and 8% at pretest and 6% at posttest for the nonintervention group. Frequency of no response for Item 2b was 5% at pretest and 0% at posttest for the intervention group and 11% at pretest and 2% at posttest for the nonintervention group.

By posttest, however, students in the intervention group used more sophisticated strategies in response to these items than they had at pretest. For Item 1a, only 13% of the intervention group continued to use an operational strategy at posttest. Moreover, 77% now used a correct approach—61% used a computational strategy, and 16% used a structural strategy. In contrast, 90% of students who did not receive the intervention continued to use an operational strategy at posttest on Item 1a, only 2% of students used a computational strategy, and no student used a structural strategy. For Item 2b, only 8% of the intervention group continued to use an operational strategy at posttest. Instead, 37% used a computational strategy, and 29% used a structural strategy. About 50% of students who did not receive the intervention, however, used an operational strategy for Item 2b at posttest, whereas only 2% used a computational strategy, and no student used a structural strategy.

[7] See Figure 1 for a description of the strategies. Recall, also, that not all codes are reported here. Those codes that do not provide information on the types of strategies used (e.g., *answer only*) are not included.

These results highlight the ability of students in the intervention group, after early algebra instruction, to correctly interpret the equal sign as a relational symbol and solve tasks that require this understanding using either computational or structural strategies. Students in the nonintervention group, on the other hand, continued to exhibit operational thinking about the equal sign and showed no evidence of the development of a relational understanding of the equal sign, a critical benchmark in the development of algebraic thinking. Moreover, the gains made by students in the intervention group in their use of a structural strategy from pretest to posttest (16% on Item 1a; 29% on Item 2b), along with the absence of the use of a structural strategy at both pretest and posttest by students who did not receive the intervention, suggest that the intervention served to increase students' ability to reason algebraically (structurally) about these equations.

Finally, we note what seems to be an inconsistency between students' strategy use on these two items. One might expect that strategy use on these items would be comparable because the items are quite similar. At first glance, Figure 3 seems to suggest that, although students in the intervention group improved on both items, they did better at posttest on Item 1a than 2b—that is, they used correct strategies more frequently for Item 1a than 2b. We think this difference is likely due to the fact that some students provided an answer of "true" or "false" for Item 2b without providing an explanation from which we could discern the strategy they used. Such responses were coded as *answer only*. For the students who did not receive the intervention, 35% gave a response of "false" with no explanation on the pretest, whereas 30% did so on the posttest. Of the intervention group, 24% gave a response of "false" with no explanation on the pretest, whereas only 3% did so on the posttest. We think it is likely that students who responded in this way used an operational strategy, reasoning that the equation is false because $57 + 22 \neq 58$. If this was in fact the case, our results across the two items are comparable.

Students' understanding of equations. Although students' understanding of the equal sign is undoubtedly a critical aspect of understanding equations, we wanted to further examine students' ability to solve simple linear equations. First, we emphasize that our intervention (and EALP more broadly) intentionally does not focus on or include instruction on procedural rules for solving equations in the elementary grades. Our goal, instead, is for children to bring together emergent understandings of the equal sign, algebraic notation, and inverse operations to reason structurally about equations. In our view, this type of thinking supports students' interpretation of quantities and expressions as objects and can strengthen their ability to solve equations by noticing underlying structure rather than relying solely on procedural rules. For example, students might solve an equation such as $3(x + 5) = 36$ by applying formal algebraic rules in which they first multiply the quantity $x + 5$ by 3. However, if they see $x + 5$ as an object, it is easier (and arguably more meaningful) to notice an underlying structure where when an indicated product results in 36 and one of the factors is 3, the remaining factor, $x + 5$, must be 12.

Figure 4 presents assessment Item 9, a description of the codes used to categorize student responses, and an example from students' written work of a response that warranted a given code.

Item 9 Find the value of n in the following equation. How did you get your answer? $3 \times n + 2 = 8$		
Strategy Code	**Description**	**Example**
Guess and Test	Student works through equation in a forward manner, substitutes value(s) in for the variable and computes to see if the value is correct. Often makes only one "guess."	$3 \times 3 + 2 = 11$ $3 \times \underline{2} + 2 = 8$
Unwind	Student works backward through constraints in the equation, inverting operations.	$8 - 2 = 6$, $6/3 = 2$

Figure 4. Coding scheme for Item 9.

Figure 5 shows pretest to posttest performance scores, in terms of percentage correct, for Item 9. The significant improvement the intervention group made in interpreting and correctly solving an equation containing a variable expression—from 13% correct at pretest to 53% correct at posttest—is noteworthy. However, Figure 6 illustrates perhaps a more important point. Thirty-nine percent of the intervention group correctly used a *guess-and-test* strategy, and 11% correctly used an *unwind* strategy at posttest; however, only 24% of the nonintervention group correctly used a *guess-and-test* strategy, and no one correctly used an *unwind* strategy. We find it significant that an *unwind* strategy occurred only with the intervention group and only at posttest because we see this strategy as a more strategic approach to solving equations in elementary grades. As Carraher and Schliemann (2007) pointed out, the process of unwinding (or undoing) is critically connected to inverse operations, so we believe it is important that some of the students in the intervention group were beginning to use this strategy intuitively, even though it was not formally taught in the intervention.

Students' understanding of algebraic expressions. We conclude our discussion of results related to the big idea of equivalence, expressions, equations, and inequalities by examining shifts in students' understanding of expressions. Students' ability to represent unknown quantities is a critical component of

algebraic thinking. This can be particularly difficult in the elementary grades because children are often unwittingly enculturated to think that "answers" come in the form of single numerical values. For example, it is difficult for young children to accept that the expression 3 + 7 might be a valid solution and, instead, will insist that the answer must be 10. This creates a challenge in later grades in working with variable expressions, where the form of a solution often necessarily includes an indicated computation (e.g., $x + 7$).

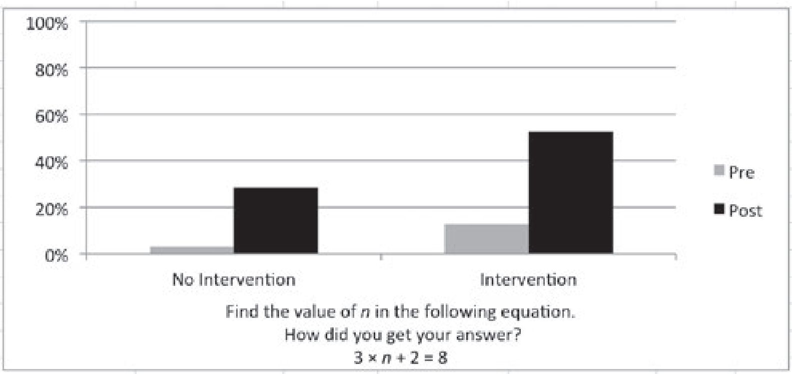


Figure 5. Percentage of students who provided correct responses on Item 9.

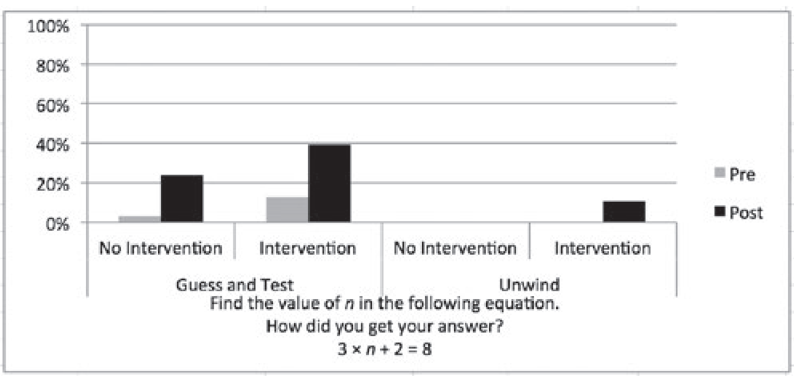


Figure 6. Percentage of students who used the indicated strategy on Item 9 to solve the equation correctly. Frequency of no response was 62% at pretest and 16% at posttest for the intervention group and 85% at pretest and 46% at posttest for the nonintervention group.

Item 7 was adapted from research-based tasks designed to elicit children's ability to represent unknown quantities with variable expressions (Carraher et al., 2008). Figure 7 provides the item, a description of the codes used to categorize student responses, and an example from students' written work of a response that warranted a given code.

<table>
<tr><td colspan="3">Item 7
Tim and Angela each have a piggy bank. They know that their piggy banks each contain the same number of pennies, but they don't know how many. Angela also has 8 pennies in her hand.</td></tr>
<tr><td colspan="3">(a) How would you describe the number of pennies Tim has?</td></tr>
<tr><td>Strategy Code</td><td>Description</td><td>Example</td></tr>
<tr><td>Letter</td><td>Student assigns a (letter) variable to this unknown.</td><td>Tim has p pennies.</td></tr>
<tr><td>Comparison</td><td>Student compares the quantities in the task.</td><td>Tim has 8 fewer than Angela.</td></tr>
<tr><td>Value</td><td>Student assigns a specific numerical value to this unknown.</td><td>Tim has 10 pennies.</td></tr>
<tr><td colspan="3">(b) How would you describe the total number of pennies Angela has?</td></tr>
<tr><td>Strategy Code</td><td>Description</td><td>Example</td></tr>
<tr><td>Letter-Related</td><td>Student writes a letter (variable) expression that is related to the variable used to represent the number of Tim's pennies in 7a.</td><td>Angela has p + 8 pennies.</td></tr>
<tr><td>Letter- New</td><td>Student assigns a letter (variable) to this unknown that is not related to the variable used to represent the number of Tim's pennies in 7a.</td><td>Angela has p pennies.</td></tr>
<tr><td>Comparison</td><td>Student compares the quantities in the task.</td><td>Angela has 8 more than Tim.</td></tr>
<tr><td>Value</td><td>Student assigns a specific numerical value to this unknown.</td><td>Angela has 18 pennies.</td></tr>
<tr><td colspan="3">(c) Angela and Tim combine all of their pennies to buy some candy. How would you describe the total number of pennies they have?</td></tr>
<tr><td>Strategy Code</td><td>Description</td><td>Example</td></tr>
<tr><td>Letter-Related</td><td>Student writes a letter (variable) expression that is the sum of responses in 7a and 7b.</td><td>They have p + p + 8 pennies.</td></tr>
<tr><td>Letter-New</td><td>Student assigns a letter (variable) that is not the sum of responses in 7a and 7b.</td><td>They have p pennies.</td></tr>
<tr><td>Value</td><td>Student assigns a specific numerical value to this unknown.</td><td>They have 28 pennies.</td></tr>
</table>

Figure 7. Coding scheme for Item 7.

Figure 8 illustrates students' performance (correctness) from pretest to posttest for Item 7. Responses in which students used variable notation appropriately (see *letter* for part a and *letter-related* for parts b and c in Figure 7) or made an accurate

statement comparing the number of pennies Tim and Angela had (for parts a and b in Figure 7) were considered correct.

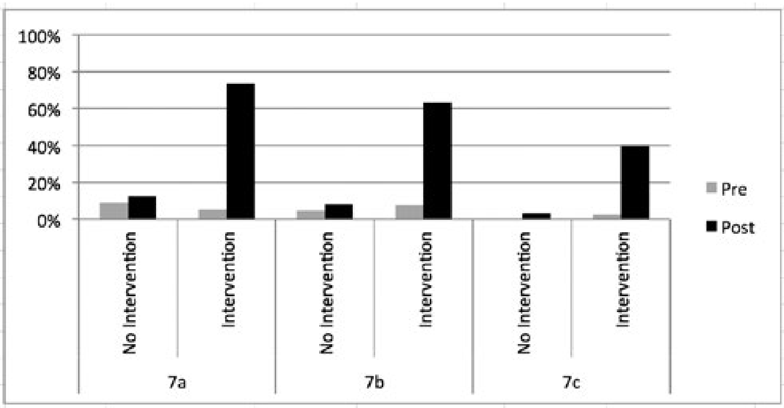


Figure 8. Percentage of students who provided correct responses on Item 7.

We gained further insight from student responses on this item by examining the strategies students used in representing unknown quantities (see Figure 9). We focused on those strategies in which students used a letter to represent a variable and those in which students assigned a particular value to the unknown quantity. As we expected with Item 7a, at pretest students in both groups chose a numerical value to represent Tim's number of pennies, although the task specifically says the quantity is not known. No student used variable notation to represent the unknown quantity at pretest. However, by posttest, no student in the intervention group assigned a specific numerical value to the quantity, and 74% used a letter to represent the quantity. Students who did not receive the intervention, however, continued to assign specific values to the unknown quantity and did not represent the quantity with a letter.

We found similar results for Items 7b and 7c. That is, at pretest, no student represented either quantity as a variable expression. Of the strategies considered here, the only strategy students used at pretest was to assign specific numerical values to the unknown quantities. By posttest, however, 74% of the intervention group used a letter to represent Angela's number of pennies in Item 7b, and 58% produced variable expressions to represent the quantity in 7c. Students who did not receive the intervention did not do this in either 7b or 7c.

What is even more notable here is that, not only did the majority of students in the intervention group represent unknown quantities with variable expressions by posttest on all of Items 7a–c, but most also connected their representations in Items 7b and 7c to their representation of Tim's number of pennies in 7a. That is, for Item 7b, 63% of the intervention group related their representation for Angela's number of pennies to that of Tim's (see the strategy *letter-related* in Figure 7). In

other words, these students understood that if *b* represented Tim's number of pennies, then Angela's number of pennies could best be represented by $b + 8$. Similarly, 39% of the intervention group related their representation in 7c to those in Items 7a and 7b. For example, if students represented Tim's number of pennies as *b* in 7a, then these students might represent the combined number of pennies for Tim and Angela as $b + b + 8$ in Item 7c. We take this to indicate that these students were using variable notation with some level of understanding, not just in a rote or meaningless fashion.

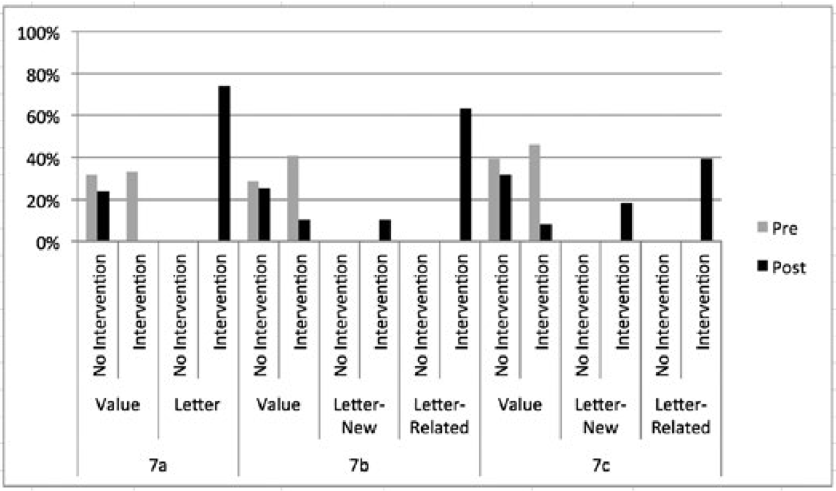


Figure 9. Percentage of students who used the indicated strategy on Item 7. Frequency of no response for Item 7a was 54% at pretest and 5% at posttest for the intervention group and 47% at pretest and 49% at posttest for the nonintervention group. Frequency of no response for Item 7b was 41% at pretest and 5% at posttest for the intervention group and 56% at pretest and 44% at posttest for the nonintervention group. Frequency of no response for Item 7c was 38% at pretest and 11% at posttest for the intervention group and 52% at pretest and 46% at posttest for the nonintervention group.

Generalized arithmetic. One of the core strands of early algebra is generalized arithmetic, the study of which allows students to deepen their arithmetic understanding by noticing and representing regularity and structure in their operations on numbers as well as justifying and reasoning with these generalizations. Although our EALP addresses arithmetic generalizations more broadly, the assessment items specifically addressed shifts in students' understanding of generalizations about fundamental properties of number and operation (e.g., the Commutative Property of Addition).

Figure 10 provides Item 2c, a description of the codes used to categorize student responses, and an example from students' written work of a response that warranted a given code. This item was primarily intended to elicit children's

understanding of the equal sign. However, it was also designed to reflect the Commutative Property of Addition so that we could examine whether students noticed this structure in reasoning about the equation.

As Figure 11 illustrates, students in the intervention group exhibited a 74% gain from pretest to posttest in terms of correctly responding "true," whereas students who did not receive the intervention exhibited only an 8% gain. Of interest here from a generalized arithmetic perspective, however, is whether students provided justifications that suggested they analyzed the structure of the equation and evoked the Commutative Property of Addition as the basis for their justification. In this respect, Figure 12 shows clear differences in strategies used by each of the student groups. In particular, at pretest, 50% of students who received the intervention and 29% of those who did not used an (incorrect) operational strategy, whereas no students from either group used a structural or computational strategy. This continued at posttest for students who did not receive the intervention, with 46% using an operational strategy and only 5% using either structural or computational strategies. However, 77% of the intervention group used either a computational or structural strategy at posttest, and only 3% used an operational strategy.

Item 2c
Circle True or False and explain your choice.
39 + 121 = 121 + 39 True False How do you know?

Strategy Code	Description	Example
Structural	Student notices the equation illustrates the underlying structure of the Commutative Property of Addition and determines equivalence without computing.	*True, because 121 + 39 is just 39 + 121 in reverse.*
Computational	Student computes to determine if the two sides of the equation have the same sum.	*True, because they both equal 160.*
Operational	Student shows an operational understanding of the equal sign, adding the numbers on the left side or adding all of the numbers given in the equation.	*False because I added 39 + 121 and it does not equal 121.*

Figure 10. Coding scheme for Item 2c.

What is interesting here is that, although neither group of students used a structural strategy at pretest, by posttest 66% of the intervention group used this strategy. We emphasize that in using a structural strategy, students evoked some form of the Commutative Property of Addition as the basis for their reasoning (see Figure 10). Results on this item, therefore, support our claim that the intervention

helped students think about equations algebraically, rather than just arithmetically, because they developed arguments that were predominantly based on their recognition of a generalized property of arithmetic at play. Students in the nonintervention group rarely developed such arguments (3% at posttest).

As was the case with Item 2b (discussed previously; see Figure 3), some students provided an answer of "true" or "false" without providing an explanation from which we could discern the strategy they used. In the intervention group, this was the case for 37% of students at pretest and 5% at posttest. Among students who did not receive the intervention, this was the case for 42% of students at pretest

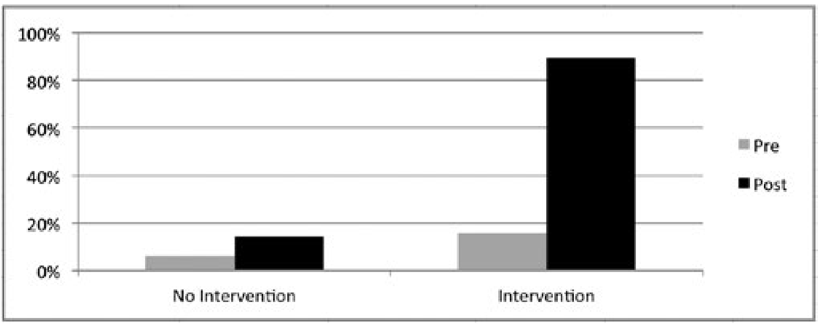


Figure 11. Percentage of students who provided correct responses on Item 2c.

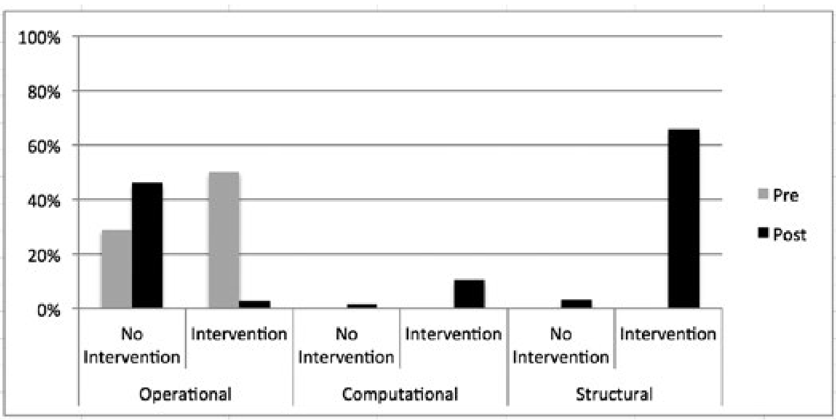


Figure 12. Percentage of students who used the indicated strategy on Item 2c. Frequency of no response was 8% at pretest and 3% at posttest for the intervention group and 17% at pretest and 5% at posttest for the nonintervention group.

and 35% of students at posttest. We think it is likely that students who responded that the equation was false (the vast majority of those receiving an *answer only* code) used an operational strategy, reasoning that the equation is false because $39 + 121 \neq 121$. Students in the intervention group essentially no longer used this reasoning on the posttest, whereas students in the nonintervention group did.

A second assessment item from the big idea of generalized arithmetic was likewise designed to ascertain whether students noticed a fundamental property as an underlying structure in an arithmetic task and, additionally, whether they believed the structure they noticed would hold for all numbers. Figure 13 provides assessment Item 4, a description of the codes used to categorize student responses, and an example from students' written work of a response that warranted a given code.

Item 4 Marcy's teacher asks her to figure out "23 + 15." She adds the two numbers and gets 38. The teacher then asks her to figure out "15 + 23." Marcy already knows the answer. a) How does she know? b) Do you think this will work for all numbers? If so, how do you know?		
Strategy Code	**Description**	**Example**
Structural	Student names the "Commutative Property of Addition" or otherwise describes the relationship in words.	*She knows it because it is the same as 23 + 15, she just switched around the numbers.*
Computational	Student computes each sum separately.	*23 + 15 = 38 and 15 + 23 = 38.*

Figure 13. Coding scheme for Item 4.

We focus here on the two issues mentioned above that Item 4 addresses: (a) as with Item 2c, whether students noticed an underlying structure as a basis for their argument about Marcy's thinking and (b) whether students understood that a generalization might hold across a broader domain of numbers, not just a particular instance. Figure 14 shows the types of strategies students used for Items 4a and 4b. Any response indicating that students noticed the underlying structure of the Commutative Property of Addition was coded as *structural*. These included responses where students cited the Commutative Property by name, referred to the property as a "turn-around fact" (typical language in elementary classrooms in the participating district), described the property in words, or noticed that the equation had the "same numbers" on either side. For the response "same numbers," we think it is a fair assumption that students did not disregard the operation. Although we could not confirm this from their written responses, our participants did not have experiences in our intervention with any other operations (−, ×, or ÷) in which they explored whether the operation was commutative. Moreover, they had not formally

studied the operations of multiplication or division at this point in their regular curriculum.

The results shown in Figure 14 suggest that by posttest, students in the intervention group used a structural approach in their arguments more frequently than a computational approach. In particular, 74% of students in the intervention group used a structural argument for Item 4a, and 55% used one for Item 4b. In the nonintervention group, only 59% of students used a structural argument for Item 4a, and 29% used one for Item 4b. These results, combined with those of Item 2c, suggest that the intervention had a positive effect on students' ability to notice underlying structure in equations and operations on numbers.

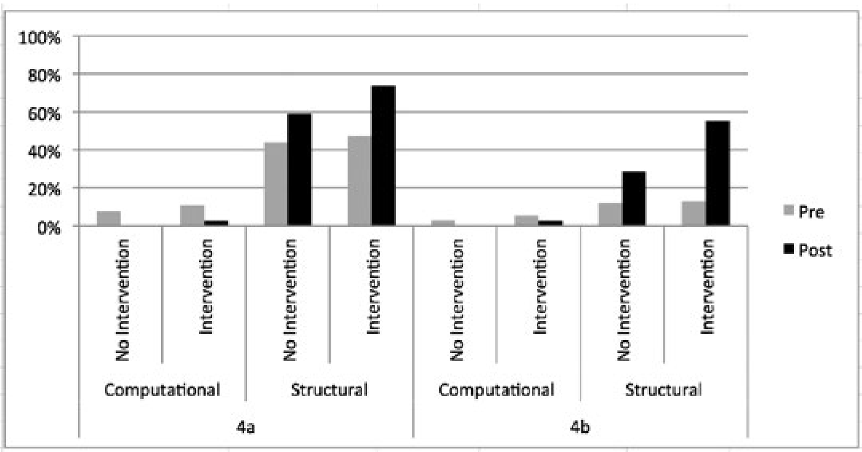


Figure 14. Percentage of students who used the indicated strategy on Item 4. Frequency of no response for Item 4a was 13% at pretest and 11% at posttest for the intervention group and 11% at both pretest and posttest for the nonintervention group. Frequency of no response for Item 4b was 42% at pretest and 16% at posttest for the intervention group and 52% at pretest and 44% at posttest for the nonintervention group.

Students' responses to the first question posed in Item 4b—"Do you think this will work for all numbers?"—provides insight into their understanding that a generalization might hold across a large domain of numbers, not just a single instance. Figure 15 shows that at pretest, responses from both student groups were comparable: 37% of students who received the intervention and 33% of those who did not thought that Marcy's thinking would work for all numbers. At posttest, however, 74% of the intervention group thought Marcy's thinking would hold for all numbers, and only 11% thought it would not hold, suggesting that the intervention helped students develop an ability to think beyond particular instances about generalizations across broad domains of numbers. There was essentially no change in thinking for students who did not receive the intervention: Only 33% at pretest and 35% at posttest believed Marcy's thinking would hold for all numbers.

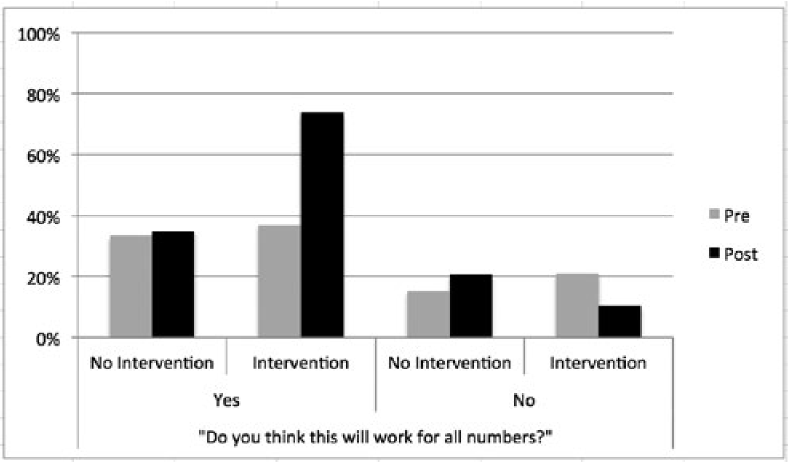
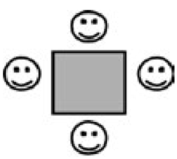


Figure 15. Percentage of students who provided the indicated responses on Item 4b.

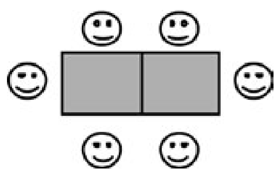
Item 10

Brady is having his friends over for a birthday party. He wants to make sure he has a seat for everyone. He has square tables.

He can seat 4 people at one square table in the following way:



If he joins another square table to the first one, he can seat 6 people:



(a) If Brady keeps joining square tables in this way, how many people can sit at 3 tables? 4 tables? 5 tables? Record your responses in the table below and fill in any missing information:

Number of tables	Number of people
1	
2	
3	
4	
5	
6	
7	

Figure 16. Coding scheme for Item 10 (*continued on page 67*).

(b) Do you see any patterns in the table? Describe them. (c) Find a rule that describes the relationship between the number of tables and the number of people who can sit at the tables. Describe your rule in words. (d) Describe your relationship using variables. What do your variables represent?		
Strategy code	**Description**	**Example**
Recursive	Student describes a recursive pattern.	*I see plus two going down the people side.*
Covariational	Student describes the covariational relationship between the number of tables and the number of people.	*Each table you add adds two people.*
Functional rule		
in words	Student describes the correspondence rule relating the number of tables and the number of people in words.	*Number of tables times two plus two equals number of people.*
in symbols	Student describes the correspondence rule relating the number of tables and the number of people symbolically.	$A \times 2 + 2 = B$; *A for the number of tables, B for the number of people*
(e) If Brady has 10 tables, how many people can he seat? Show how you got your answer.		
Drawing	Student draws tables and counts the number of people who can be seated.	[*Student draws 10 tables and counts 22 spaces for people to sit*]
Recursive	Student extends the pattern found in the table to 10 tables.	[*Student extends table from 10a until reaching entry (10, 22)*]
Functional rule	Student uses the functional rule to find a solution.	$2 \times 10 + 2 = 22$ *people*

Figure 16. Coding scheme for Item 10 (*continued*).

Finally, Item 6 (see Appendix B) was a multiple-choice item designed to see if students would notice the underlying structure of a fundamental property and identify a symbolic representation of the property. This item is based on the Additive Inverse Property and required students to select a symbolic representation to correspond to a child's (Evelyn's) thinking described in words. Students in the intervention and nonintervention groups were fairly comparable at pretest with 55% and 41%, respectively, choosing the correct symbolic representation. However, by posttest, 89% of the intervention group identified the correct representation, whereas only 57% of the nonintervention group did. In addition to providing information

about students' ability to recognize an underlying fundamental property, results from this item also suggest that students in the intervention group were better able to interpret an equation containing variable expressions and connect this representation to its natural language counterpart at posttest than at pretest. This type of representational fluency—the ability to navigate between different representations of a mathematical generalization—is another foundational aspect of algebraic thinking (Brizuela & Earnest, 2008; Kaput, 2008).

Functional thinking. We conclude by examining students' performance and strategy use on an item designed to explore their understanding of functional relationships. Figure 16 (on pp. 66–67) provides Item 10, a description of the codes used to categorize student responses for 10b–e, and an example from students' written work of a response that warranted a given code.

Figure 17 shows students' performance from pretest to posttest for Item 10. As noted earlier, gains in performance on each part of Item 10 were significantly higher for the intervention group than for the nonintervention group, with the exception of 10e. Gains on 10e for the intervention group were higher, but not significantly higher, than those for the nonintervention group. We think this exception might be partly explained by the fact that Item 10e could be answered arithmetically or without knowing the function rule. For example, because students were asked to find a relatively near function value (the number of people that could be seated at 10 tables), they might easily draw 10 tables at which Brady's friends might be seated and count the number of seats or extend the table given in 10a until reaching 10 tables. Choosing a large value for which students might not use these methods (for example, 200 tables) might provide more insight into children's algebraic thinking when predicting far function values. What we do know from student responses is that, although some students who did not receive the intervention were able to use strategies such as drawing tables or extending the recursive pattern in the table in Item 10e (14% and 19%, respectively), no student in this group was able to use a function rule to find the value. However, 8% of students in the intervention group used a function rule to find the value in Item 10e.

The posttest performance on Items 10a and 10b for students in the intervention group might be anticipated, given that completing tables and finding recursive patterns have been advocated as attainable national standards for upper elementary grades (NCTM, 2000; NGA & CCSSO, 2010). That is, given appropriate instruction, it is reasonable that students might improve significantly in their ability to meet such standards, as students in our intervention group did. We therefore find their performance on Items 10c and 10d, which required them to identify a relationship between two covarying quantities and describe this relationship in words and variable notation, more noteworthy. To understand this further, it is helpful to examine the strategies both groups of students used in responding to these items.

A major goal of Item 10, captured by 10c and 10d, was to examine whether students were able to identify and represent relationships between covarying quantities in either words or variable notation. Although we coded a covariational

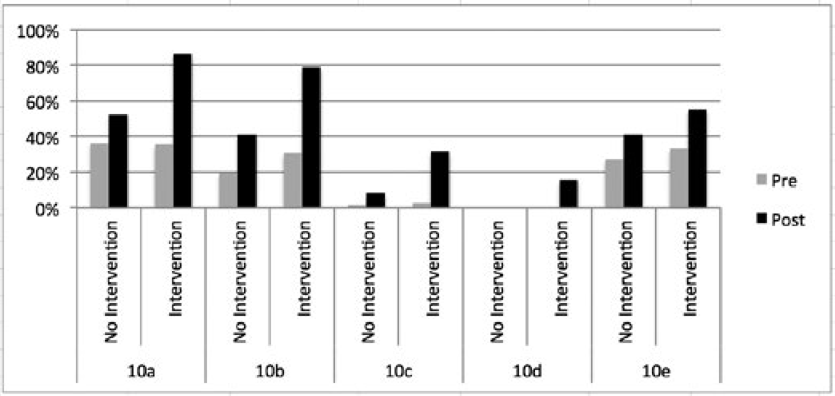


Figure 17. Percentage of students who provided correct responses on Item 10.

relationship (e.g., "every time you add one more table, you add two more people") as correct for Item 10c because it accounts for how two quantities covary, we did not code a recursive pattern such as "add 2" correct because it only takes the growth of one quantity (such as the number of people) into account. Figure 18 shows that at pretest, only 3% of students from the intervention group and 2% of students from the nonintervention group could describe a covariational relationship (Item 10c). However, at posttest, 24% of the intervention group could provide a covariational relationship, whereas only 8% of the nonintervention group could. More importantly, although neither group described a function rule in words or variables at pretest, by posttest, 8% of the intervention group represented a function rule in words (Item 10c), and 16% did so with variable notation (Item 10d). None of the students in the nonintervention group did this at posttest.

It is also noteworthy to us that more students in the intervention group were able to represent their function rule with variable notation (16%) than in words (8%). We view this as important because it is sometimes presumed that young children cannot understand variable notation and can only be expected to use natural language to describe mathematical generalizations, including functional relationships. However, at least with this item, the students in the intervention group were more successful representing the generalization with variable notation than with words.

Although results of Item 10 suggest that, overall, the intervention had a statistically significant effect on the ability of students in the intervention group to identify and represent generalizations about covarying quantities, results on this item are more modest than those on the other items discussed here. We suggest several reasons for why this might be the case. First, with numerous parts and a problem context that required (for third-grade students) a significant amount of reading comprehension, Item 10 was the most detailed and time-consuming problem for students. Moreover, we addressed functional thinking at the end of our intervention (approximately the last 30% of lessons), so students' experiences with ideas about covarying relationships were relatively brief in comparison to other concepts we addressed. For example,

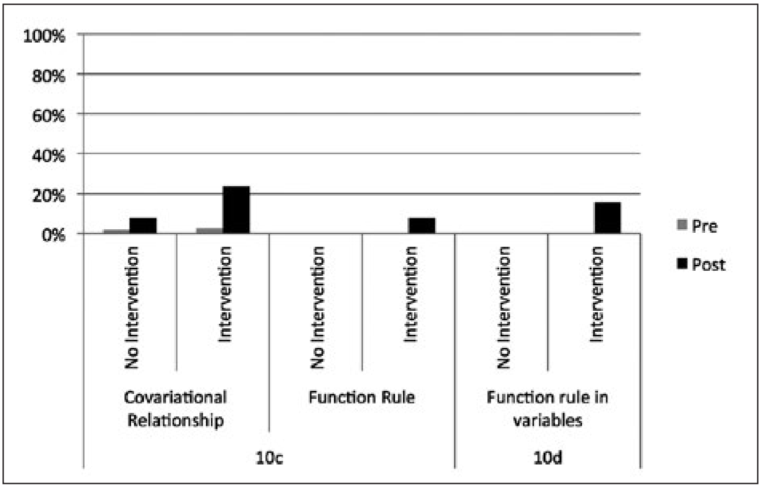


Figure 18. Percentage of students who used the indicated strategy on Items 10c and 10d. Frequency of *no response* for Item 10c was 62% at pretest and 8% at posttest for the intervention group and 79% at pretest and 51% at posttest for the nonintervention group. Frequency of *no response* for Item 10d was 95% at pretest and 13% at posttest for the intervention group and 91% at pretest and 83% at posttest for the nonintervention group.

concepts such as a relational understanding of the equal sign and generalizing fundamental properties of number and operation were addressed at the beginning of the intervention and revisited throughout the year. Thus, for other big ideas (e.g., generalized arithmetic), students had more time to develop an understanding of the associated concepts. Finally, the concepts associated with functional thinking are themselves further removed from typical tasks in elementary grades and were, in our view, more complex than the concepts addressed by other items in our assessment.

Students' difficulties with the concept of function at even middle and secondary grades is well documented (Artigue, 1992; Confrey & Smith, 1995; Schoenfeld, Smith, & Arcavi, 1993; Sfard, 1992) and includes their limited understanding of and flexibility with representations such as variable notation (Freudenthal, 1982; Usiskin, 1988), tables, equations, and verbal descriptions (Goldenberg, 1988; Knuth, 2000; Moschkovich, Schoenfeld, & Arcavi, 1993). Thus, for the intervention group, we find gains on Item 10 particularly impressive and consistent with findings from other early algebra research regarding children's understanding of functions (Blanton et al., 2014; Brizuela, Blanton, Sawrey, Newman-Owens, & Gardiner, in press; Moss et al., 2008), suggesting that exposure to these concepts in elementary grades is warranted.

Discussion

In this study, we found that students who experienced our early algebra intervention demonstrated statistically significantly greater improvements in their performance on an assessment designed to measure various aspects of their understanding

of algebraic concepts and practices than did students who did not receive the intervention. In particular, we found that students in the intervention group improved in their ability to:

- Think relationally about the equal sign,
- Represent unknown quantities in meaningful ways with variable notation,
- Recognize the underlying structure of fundamental properties in equations and use this to justify their thinking,
- Think beyond particular instances to consider whether generalizations were true across a broad domain of numbers,
- Both produce and comprehend variable representations of generalized claims, and
- Generalize and symbolically represent functional relationships between covarying quantities.

It might not be surprising that students who received the early algebra intervention performed as they did. However, the performance of these students highlights a particularly compelling aspect of our findings about early algebra: As early as Grade 3, children can successfully develop critical components of algebraic thinking skills that are foundational to the successful study of algebra in the later grades. We also found that students who experienced a business-as-usual approach to elementary school mathematics demonstrated very little gain in their performance on the early algebra assessment and, in fact, seemed to learn very little algebra overall. The performance of these students highlights another compelling aspect of our findings: Typical arithmetic-based elementary school mathematics curricula and instruction does little to prepare students for the successful study of algebra in the later grades.

We interpret these findings not only as an indication that the intervention was appropriate for the third-grade students in our study, but also as a strong indicator regarding children's potential for learning algebra. In particular, the substantial gains students made during 1 year of an early algebra intervention provide compelling evidence that a comprehensive, longitudinal early algebra experience is likely to significantly affect children's algebra understanding in the elementary grades. Moreover, our findings suggest that an early algebra education can potentially ameliorate some of the difficulties students have with algebra in later grades. As Carraher and Schliemann (2007) pointed out, many of the difficulties students have with algebra—and that our intervention addressed—are based on conceptions that include an operational view of the equal sign (e.g., Kieran, 1981), a focus on particular (numerical) answers rather than generalized claims (Booth, 1984), a lack of recognition of fundamental properties of number and operation at play in arithmetic (MacGregor, 1996), and a lack of understanding of variable notation as a way to represent relationships between quantities (Bednarz, 2001).

In what follows, we discuss two particular aspects of our findings regarding children's ability to think structurally and to use variable notation—characteristics of algebraic thinking that are critical to success in algebra in later grades.

Development of Algebraic Thinking Practices

An analysis of the shifts in strategies students used from pretest to posttest showed that students who received the intervention—unlike those who did not—increasingly engaged in algebraic thinking. For example, as discussed earlier, although neither student group interpreted the equations in Items 1a and 2b in a structural way at pretest, students in the intervention group began to do so by posttest, using a structural approach to find a missing value and to reason about the truth of an equation. By posttest, some students in the intervention group—unlike the students in the nonintervention group—also began to use an unwind strategy as a way to solve equations (Item 9). In addition, they made gains in their ability to recognize the underlying structure of fundamental properties and to use this as a basis for justifying a generalization for a class of numbers (Items 2c, 4, and 6).

The ability to think structurally is an important aspect of algebraic thinking (Kieran, 2007; Knuth et al., 2006; Pimm, 1995), yet students often have difficulty with this type of thinking in middle school and beyond. Apart from early algebra instruction, this type of thinking is rarely cultivated in elementary grades. However, results from the study reported here suggest that not only did students in the intervention group make significant gains in correctly solving algebra items, they also exhibited an emerging ability to use algebraic approaches—to think more structurally—in their problem solving.

Booth (1989) argued that algebraic transformations—a significant part of algebra in middle and secondary grades—require "that we first understand the structural properties of mathematical operations and relations which distinguish allowable transformations from those that are not" (pp. 57–58). In our view, a starting point for this understanding entails the emergent forms of thinking exhibited by students in the intervention group in this study whereby, for example, they noticed structural forms of fundamental properties implicit in equations or noticed—without the use of arithmetic—how transforming one quantity affected another equivalent quantity. That third-grade students exhibited a clear development in their ability to think structurally about algebraic tasks after a 1-year intervention (with only about 20 hours of instruction) suggests that a sustained, multiyear approach to algebra instruction beginning in the elementary grades could have significant effects on students' ability to generalize, represent, justify, and reason with mathematical structure—all fundamental practices of algebraic thinking (Kaput, 2008).

Children's Use of Variable Notation

Another critical component of the assessment items discussed here—and a distinct big idea included in our EALP—deals with children's understanding of variable and use of variable notation. Algebra is fundamentally linked to a "mathematical language that combines operations, variables, and numbers to express mathematical structure and relationships in succinct forms" (Blanton et al., 2011, p. 67). Understanding variable and variable notation is, therefore, central to understanding algebra.

Research on adolescents highlights their inability to use variable notation to represent quantities or relationships between them (Bednarz, 2001; McNeil et al., 2010; Vergnaud, 1985). However, we found that at posttest, students in the intervention group were able to represent their ideas using variable notation across all algebraic contexts addressed in the intervention. In particular, they were able to produce meaningful variable representations to model multistep linear problem situations (Item 7) and to interpret and solve linear equations containing variable expressions (Item 9). It is well documented that an unintended result of arithmetic-focused instruction is that children have difficulty accepting the ambiguity of unknown quantities and, instead, have a deeply held tendency to assign specific numerical values to unknown quantities even if it violates the problem situation (e.g., Carraher et al., 2008; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). However, we found that not only were students in the intervention group able to correctly represent unknown quantities with variable notation, but they were also able to connect their choice of notation across problem situations to represent different unknown quantities in meaningful ways (Item 7). We also found that they were able to navigate between representations of arithmetic generalizations using both natural language and variable notation (Item 6). Finally, not only were students in the intervention group able to identify functional relationships between two quantities, but they also used variable notation more frequently than words to represent these relationships (Item 10).

Although historically educators have been tentative about the use of variable notation in the elementary grades, we find that early algebra research in general, and the study reported here in particular, does not warrant withholding such notation as a linguistic tool for children. Our findings are consistent with other recent findings in the elementary grades (e.g., Brizuela et al., in press; Dougherty, 2008) that suggest young children are far more capable of using variable notation as a representational tool than previously thought; therefore, introducing these ideas in the elementary grades holds promise for addressing the documented difficulties older students have with variable notation. As Carraher and Schliemann (2007) reported, "Some have argued that algebraic notation makes it easier for adults *as well as young learners* [emphasis added] to give expression to . . . mathematical generalizations" (p. 671). Indeed, as we noted earlier, with the most complex ideas encountered in our intervention (functional relationships), students used variable notation more frequently than words to represent their thinking.

Directions for Further Research

Early algebra research has done much to clarify our understanding of how children think algebraically. Thus, we are now in a position to begin to integrate findings from different programs of early algebra research to examine the impact of a comprehensive early algebra education. In this article, we reported results of a 1-year, early algebra intervention in third grade. An important feature of our work is that it is based on a learning progressions approach (e.g., Stevens, Shin,

& Krajcik, 2009) by which we can draw from and build on empirical research through a feedback mechanism that keeps the instructional content close to children's mathematical (algebraic) thinking. A second important feature is that it intentionally integrates several big ideas of early algebra into a comprehensive, sustained intervention. Much of early algebra research has necessarily focused on one big idea in isolation in order to detail the nuances of how children think about a specific concept (e.g., mathematical equivalence). Our broader goal has been to pull these findings together into a curricular progression that is accessible to students and that can be shown to positively affect their algebra understanding.

We appreciate the need to more tightly define the conditions under which an early algebra intervention occurs to strengthen the argument that the intervention itself explains growth in children's algebraic thinking. Future studies that include the random assignment of groups to instructional conditions and that closely monitor comparison group instructional practices can further clarify our understanding of the effect of the early algebra intervention. We view the study reported here as an essential first step in our broader objective of understanding the effect of early algebra through experimental design research. In our view, and because early algebra itself is not business as usual in elementary school mathematics, efficacy studies such as the one reported here are prerequisites for the significantly more resource-intensive randomized studies required to more fully understand the effect of early algebra on student understanding.

We also acknowledge again that the intervention used in our study—like the EALP on which it is based—represents one approach to early algebra. We do not claim that the instructional sequence we developed is the only possible sequence nor that the way we parsed algebraic content into big ideas is unique. We also appreciate that, as is characteristic of learning progressions research, a different sequence could lead to somewhat different results.

Moreover, in focusing here on crucial quantitative data, we necessarily miss other important parts of the picture, parts that might further detail how children thought about the concepts addressed in our intervention. Future studies that examine the nature of students' algebraic thinking through qualitative methods such as in-depth interviews, used in conjunction with written assessments of growth in students' thinking, can potentially shed more light onto findings gleaned from written responses. We can report, however, that our observations of the intervention group during classroom instruction were consistent with these students' performance on the pretest and posttest (for a description of students' engagement with concepts addressed in our intervention, see Isler et al., 2014/2015; Marum et al., 2011; Stephens et al., 2012). That is, overall we felt that our intervention did appropriately target third-grade students, and we made only minor modifications to our instructional plan and assessment instrument as a result of our intervention.

The duration and grade level of our intervention—1 year, third grade—necessarily raises the question of how students' thinking might develop in other grades or to what extent their thinking might develop given a longer intervention. To this

end, we are currently conducting a 3-year longitudinal study that follows a cohort of students from the start of third grade to the end of fifth grade. We expect this study to contribute to our understanding of how children's algebraic thinking emerges across an extended span of time and how they are able to transition into a more formal study of algebra.

We also recognize the particular context in which our study took place. A member of our research team—recently, an elementary teacher herself—taught the intervention. It might be suggested that our teacher–researcher provided instruction that "looked like" the assessment and biased the outcomes. Although our assessment was intentionally aligned with our early algebra lessons and EALP, it is important to note that students' only exposure to the assessment items was at pretest and posttest.

This raises a further question about the ability of classroom teachers to deliver this kind of early algebra instruction. Although elementary teachers are in the critical path to algebra reform, they have not been provided with adequate experiences to teach the rich and connected kinds of algebraic thinking that constitute early algebra. Instead, they are more likely to have experienced the type of algebra instruction we need to replace (e.g., Greenberg & Walsh, 2008; Kaput, 1999; Kaput & Blanton, 2005). This situation, coupled with the fact that elementary teachers often have deep anxieties about mathematics (Battista, 1986; Haycock, 2001; Levine, 1995), particularly algebra, can interfere with their capacity to teach new, more complex content. To address this, we are currently examining the impact of a comprehensive, sustained, teacher-led intervention on children's algebraic thinking.

Another question that warrants closer inspection is: How does early algebra education influence students' understanding of arithmetic? As Booth (1988) argued, "the difficulties that students experience in algebra are not so much difficulties in algebra itself as problems in arithmetic that remain uncorrected" (p. 29, as cited in Carraher & Schliemann, 2007, p. 675). Because early algebra intentionally builds students' understanding of such fundamental concepts as relationships between number and operations, quantities and their equivalence, and relationships between quantities that covary—all within problem-based tasks that rely heavily on operations on numbers—it stands to reason that early algebra education not only would improve children's algebraic thinking but also could strengthen their understanding of arithmetic as well. Although this is consistent with our observation that children develop increasing fluency with arithmetic in the service of solving algebra tasks, the question of how we characterize and measure the relationship between the development of children's algebraic understanding and their arithmetic understanding needs to be examined further.

Conclusion

The overarching results of our study indicate that (a) elementary students are capable of engaging in rather sophisticated algebraic reasoning that has traditionally been delayed until middle school or later and (b) typical elementary mathematics

curricula and instruction may not adequately prepare students to successfully navigate the significant transition from the concrete, arithmetic reasoning of elementary school to the increasingly complex, abstract algebraic reasoning required for middle school and beyond. We believe these results are important for researchers, teachers, curriculum developers, and policy makers to consider.

Our finding that children are capable of thinking algebraically across a broad and diverse set of big ideas—from generalized arithmetic to functional thinking—highlights the feasibility of a comprehensive approach to early algebra education. We are concerned that a more narrow interpretation of early algebra in light of recent national content standards (NGA & CCSSO, 2010) might constrain the role of algebra in the elementary grades as a subset of operations, with an exclusive focus on concepts perhaps more closely associated with generalized arithmetic. Generalized arithmetic is a critical aspect of early algebra, but it is also important for students to gain exposure to a comprehensive early algebra approach that includes all content areas where algebraic thinking practices can occur. For example, the concept of variable is rich and diverse and takes on different meanings in different contexts—including as a varying quantity in covariational relationships, as a generalized number in generalizations about fundamental properties of number and operation, or as a fixed unknown in problem situations involving linear equations (Blanton et al., 2011). In our view, children need experiences with variable and variable notation in all of these areas so that, over time, they develop a rich and nuanced understanding of variable and are able to use variable notation to represent and reason with mathematical generalizations across multiple algebraic domains. Moreover, children's study of functional thinking in the elementary grades, to the depth described here, can support their later engagement with crucial content standards regarding functional thinking in middle grades. A narrow interpretation of algebra in the elementary grades as a subset of operations could hinder this.

In contrast, however, the Mathematical Practices outlined in the *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010) support a comprehensive approach to early algebra—such as that advocated here—that goes beyond operations on numbers. In fact, we argue that a comprehensive approach to early algebra can be a rich opportunity for elementary schools to develop these mathematical practices. Thus, we see the work of early algebra as both synergistic with the development of children's mathematical practices and a promising opportunity of the *Common Core State Standards for Mathematics*.

Because it still remains to be seen how early algebra will be interpreted in educational settings—either in the daily work of classroom instruction or in the more formal practices of standardized testing—as a result of initiatives such as the *Common Core State Standards for Mathematics*, research such as that reported here is urgently needed in early algebra. Although there are still important open questions regarding how children think algebraically about specific concepts (Carraher & Schliemann, 2007), we are in an educo-political environment where remnants of the traditional "arithmetic-then-algebra" approach still threaten the

development of children's mathematical thinking across a diverse range of big algebraic ideas, beginning in kindergarten. In our view, it is critical that we continue to document not only that early algebra is possible, but also that it positively affects children's algebra readiness in middle grades. We see this study as a contribution to that effort.

References

Artigue, M. (1992). Functions from an algebraic and graphic point of view: Cognitive difficulties and teaching practices. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 109–132). Washington, DC: Mathematical Association of America.

Bastable, V., & Schifter, D. (2008). Classroom stories: Examples of elementary students engaged in early algebra. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 165–184). New York, NY: Lawrence Erlbaum Associates.

Battista, M. T. (1986). The relationship of mathematics anxiety and mathematical knowledge to the learning of mathematical pedagogy by preservice elementary teachers. *School Science and Mathematics, 86*(1), 10–19. doi:10.1111/j.1949-8594.1986.tb11580.x

Bednarz, N. (2001). A problem-solving approach to algebra: Accounting for the reasonings and notations developed by students. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra: Proceedings of the 12th ICMI Study Conference* (Vol. 1, pp. 69–78). Victoria, Australia: University of Melbourne.

Blanton, M. L. (2008). *Algebra in elementary classrooms: Transforming thinking, transforming practice*. Portsmouth, NH: Heinemann.

Blanton, M., Brizuela, B., Gardiner, A., Sawrey, K., & Newman-Owens, A. (in press). A learning trajectory in six-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education*.

Blanton, M., Levi, L., Crites, T., & Dougherty, B. (2011). Developing essential understanding of algebraic thinking for teaching mathematics in grades 3–5. In R. M. Zbiek (Series Ed.), *Essential understanding series*. Reston, VA: National Council of Teachers of Mathematics.

Blanton, M., Stephens, A., Knuth, E., Gardiner, A., & Isler, I. (2014). *Progressions of learning in children's algebraic thinking.* Manuscript in preparation.

Booth, L. R. (1984). *Algebra: Children's strategies and errors*. Windsor, United Kingdom: NFER-NELSON.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K–12: 1988 yearbook* (pp. 20–32). Reston, VA: National Council of Teachers of Mathematics.

Booth, L. R. (1989). A question of structure or a reaction to: "The early learning of algebra: A structural perspective." In S. Wagner & C. Kieran (Eds.), *Research agenda for mathematics education: Research issues in the learning and teaching of algebra* (Vol. 4, pp. 57–59). Reston, VA: National Council of Teachers of Mathematics.

Britt, M. S., & Irwin, K. C. (2008). Algebraic thinking with and without algebraic representation: A three-year longitudinal study. *ZDM, 40*(1), 39–53. doi:10.1007/s11858-007-0064-x

Brizuela, B., Blanton, M., Sawrey, K., Newman-Owens, A., & Gardiner, A. (in press). Children's use of variable notation to represent their algebraic ideas. *Mathematical Thinking and Learning.*

Brizuela, B. M., & Earnest, D. (2008). Multiple notational systems and algebraic understandings: The case of the "best deal" problem. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 273–301). New York, NY: Lawrence Erlbaum Associates.

Cai, J., & Knuth, E. (Eds.). (2011). *Early algebraization: A global dialogue from multiple perspectives*. Heidelberg, Germany: Springer. doi:10.1007/978-3-642-17735-4

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Carpenter, T. P., Levi, L., Berman, P. W., & Pligge, M. (2005). Developing algebraic reasoning in the elementary school. In T. A. Romberg, T. P. Carpenter, & F. Dremock (Eds.), *Understanding mathematics and science matters* (pp. 81–98). Mahwah, NJ: Lawrence Erlbaum Associates.

Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 669–705). Charlotte, NC: Information Age.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education, 37*(2), 87–115.

Carraher, D. W., Schliemann, A. D., & Schwartz, J. L. (2008). Early algebra is not the same as algebra early. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). New York, NY: Lawrence Erlbaum Associates.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66–86. doi:10.2307/749228

Cooper, T. J., & Warren, E. (2011). Years 2 to 6 students' ability to generalise: Models, representations and theory. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 187–214). Heidelberg, Germany: Springer. doi:10.1007/978-3-642-17735-4_12

Dougherty, B. J. (2003). Voyaging from theory to practice in learning: Measure up. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the Joint Meeting of the 27th Conference of the International Group for the Psychology of Mathematics Education and 25th Conference of PME-NA* (Vol. 1, pp. 17–23). Honolulu, HI: University of Hawaii.

Dougherty, B. (2008). Measure up: A quantitative view of early algebra. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 389–412). New York, NY: Lawrence Erlbaum Associates.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6*(4), 232–236.

Freudenthal, H. (1982). Variables and functions. In G. V. Barnveld & H. Krabbendam (Eds.), *Proceedings of the Conference on Functions* (pp. 7–20). Enschede, the Netherlands: National Institute for Curriculum Development.

Goldenberg, E. P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representations of functions. *Journal of Mathematical Behavior, 7*(2), 135–173.

Greenberg, J., & Walsh, K. (2008, June). No common denominator: The preparation of elementary teachers in mathematics by America's education schools. Washington, DC: National Council on Teacher Quality. Retrieved from http://www.nctq.org/dmsView/No_Common_Denominator_NCTQ_Report

Haycock, K. (2001). Closing the achievement gap. *Educational Leadership, 58*(6), 6–11.

Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B., Garnier, H., Smith, M., . . . Gallimore, R. (2005). Mathematics teaching in the United States today (and tomorrow): Results from the TIMSS 1999 video study. *Educational Evaluation and Policy Analysis, 27*(2), 111–132. doi:10.3102/01623737027002111

Irons, C. (2003). *Growing with mathematics*. Guilford, CT: Wright Group/McGraw Hill.

Isler, I., Marum, T., Stephens, A., Blanton, M., Knuth, E., & Gardiner, A. (2014/2015). The string task: Not just for high school. *Teaching Children Mathematics*, *21*(5), 282–292.

Kaput, J. J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K–12 curriculum. In National Council of Teachers of Mathematics, Mathematical Sciences Education Board, & National Research Council (Eds.), *The nature and role of algebra in the K–14 curriculum: Proceedings of a National Symposium* (pp. 25–26). Washington, DC: National Academies Press.

Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. A. Romberg (Eds.), *Mathematical classrooms that promote understanding* (pp. 133–155). Mahwah, NJ: Lawrence Erlbaum Associates.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). New York, NY: Lawrence Erlbaum Associates.

Kaput, J. J., & Blanton, M. L. (2005). A teacher-centered approach to algebrafying elementary mathematics. In T. A. Romberg, T. P. Carpenter, & F. Dremock (Eds.), *Understanding mathematics and science matters* (pp. 99–125). Mahwah, NJ: Lawrence Erlbaum Associates.

Kaput, J. J., Carraher, D. W., & Blanton, M. L. (Eds.). (2008). *Algebra in the early grades.* New York, NY: Lawrence Erlbaum Associates.

Katz, V. (Ed.). (2007). *Algebra: Gateway to a technological future*. Washington, DC: Mathematical Association of America.

Kenney, P. A., Lindquist, M. M., & Heffernan, C. L. (2002). Butterflies and caterpillars: Multiplicative and proportional reasoning in the early grades. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions: 2002 yearbook* (pp. 87–99). Reston, VA: National Council of Teachers of Mathematics.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317–326. doi:10.1007/bf00311062

Kieran, C. (2007). Learning and teaching of algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 707–762). Charlotte, NC: Information Age.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Kline, M. (1972). *Mathematical thought from ancient to modern times* (Vols. 1–3). New York, NY: Oxford University Press.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500–507. doi:10.2307/749655

Knuth, E. J., Alibali, M. W., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2008). The importance of equal sign understanding in the middle grades. *Mathematics Teaching in the Middle School, 13*(9), 514–519.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: Equality & variable. *ZDM, 37*(1), 68–76. doi:10.1007/bf02655899

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297–312.

Levine, G. (1995). Closing the gender gap: Focus on mathematics anxiety. *Contemporary Education, 67*(1), 42–45.

Lins, R., & Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of algebra: The 12th ICMI study* (pp. 47–70). Norwell, MA: Kluwer Academic Publishers. doi:10.1007/1-4020-8131-6_4

Lloyd, G., Herbel-Eisenmann, B., & Star, J. (2011). Developing essential understanding of expressions, equations, and functions for teaching mathematics in grades 6–8. In R. M. Zbiek (Series Ed.), *Essential understanding series*. Reston, VA: National Council of Teachers of Mathematics.

MacGregor, M. (1996). Curricular aspects of arithmetic and algebra. In J. Gimenez, R. C. Lins, & B. Gomez (Eds.), *Arithmetics and algebra education: Searching for the future* (pp. 50–54). Tarragona, Spain: Universitat Rovira i Virgili.

Marum, T., Isler, I., Stephens, A., Gardiner, A., Blanton, M., & Knuth, E. (2011). From specific value to variable: Developing students' abilities to represent unknowns. In L. R. Wiest & T. Lamberg (Eds.), Proce*edings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 106–113). Reno, NV: University of Nevada, Reno.

McNeil, N. M., Weinberg, A., Hattikudar, S., Stephens, A. C., Asquith, P., Knuth, E. J., & Alibali, M. W. (2010). A is for apple: Mnemonic symbols hinder the interpretation of algebraic expressions. *Journal of Educational Psychology, 102*(3), 625–634. doi:10.1037/a0019105

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 69–100). Mahwah, NJ: Lawrence Erlbaum Associates.

Moses, R. P., & Cobb, C. E., Jr. (2001). *Radical equations: Math literacy and civil rights*. Boston, MA: Beacon Press.

Moss, J., Beatty, R., Barkin, S., & Shillolo, G. (2008). "What is your theory? What is your rule?": Fourth graders build an understanding of functions through patterns and generalizing problems. In C.

Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics: 70th yearbook* (pp. 155–168). Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Pimm, D. (1995). *Symbols and meanings in school mathematics*. London, United Kingdom: Routledge. doi:10.4324/9780203428610

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: A strategic research and development program in mathematics education*. Washington, DC: U.S. Department of Education, Office of Educational Research and Improvement.

Schifter, D. (1999). Reasoning about operations: Early algebraic thinking in grades K–6. In L. V. Stiff & F. R. Curio (Eds.), *Developing mathematical reasoning in grades K–12: 1999 yearbook* (pp. 62–81). Reston, VA: National Council of Teachers of Mathematics.

Schifter, D. (2009). Representation-based proof in the elementary grades. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across grades: A K–16 perspective* (pp. 71–86). New York, NY: Routledge.

Schifter, D., Monk, S., Russell, S. J., & Bastable, V. (2008). Early algebra: What does understanding the laws of arithmetic mean in the elementary grades? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 413–447). New York, NY: Lawrence Erlbaum Associates.

Schliemann, A. D., Carraher, D. W., & Brizuela, B. M. (in press). Algebra in elementary school and its impact on middle school learning. *Recherches en Didactique des Mathématiques*, Paris, France.

Schoenfeld, A. H. (1995). Is thinking about 'algebra' a misdirection? In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The algebra initiative colloquium. Volume 2: Working group papers* (pp. 83–86). Washington, DC: U.S. Department of Education, Office of Educational Research and Improvement, National Institute on Student Achievement, Curriculum, and, Assessment. Retrieved from http://files.eric.ed.gov/fulltext/ED385437.pdf

Schoenfeld, A. H., Smith, J. P., III, & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 55–175). Hillsdale, NJ: Lawrence ErIbaum.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 59–84). Washington, DC: Mathematical Association of America.

Shin, N., Stevens, S. Y., Short, H., & Krajcik, J. (2009). *Learning progressions to support coherence curricula in instructional material, instruction, and assessment design*. Paper presented at the Learning Progressions in Science (LeaPS) Conference, Iowa City, IA. Retrieved from http://www.education.msu.edu/projects/leaps/proceedings/Shin.pdf

Stephens, A. C., Isler, I., Marum, T., Blanton, M. L., Knuth, E. J., & Gardiner, A. M. (2012). From recursive pattern to correspondence rule: Developing students' abilities to engage in functional thinking. In L. R. Van Zoest, J.-J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 821–828). Kalamazoo, MI: Western Michigan University. Retrieved from http://www.pmena.org/proceedings/PMENA%2034%202012%20Proceedings.pdf

Stevens, S. Y., Shin, N., & Krajcik, J. S. (2009). *Towards a model for the development of an empirically tested learning progression*. Paper presented at the Learning Progressions in Science (LeaPS) Conference, Iowa City, IA. Retrieved from http://www.education.msu.edu/projects/leaps/proceedings/Stevens.pdf

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (with Derghazarian, E., Huber, G., Ichioka, F., & Kersting, N.) (1999, February). *The TIMSS videotape classroom study: Methods and findings from an exploratory research project on eighth-grade mathematics instruction in Germany, Japan, and the United States* (NCES 99–074). Washington, DC: U.S. Department of Education, Of-

fice of Educational Research and Improvement, National Center for Educational Statistics. Retrieved from http://nces.ed.gov/pubs99/1999074.pdf

U.S. Department of Education. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: Author. Retrieved from http://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf

U.S. Department of Education, Office of Educational Research and Improvement, National Center for Education Statistics. (1996, November). *Pursuing excellence: A study of U.S. eighth-grade mathematics and science teaching, learning, curriculum, and achievement in international context: Initial findings from the Third International Mathematics and Science Study* (NCES 97-198). Washington, DC: Author. Retrieved from http://nces.ed.gov/pubs97/97198.pdf

U.S. Department of Education, Office of Educational Research and Improvement, National Center for Education Statistics. (1997, June). *Pursuing excellence: A study of U.S. fourth-grade mathematics and science achievement in international context* (NCES 97-255). Washington, DC: Author. Retrieved from http://nces.ed.gov/pubs97/97255.pdf

U.S. Department of Education, Office of Educational Research and Improvement, National Center for Education Statistics. (1998, August). *Pursuing excellence: A study of U.S. twelfth-grade mathematics and science achievement in international context*: *Initial findings from the Third International Mathematics and Science Study* (Revised, NCES 98-049). Washington, DC: Author. Retrieved from http://nces.ed.gov/pubs98/98049.pdf

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford & A. P. Schulte (Eds.), *Ideas of algebra, K–12: 1988 yearbook* (pp. 8–19). Reston, VA: National Council of Teachers of Mathematics.

Vergnaud, G. (1985). Understanding mathematics at the secondary-school level. In A. W. Bell, B. Low, & J. Kilpatrick (Eds.), *Theory, research, and practice in mathematical education: Working group reports and collected papers* (pp. 27–45). Nottingham, United Kingdom: University of Nottingham, Shell Center for Mathematical Education.

Authors

Maria Blanton, TERC, 2067 Massachusetts Avenue, Cambridge, MA 02140; Maria_Blanton@terc.edu

Ana Stephens, Wisconsin Center for Education Research, University of Wisconsin Madison, 1025 W. Johnson Street, Room 683, Madison, WI 53706; acstephens@wisc.edu

Eric Knuth, Department of Curriculum and Instruction, University of Wisconsin Madison, Teacher Education Building 476c, 225 N. Mills Street, Madison, WI 53706; knuth@education.wisc.edu

Angela Murphy Gardiner, TERC, 2067 Massachusetts Avenue, Cambridge, MA 02140; Angela_Gardiner@terc.edu

Isil Isler, Wisconsin Center for Education Research, University of Wisconsin Madison, 1025 W. Johnson Street, Room 690, Madison, WI 53706; isler@wisc.edu

Jee-Seon Kim, Department of Educational Psychology, University of Wisconsin Madison, 1025 W. Johnson Street, Room 1075e, Madison, WI 53706; jeeseonkim@wisc.edu

Submitted February 7, 2014

Accepted May 4, 2014

APPENDIX A: SAMPLE LESSON

Lesson 8: Modeling Problem Situations with Linear Equations and Inequalities

Lesson objectives

- Understand how to represent a quantity in a problem situation as a (linear) algebraic expression.
- Understand how to interpret variables and algebraic expressions within a problem context.
- Understand how to relate two expressions in an equation or inequality.

Jumpstart

1. $50 + 10 = 55 + b$; What value of b will make the equation true? How do you know?
2. 28 + ___ = ___ + 28; What numbers would make the equation true?
3. Kevin adds three even numbers together. Is his answer an even or odd number? How do you know? Use cubes or draw a picture to explain your thinking.
4. Jackson has 24 cookies. His cousin Rosie has 28 cookies. How would you represent the relationship between the number of cookies they each have? Using the same numbers, can you represent your relationship in a different way?

The Candy Problem (Adapted from Blanton, 2008; Carraher, Schliemann, & Schwartz, 2008)

A. Jack and Ava each have a box of candies. Their boxes contain the same number of pieces of candy. Ava has 4 additional pieces of candy in her hand. Draw a picture to illustrate this situation.

B. How would you represent the number of pieces of candy Jack has? How would you represent the number of pieces of candy Ava has? Using the same variable and number, can you represent the number of pieces of candy Ava has in a different way?

C. Who has more candy, Jack or Ava? How can you use your picture to explain your answer?

D. How would you represent the relationship between the number of pieces of candy Jack has and the number of pieces Ava has in a mathematical sentence?

E. Suppose Ava counted her candy and found that she had 16 pieces. How does this new information relate to how you previously represented the number of candies Ava has? Write an equation that represents what you know about the number of pieces of candy Ava has.

APPENDIX B

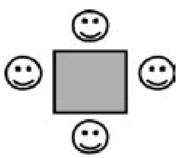
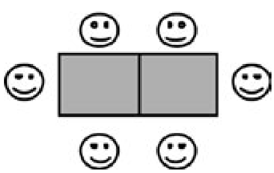
Written Assessment Items and Content Addressed

Assessment Item	Big Idea(s)	Concepts and Practices Addressed
1. Fill in the blanks with the value that makes the following number sentences true. How did you get your answer? a) 7 + 3 = ____ + 4 Why? b) 5 + 3 = ____ + 3 Why?	Equivalence, Expressions, Equations, & Inequalities	• Solve missing value problems by reasoning from the structural relationship in the equation (open number sentences)
2. Circle True or False and explain your choice. a) 12 + 3 = 15 + 4 True False How do you know? b) 57 + 22 = 58 + 21 True False How do you know? c) 39 + 121 = 121 + 39 True False How do you know?	Equivalence, Expressions, Equations, & Inequalities	• Interpret equations written in various formats (e.g., other than $a + b = c$) to correctly assess an equivalence relationship (true/false number sentences)
3. a) In the number sentence 3 + 4 = 7, what is the name of the symbol "="? b) What does the symbol "=" mean?	Equivalence, Expressions, Equations, & Inequalities	• Identify meaning of "=" as expressing a relationship between quantities
4. Marcy's teacher asks her to figure out "23 + 15." She adds the two numbers and gets 38. The teacher then asks her to figure out "15 + 23." Marcy already knows the answer. a) How does she know? b) Do you think this will work for all numbers? If so, how do you know?	Generalized Arithmetic	• Analyze information to develop a conjecture about an arithmetic relationship • Develop a justification or argument (using empirical or representation-based arguments) to support a conjecture's truth • Identify values for which a conjecture is true • Examine the characteristic that a generalization (property) is true for all values in a given number domain

APPENDIX B

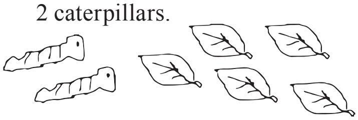
Written Assessment Items and Content Addressed (*continued*)

5. The following number sentence is true: 15 + 8 = 23. Is 15 + 8 + 12 = 23 + 12 true or false? How do you know?	Equivalence, Expressions, Equations, & Inequalities; Generalized Arithmetic	• Interpret equations written in various formats (e.g., other than $a + b = c$) to correctly assess an equivalent relationship (true/false number sentences) • Identify generalization (property) in use
6. Evelyn computes the following: 8 – 8 = ___ 12 – 12 =___ She gets an answer of 0 each time. She starts to thinks that anytime you subtract a number from itself, the answer is 0. Which of the following best describes her thinking? Circle your answer. a) $a + 0 = 0$ b) $a = b + a + b$ c) $a - a = 0$ d) $a \times 0 = 0$	Generalized Arithmetic; Variable	• Analyze information to develop a conjecture about an arithmetic relationship • Express a generalization (property) using variables • Examine meaning of repeated variable or different variables in same equation
7. Tim and Angela each have a piggy bank. They know that their piggy banks each contain the same number of pennies, but they don't know how many. Angela also has 8 pennies in her hand. a) How would you describe the number of pennies Tim has? b) How would you describe the total number of pennies Angela has? c) Angela and Tim combine all of their pennies to buy some candy. How would you describe the total number of pennies they have?	Equivalence, Expressions, Equations, & Inequalities; Variable	• Identify variable to represent an unknown quantity • Represent a quantity as an algebraic expression using variables

8. a) Josh's friend Jack has 3 times the number of video games that Josh has. If Josh has n video games, which of the following describes the number of video games Jack has? Circle your answer. a) $n + 3$ b) $3 \times n$ c) n d) 3 b) How do you know?	Equivalence, Expressions, Equations, & Inequalities; Variable	• Represent a quantity as an algebraic expression using variables
9. Find the value of n in the following equation. How did you get your answer? $3 \times n + 2 = 8$	Equivalence, Expressions, Equations, & Inequalities; Variable	• Analyze the structure of an equation to determine value of variable • Check the solution or determine if the solution is reasonable given the context of the problem
10. Brady is having his friends over for a birthday party. He wants to make sure he has a seat for everyone. He has square tables. He can seat 4 people at one square table in the following way:  If he joins another square table to the first one, he can seat 6 people 	Functional Thinking; Variable	• Generate data and organize in a function table • Identify a recursive pattern and describe in words; use pattern to predict near data • Identify a covariational relationship and describe in words • Identify a function rule and describe in words and variables • Use function rule to predict far function values

APPENDIX B

Written Assessment Items and Content Addressed **(***continued***)**

<table>
<tr>
<td>
a) If Brady keeps joining square tables in this way, how many people can sit at:
3 tables?
4 tables?
5 tables?
Record your responses in the table below and fill in any missing information:
<table>
<tr><th>Number of tables</th><th>Number of people</th></tr>
<tr><td>1</td><td></td></tr>
<tr><td>2</td><td></td></tr>
<tr><td>3</td><td></td></tr>
<tr><td>4</td><td></td></tr>
<tr><td>5</td><td></td></tr>
<tr><td>6</td><td></td></tr>
</table>
b) Do you see any patterns in the table? Describe them.
c) Find a rule that describes the relationship between the number of tables and the number of people who can sit at the tables. Describe your rule in words.
d) Describe your relationship using variables. What do your variables represent?
e) If Brady has 10 tables, how many people can he seat? Show how you got your answer
</td>
<td></td>
<td></td>
</tr>
<tr>
<td>
11. A fourth-grade class needs 5 leaves each day to feed its 2 caterpillars.

How many leaves would they need each day for 12 caterpillars? Explain how you got your answer.
</td>
<td>Proportional Reasoning</td>
<td>• Use multiplicative relationships to reason proportionally about data (e.g., If 2 pieces of candy cost 10 cents, how much would 4 pieces cost?)</td>
</tr>
</table>

APPENDIX C

Statistical Results of Item Analysis

Item	Pretest proportion correct (Nonintervention)	Pretest proportion correct (Intervention)	Fisher's *p* value	Posttest proportion correct (Nonintervention)	Posttest proportion correct (Intervention)	Fisher's *p* value
1a	0.030	0.000	0.532	0.032	0.842	0.000
1b	0.015	0.000	1.000	0.016	0.842	0.000
2a	0.258	0.132	0.145	0.317	0.868	0.000
2b	0.076	0.132	0.491	0.095	0.842	0.000
2c	0.061	0.158	0.165	0.143	0.895	0.000
4b	0.333	0.368	0.831	0.349	0.737	0.000
5	0.076	0.132	0.491	0.048	0.368	0.000
6	0.409	0.553	0.220	0.571	0.895	0.001
7a	0.091	0.051	0.707	0.127	0.737	0.000
7b	0.045	0.077	0.668	0.079	0.632	0.000
7c	0.000	0.026	0.371	0.032	0.395	0.000
8a	0.424	0.513	0.422	0.492	0.895	0.000
9	0.030	0.128	0.099	0.286	0.526	0.020
10a	0.364	0.359	1.000	0.524	0.868	0.000
10b	0.197	0.308	0.239	0.413	0.789	0.000
10c	0.015	0.026	1.000	0.079	0.316	0.005
10d	0.000	0.000	1.000	0.000	0.158	0.002
10e	0.273	0.333	0.516	0.413	0.553	0.218*
11	0.136	0.179	0.583	0.159	0.289	0.135*

*Not statistically significant at posttest

Note. Items 3a, 3b, 4a, 8b were coded for strategy use only (not correctness).