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Conceptualizing Mathematically Significant Pedagogical Opportunities to Build on Student Thinking

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The mathematics education community values using student thinking to develop mathematical concepts, but the nuances of this practice are not clearly understood. We conceptualize an important group of instances in classroom lessons that occur at the intersection of student thinking, significant mathematics, and pedagogical opportunities—what we call **M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking. We analyze dialogue to illustrate a process for determining whether a classroom instance offers such an opportunity and to demonstrate the usefulness of the construct in examining classroom discourse. This construct contributes to research and professional development related to teachers' mathematically productive use of student thinking by providing a lens and generating a common language for recognizing and agreeing on a critical core of student mathematical thinking that researchers can attend to as they study classroom practice and that teachers can aspire to notice and build upon when it occurs in their classrooms.

Key words: Classroom mathematics discourse; Professional development; Student mathematical thinking; Teachable moments; Teaching practice

Skilled teachers and teacher educators often recognize when important mathematical moments occur during a lesson. Although they have not been an explicit focus of research, such moments have been mentioned in a number of different ways in the literature. For example, Walshaw and Anthony (2008) referred to "critical moments in the classroom when students created a moment of choice or opportunity" (p. 527) for the teacher. Davis (1997) described a moment when a teacher could have used student thinking as "a potentially powerful learning opportunity" (p. 360). Davies and Walker (2005) used the phrase "significant mathematical instances"

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(p. 275), and Thames and Ball (2013) used "crucial mathematic hinge moment[s]" (p. 31). Schoenfeld (2008) referred to moments that contained "the fodder for a content-related conversation" (p. 57) and "an issue that the teacher judges to be a candidate for classroom discussion" (p. 65).

It is clear from the literature that these moments, whatever they are called, have the potential to contribute in important ways to mathematics teaching and learning. Without a clear language to describe them, however, it is difficult to discuss such moments with teachers or even among researchers. Although we acknowledge that "the 'real' classroom experience is elusive" in that "each moment is experienced differently by the actors involved" and "the choices of what to focus on, which story to follow, are endless" (Lewis, 2008, p. 5), we are nevertheless convinced that there are moments that can be recognized and agreed upon by researchers, teachers, and teacher educators as being opportune for student mathematics learning. One critical role that research in mathematics education can play is to provide lenses, informed by research and advocated by the community at large, to facilitate this mutual recognition and agreement. Based on studying references to such moments in the literature and reflecting on our own classroom and research experiences, we have developed a framework that is designed to be such a lens. Specifically, this framework is a tool for analyzing the mathematical and pedagogical potential of student mathematical thinking that emerges during classroom instruction.

This work was prompted, in part, by identified limitations in existing research on teachers' use of student thinking. Although research in mathematics teacher education suggests that students benefit from instructional practices that build on their mathematical thinking (e.g., Fennema et al., 1996; Stein & Lane, 1996), such practices have been found to be complex and difficult both to understand and to enact (Ball & Cohen, 1999; Feiman-Nemser, 2001; Sherin, 2002; Silver, Ghousseini, Gosen, Charalambous, & Font Strawhun, 2005). Often, opportunities to use student thinking to further mathematical understanding either go unnoticed or are not acted upon by teachers, particularly novices (Peterson & Leatham, 2009; Stockero & Van Zoest, 2013). One reason learning to use student mathematical thinking is so difficult seems to be the complexity of recognizing and interpreting such thinking. Much existing research on teachers' use of student thinking tends to focus on how teachers respond to student thinking rather than on identifying attributes of student thinking that teachers need to understand to make effective decisions about which thinking is most productive to pursue. The framework introduced here addresses this gap in the literature.

Our work is part of ongoing efforts to understand the practice of productively using student mathematical thinking during whole-group instruction. For example, Stein, Smith, and colleagues' work on the cognitive demand of tasks (e.g., Stein, Smith, Henningsen, & Silver, 2009) and orchestrating classroom discussions (e.g., Smith & Stein, 2011) has been instrumental in focusing attention on the elicitation and use of student mathematical thinking in classrooms. Specifically, their work centers on using a particular kind of student thinking—that which emerges when doing a mathematical task—as the basis for a whole-class discussion of the

mathematical ideas that the task is designed to elicit. Although necessarily related to the precursor practice of *eliciting* student thinking and the follow-up practice of *responding* to student thinking, our framework complements Stein and Smith's work by focusing on the practice of *recognizing* potentially productive student thinking once it has occurred.

There are two other important differences between our work and that of Stein, Smith, and colleagues (e.g., Smith & Stein, 2011; Stein, Smith, Henningsen, & Silver, 2009). First, our work includes a broader range of student actions, such as student questions that arise during a lecture or student comments that emerge during a discussion of homework. Although we agree that high cognitive demand tasks create particularly fertile ground for student thinking to occur, we have also found instances of student mathematical thinking with considerable potential at a given moment in classrooms that lack rich tasks. Our intent was to conceptualize a construct that would apply across the spectrum of instructional situations one might encounter in mathematics classrooms. Second, our work differs in that it focuses on when it is productive to act on student thinking at the moment in which it occurs during a lesson as opposed to monitoring and selecting student work and then purposefully sequencing the presentation of that work in a later stage of a lesson (see Smith & Stein, 2011). Although we also value the use of student thinking at a point after it has occurred—and feel our framework can provide insight into evaluating all types of student thinking—we are convinced that there are certain expressions of student thinking that lose their instructional value if they are not acted on in the moment.

Our work is also closely connected to the growing body of mathematics education research focused on teacher *noticing* (e.g., Sherin, Jacobs, & Philipp, 2011). Van Es and Sherin (2002) conceptualized noticing as being composed of three interrelated skills: (a) identifying what is important in a classroom situation, (b) making connections between the particulars of the situation and broader educational principles, and (c) reasoning about the situation in context. Our framework has the potential to provide a mechanism for analyzing and developing these three skills in relation to noticing in-the-moment opportunities to further student mathematical understanding by (a) providing a means for identifying instances of student thinking that might be judged as "mathematically important" to notice in a given lesson; (b) making a connection between a particular instance of student mathematical thinking and the broader educational principle of building on student thinking—that is, determining whether a particular instance of student thinking is one that might be productively built upon during a lesson; and (c) taking into account the classroom context when determining whether an instance is one that might provide leverage for moving the students in the class forward in their mathematical understanding.

We refer to instances of student thinking that have considerable potential at a given moment to become the object of rich discussion about important mathematical ideas as **M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking (MOSTs). We see these instances as opportunities to engage students in making sense of mathematical ideas that have originated with students—that is,

opportunities to *build* on student mathematical thinking by making it the object of rich mathematical discussion.[1] To illustrate how a teacher might capitalize on a **M**athematically Significant Pedagogical **O**pportunity to Build on **S**tudent **T**hinking (MOST), consider a trigonometry lesson in which students are solving a problem that asks them to find an angle of depression. A student raises his hand and says, "I solved it by finding the angle at the bottom [the angle of elevation] and that gave me the right answer." This example illustrates student mathematical thinking that could become the object of a classroom discussion that focuses on the relationship between the angle of elevation and angle of depression. Were a teacher to engage the rest of the class in reasoning about the student's solution and, as a result, develop students' understanding of these mathematical relationships, we would characterize that use of student mathematical thinking as building on student thinking.

In this article, we define MOSTs, analyze dialogue drawn from the literature to illustrate a process for determining whether a classroom instance is a MOST, and discuss how the construct can contribute to understanding classroom mathematics discourse. In doing so, we lay the foundation for future work that focuses both on understanding the practice of capitalizing on MOSTs and on using the MOST construct as a tool to support teacher learning.

Characteristics of MOSTs

Our conceptual framework locates MOSTs in the intersection of three characteristics of important moments commonly seen in the literature: student mathematical thinking, mathematically significant, and pedagogical opportunity. Figure 1 illustrates this intersection and the pathway we use in defining the construct.

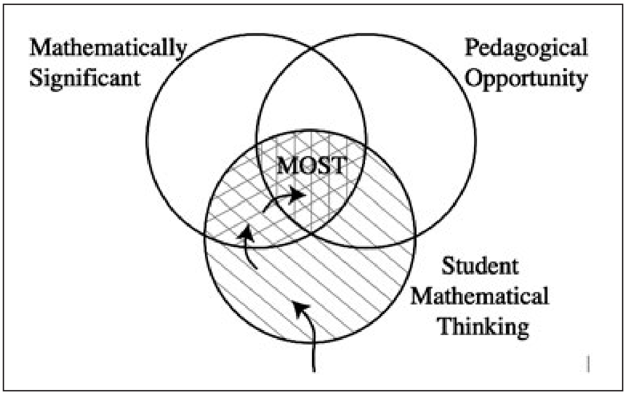

Figure 1. The MOST analytic process. The arrows indicate the pathway followed in the analysis. The shaded regions indicate a honing in from all student contributions to those that are MOSTs.

[1] We use *discussion* broadly to include verbal and nonverbal consideration of the student thinking; however, using student thinking to further mathematical understanding typically involves verbal interactions.

Although we could talk about these three characteristics independently, we have found it more useful for our work to define them such that they build on one another (see Figure 1). Because our goal is to better understand the productive use of student thinking in mathematics classrooms, we consider the foundational characteristic of a MOST to be *student mathematical thinking*. We then focus on whether building on the student mathematical thinking is likely to advance students' development of important mathematical ideas—whether the student thinking is *mathematically significant*. Finally, we consider whether student mathematical thinking can and should be built on to develop students' understanding of significant mathematics at the moment it becomes public—whether there is a *pedagogical opportunity*.

We follow the pathway (illustrated in Figure 1) to define MOSTs and, in doing so, discuss the analytic process for identifying them. We note that this analytic process is not intended to model the actual decision-making process a teacher might go through while teaching a lesson. Rather, this analytic process provides a systematic way to unpack and articulate the various characteristics and criteria of MOSTs. That said, explicating these characteristics and what we see as a productive process for analyzing them has the potential to inform future in-the-moment decisions by helping teachers more deliberately consider whether student mathematical thinking can and should become the object of discussion to work toward a mathematical goal.

In our analytic process, the unit of analysis is an *instance*—an observable student action or small collection of connected actions (such as a verbal expression combined with a gesture). In essence, an instance is an *idea unit* (Stockero, 2008). Typically an idea unit is one conversational turn or physical expression (such as writing a solution on the board), but it can involve multiple turns. For example, if the expression of an idea were interrupted by another speaker with a comment that merely encouraged the initial speaker (e.g., "yeah," "okay," or "um-hum"), the speakers' initial idea and the continuation of it would be considered a single instance.

Student Mathematical Thinking

For an instance to be a MOST, it must be grounded in student thinking—in particular, student thinking that is mathematical in nature. The instance must meet two criteria to be characterized as embodying *student mathematical thinking*: (a) one can observe student action that provides sufficient evidence to make reasonable inferences about *student mathematics* and (b) one can articulate a mathematical idea that is closely related to the student mathematics of the instance—what we call a *mathematical point*. We discuss each of these criteria in turn, but first note that we make a distinction between observable and observed. There are many cases, particularly with novice teachers, in which student thinking is observable but not observed by the teacher (e.g., Berliner, 2001; Peterson & Leatham, 2009; Stockero & Van Zoest, 2013). One explanation for this phenomenon is inattentional blindness, a failure to focus attention on unexpected events (see, for example, Simons, 2000). In addition, this phenomenon is closely tied to teacher noticing (e.g., Sherin et al., 2011)—what a teacher attends to (or fails to attend to) during a lesson. Thus, a teacher's failure to observe student thinking (different from a failure to act on that

thinking, which may be deliberate) does not mean that there is no observable evidence of student thinking. For the purposes of our work, *observable* refers to student thinking that could be observed by the teacher, as noted by someone (e.g., the teacher, other students, a researcher) who witnessed the instance, either by being present or by engaging with a record of the interactions.

Student mathematics. An instance meets the *student mathematics* criterion if an observer can infer what the student is expressing mathematically. Although we recognize the impossibility of directly accessing the thoughts of students, teachers often make inferences based on their observations of what students say and do. For a reasonable inference to be made, a student's actions must provide sufficient evidence about what he or she is expressing mathematically (regardless of its correctness) for an observer to confidently articulate and provide a reasoned argument for that inference. In the classroom setting, this evidence is most commonly visible in student actions such as verbal utterances, gestures, or written work. We refer to a clearly articulated inference of what the student is expressing mathematically as the student mathematics of the instance.

For example, in the trigonometry example mentioned earlier, the student said, "I solved it by finding the angle at the bottom and that gave me the right answer." In this case, we infer the student mathematics to be, "I found the angle of elevation instead of the angle of depression, and it gave me the correct answer." In other cases, one must draw more heavily on the current context and on knowledge of common ways students think about different mathematical ideas to infer the student mathematics. For example, in the context of a discussion about the slope-intercept form of linear equations, if a student says, "Can it ever have two *y*-intercepts?" one might reasonably infer the student mathematics to be, "Can a graph of a linear equation ever have two y-intercepts?" Although it is certainly possible that this student is wondering about the multiplicity of *y*-intercepts for graphs of all types of equations, the evidence from the context suggests the likelihood that he or she is thinking about the linear context. Given our ability to articulate and provide a reasoned argument for the inferred student mathematics in these two examples, these instances meet the student mathematics criterion.

When inferring the student mathematics, we make a distinction between evidence *that* students are thinking and evidence of *what* students might be thinking mathematically. When students wave their hands furiously in the air, it provides evidence that they are thinking, but not about the content of their thoughts. In contrast, when students articulate their own mathematical thinking about a problem, it is likely that there will be sufficient evidence for an observer to infer what they are thinking mathematically. The same is true when they question the teacher's or other students' mathematical ideas or add to another student's mathematical statement. In some cases, however, student contributions may seem to be mathematical in nature and yet still require further information to make a reasonable inference about the student mathematics—for example, when a student seems to be guessing or when a student's idea is unclear. In each of these cases, the instance again provides evidence that the

student is thinking but fails to provide sufficient evidence of what the student is thinking mathematically. Because we cannot infer the student mathematics of these instances, they fail to meet the student mathematics criterion.[2]

We conclude the discussion of the student mathematics criterion by reiterating that each situation is evaluated according to its context. Consider, for example, the simple student answer, "No," given in response to the question "Do you understand?" This answer would not provide insight into what the student is thinking mathematically; thus, there would not be sufficient evidence to infer the student mathematics and this criterion would not be met. If, on the other hand, a teacher asks the class whether the equation $Ax + By = C$ is a linear equation, it could be reasonably inferred that a student who answers "No" is thinking that "The equation $Ax + By = C$ is not a linear equation," so the student mathematics criterion is met. A key requirement is that an observer must be able to infer the student mathematics—that is, articulate an evidence-based inference about what the student's actions indicate about their mathematical thinking. That said, we also emphasize that students need not (and often do not) fully and clearly articulate their ideas. If an observer is able to make a reasonable inference based on the evidence at hand, then the student mathematics criterion has been met.

Mathematical point. After an instance is deemed to satisfy the student mathematics criterion, we determine whether the instance meets the *mathematical point* criterion. An instance meets the *mathematical point* criterion if there is a mathematical idea that is closely related to the student mathematics of the instance. To be closely related to student mathematics, the idea must be one that learners could better understand by considering the student mathematics. We refer to a concise statement of this mathematical idea as the *mathematical point* of the instance.[3] The following statements are examples of mathematical points:

- Addition and subtraction are inverse operations.
- One can add fractions with a common denominator by adding their numerators and keeping the common denominator.
- When attempting to disprove a mathematical statement, it is sufficient to provide a single counterexample.
- When a higher vantage point A and a lower vantage point B are viewed relative to one another, the angle of elevation from B to A and the angle of depression from A to B are congruent to one another.

The reader might recognize the last mathematical point in the list as an idea that is closely related to the student mathematics in the previously mentioned trigonometry instance. Thus, that instance meets the mathematical point criterion.

[2] Such instances may (and often do), however, prompt a teacher move to elicit more information. If this move were effective in surfacing sufficient evidence to infer the student's mathematics, a related instance later in the discussion would then meet the student mathematics criterion.

[3] Although consistent with Sleep's use of *mathematical point* to represent what a teacher is "steering instruction toward" (Sleep, 2012, p. 935), our definition focuses on individual mathematical ideas rather than a "package of ideas" (Sleep, 2009, p. 14).

Not all student mathematics is closely related to a mathematical point. To illustrate, consider a lesson about the Pythagorean theorem in which the teacher tells students that if their solution to the problem they are working on is really big, they may have forgotten to take the square root. A student raises her hand and says, "I took the square root, but my answer is still wrong." Here, there is sufficient evidence to infer the following student mathematics: "I took the square root, but I still got the wrong answer." Although we know from the context that the student is using square roots and is likely using the Pythagorean theorem in some way, we have insufficient information about what the student might have done to determine a closely related mathematical idea. Because we cannot determine a mathematical point, this instance does not satisfy the mathematical point criterion.[4]

Summary. To determine whether an instance embodies the student mathematical thinking characteristic, we first focus on whether an observable student action allows us to make a reasonable inference about what a student is thinking mathematically in order to articulate the student mathematics. If we can infer and make a reasoned argument for the student mathematics, we then consider whether there is a mathematical idea that is closely related to that student mathematics—whether we can determine a mathematical point of the instance. Figure 2 depicts the analysis process and provides questions that encapsulate each criterion. If a criterion is not met, the instance does not embody student mathematical thinking, and the analysis ends—the instance is not a MOST.

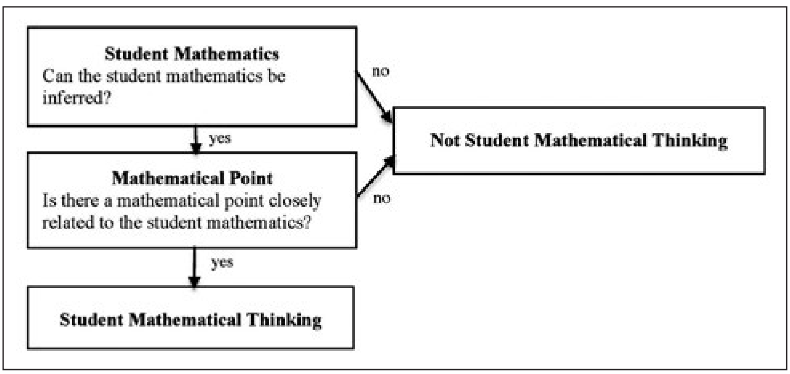

Figure 2. Analysis for the student mathematical thinking characteristic.

Mathematically Significant

In order for an instance to be a MOST, the mathematical point related to the student mathematics must warrant the use of limited instructional time; that is, the instance

[4] Here we have another instance in which a teacher move to elicit additional information—by asking the student what she did or by having her share her work—seems warranted in hopes that one could determine whether there is a mathematically significant point that could be better understood by making the student mathematics of the follow-up instance the object of discussion.

must be what we call *mathematically significant*. Such instances contain student mathematics that can be used to further students' understanding of important mathematics by being made the object of discussion. We use the term *mathematically significant* in the context of teachers engaging a class in the learning of mathematics. In other words, mathematical significance is relative to a group of students at a particular point in their mathematical development. An instance is characterized as being mathematically significant when it meets two key criteria: (a) the mathematical point is *appropriate* for the mathematical development level of the students and (b) the mathematical point is *central* to mathematical goals for their learning.

Appropriate mathematics. The first criterion for mathematically significant is that the mathematical point be appropriate for the mathematical development level of students with backgrounds similar to those in the classroom. Meeting the *appropriate mathematics* criterion requires two things. First, the mathematical point must be accessible to the students given their prior mathematical experiences; they must have adequate background knowledge to engage with the mathematical point. Second, the mathematical point should not be one that most students at this mathematical level would already understand. If they did, pursuing it would not likely move them forward in their mathematical understanding. Documents such as the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) and research on learning trajectories (e.g., Clements & Sarama, 2009; Maloney, Confrey, & Nguyen, 2014; Steffe, 2004; Stylianides, 2008) are important resources for understanding which mathematical points are accessible to, but not likely understood by, students with a particular level of mathematical experience. Knowledge gained from experience with students also contributes significantly to determining appropriateness.

To illustrate the accessibility component of appropriate mathematics, consider the mathematical point "Limits can be used to find the exact area of a shape" in two different settings. This mathematical point would be accessible in a calculus course in which the formal definition of a limit was being studied, even if limits had not yet been used to find the exact area of a region. If this same mathematical point were to arise in a classroom in which limits had not yet been introduced, however, it would not be accessible and thus not appropriate mathematics because the students would not have the background knowledge to engage with the formal definition of a limit. In the latter case, even though the instance involves student mathematics related to a mathematical point, it would fail the appropriate mathematics criterion and thus would not be considered mathematically significant.

The other component of the appropriate mathematics criterion is whether it is likely that the mathematical point is already understood, which is illustrated by considering the mathematical point "Numerals and objects have a one-to-one correspondence." When children are learning to count, the idea that their uttered cadence of "one, two, three, . . ." must have a one-to-one correspondence with the objects at which they are pointing is mathematically appropriate for their stage of

mathematical development because it is accessible but not likely to be already understood. An instance in which this same point surfaced with more mathematically advanced children would not help them move forward in their learning because they would likely already understand the idea; thus it would not meet the appropriate mathematics criterion.

Central Mathematics. The second criterion of the mathematically significant characteristic, *central mathematics*, is that the mathematical point be related to a central mathematical goal for students in that class. Mathematical goals for student learning could be determined by the teacher or by an external source, such as curriculum documents (e.g., National Council of Teachers of Mathematics [NCTM], 2000; NGA & CCSSO, 2010), or they could be inferred by an observer who is knowledgeable in the field of mathematics education. When analyzing the mathematical point in relation to the central mathematics criterion, it is important to keep in mind that mathematical goals (a) range from goals for a specific lesson to broad goals for mathematical learning in general and (b) encompass both mathematical content and mathematical practices.[5]

An instance can meet the central mathematics criterion if the mathematical point of the instance is closely related to a learning goal for students in that classroom. Centrality to the lesson and centrality to the discipline of mathematics are two ends of a continuum of goals to which a mathematical point could be related. Other places along this continuum include centrality to a unit of instruction or to a course. The further a mathematical point is from the goal of a lesson, the more central it must be to the unit, the course, or the discipline of mathematics to meet this criterion. Mathematical points that are further from the goals of the lesson must relate to higher priority goals for students' overall mathematical understanding to warrant the use of limited instructional time during a lesson in which they are not explicit instructional goals. In short, the threshold for meeting the central mathematics criterion is related to the proximity of the mathematical point to lesson goals and to how central the point is to mathematics in general.

Consider, for example, a beginning algebra lesson in which the goal is for students to understand the relationship between the graph of a linear function and its equation written in slope-intercept form, $y = mx + b$. Suppose a student asks a question related to the following mathematical point: "In a linear equation of the form $y = mx + b$, the b-value represents the y-coordinate of the y-intercept." This point is appropriate mathematics for students at this level and is central to the mathematical goal of the lesson, so an instance in this lesson in which this question arises clearly meets both criteria and thus is mathematically significant. On the other end of the central mathematics continuum, suppose a student during this same lesson asks a question related to the mathematical point "Functions have a unique output value for each

[5] Of note are goals involved with developing mathematical ways of thinking, such as the mathematical practices outlined in the *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010). These mathematical practices are important across the learning spectrum because they get at the heart of the discipline.

input value." This mathematical point is appropriate for these students. Although understanding this mathematical point is certainly a more distant goal from the day's lesson than the previous mathematical point, it is a central mathematical goal for beginning algebra. Thus, this instance is also mathematically significant.

Not all mathematical points that are appropriate, however, meet the central mathematics criterion. Suppose, for example, that during an algebra lesson about the Pythagorean theorem a student asks how one could find the square root of a number without a calculator. The mathematical point "One can find the square root of a number without technology by using an algorithm similar to the long division algorithm" is appropriate; that is, the algorithm would be accessible to students but not likely already understood. Understanding this mathematical point, however, is not a central learning goal for students, so it does not meet the central mathematics criterion. Thus, this instance would not be mathematically significant.

Summary. To characterize an instance of student mathematical thinking as mathematically significant, one sequentially measures the mathematical point of the instance against two individual criteria: appropriate mathematics and central mathematics. Figure 3 depicts the analysis process and provides questions that encapsulate each criterion. As in the student mathematical thinking analysis, if a criterion is not met, the instance is not judged to be mathematically significant, and the analysis of whether the instance is a MOST ends. When the criteria are met, we say not only that the instance is mathematically significant but also that the mathematical point of the instance is a *mathematically significant point*.

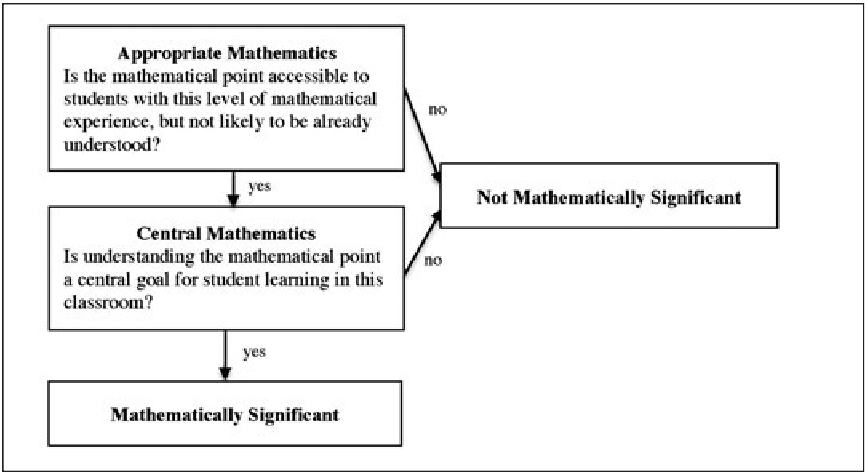

Figure 3. Analysis for the mathematically significant characteristic.

Pedagogical Opportunity

Teachers continuously seek evidence of their students' engagement with a wide variety of instructional goals. They take cues from actions big and small, making

adjustments and pushing students to elaborate, explain, and justify their thinking. Not all student actions, however, are "critical moments" (Walshaw & Anthony, 2008, p. 527) that create "potentially powerful learning opportunit[ies]" (Davis, 1997, p. 360). In the interest of differentiating student actions that meet this higher threshold, we define a *pedagogical opportunity* as an instance in classroom discourse when student thinking provides an opportunity at that moment to make a specific type of pedagogical move—a move to build on that student thinking toward the mathematically significant point of the instance. Some student actions provide opportunities to make different types of pedagogical moves, for example, to elicit more student mathematical thinking, to explain a mathematical concept, or to clarify instructions for the mathematical task at hand. Our focus is on instances in which the student mathematical thinking provides an opportunity at that moment for the class to build on that thinking toward a mathematically significant point.

In conceptualizing the notion of pedagogical opportunity, we draw on the work of Remillard and Geist (2002), who, in the context of their professional development work with teachers, used the phrase *openings in the curriculum* to refer to learners' "questions, observations, [and] challenges" (p. 13). Initially they thought of these openings as breaks in facilitators' plans but came to view them more as "opportunities for facilitators to foster learning by capitalizing on mathematical or pedagogical issues that arose" (Remillard & Geist, 2002, p. 13). For us, a pedagogical opportunity presents itself when an opening is created by an instance of student thinking at a time that is opportune to capitalize on that thinking by making it the object of discussion. Thus, an instance embodies a pedagogical opportunity when it meets two key criteria: (a) the student thinking of the instance creates an opening to build on that thinking toward the mathematical point of the instance and (b) the timing is right to take advantage of the opening at the moment the thinking surfaces during the lesson.

Opening. For students to learn mathematics with understanding, Harel (2013) argued that there must be an intellectual need to engage in sense-making activity—a need for students to "reach equilibrium by learning a new piece of knowledge" (p. 122). These needs include a need to be certain that mathematical ideas are true, a need to understand why they are true, and the need created when students "see as their own a problem in which their approach is manifestly inadequate" (von Glasersfeld, 1995, p. 15). Building on this notion, we define an *opening* as an instance in which the expression of a student's mathematical thinking seems to create, or has the potential to create, an intellectual need for students to make sense of the student mathematics, thus providing an opportunity to understand the mathematical point. In other words, an opening occurs when the expression of a student idea has the potential to create cognitive conflict for students and thus a need to resolve that conflict by engaging with the idea, leading to a better understanding of the mathematical point. We emphasize that an opening is determined by what students, not teachers, do and say. Consequently, determining whether there is an opening requires analyzing the extent to which student mathematical thinking appears to create an intellectual need for other students in the class.

Whether an instance is an opening is determined by the extent to which it appears to position students as needing to make sense of the student mathematics. Instances that are likely to create an opening include (a) a correct answer with novel reasoning, (b) an incorrect answer that involves a common or mathematically rich misconception, (c) a mathematical contradiction, (d) incomplete or incorrect reasoning, and (e) *why* or generalizing questions. In each of these types of instances, the thinking is likely to create an intellectual need for students to make sense of the mathematics—a need to determine the validity of a student's reasoning, to explain why the student's thinking is incorrect, to resolve a contradiction, or to determine the answer to a mathematical question. Note that such openings can occur in classrooms for which substantial intellectual engagement is the norm as well as in classrooms in which a student is struggling to make conceptual sense of an idea that is being presented in a purely procedural manner.

To illustrate the analysis for the opening criterion, suppose a teacher is presenting a lecture on solving systems of equations. Now, consider the contrast between the following two students' actions (in this case, student questions). Student A asks a question related to why adding two equations together does not change the solution set. Student B asks the teacher to repeat their explanation of the steps to eliminate the variable *x*. Both student questions contain student mathematical thinking related to the mathematical point "The addition property of equality," which is appropriate and central to this lesson. Both instances thus qualify as mathematically significant. The contrast we wish to highlight here is that Student A's mathematical thinking (a *why* question) creates an intellectual need to make sense of the content at hand—to understand why the procedure works mathematically. Although Student B may very well want to understand why the procedure works, their question only allows us to infer a desire to hear part of the explanation again. Hence, this action (a *repeat* question) likely does not create an intellectual need for students to make sense of the student mathematics. These students' questions have created very different situations—one in which the student's question can become an object of discussion to help the class make sense of the procedure and the other a chance for the teacher to re-explain a particular step in the procedure. Student A's question provides an opening to build on their thinking, whereas Student B's question does not.

Timing. By definition, *timing* is an element of any opportunity, as an opportunity is "a favorable juncture of circumstances" or "a good chance for advancement or progress" (Opportunity, 2014). Thus, a pedagogical opportunity occurs at a time that is opportune—a time when there is not just an opening but when taking advantage of the opening at that moment is likely to further students' understanding of the mathematically significant point of the instance. Erickson (2003) referred to teachable moments as "right times of tactical action" (p. x). He went on to state, "the teacher who knows how to ride the crests of these pedagogical waves with her students is worthy of their trust" (p. x). Anyone who has watched a surfer wait for the perfect wave (or done so themselves) knows that timing is essential to catching a wave. We contend that for a pedagogical opportunity to occur, the timing must be

right to catch a pedagogical wave. And how might someone recognize the right time? Whereas an opening is related to students' intellectual need, good timing is determined by when an opening surfaces during a lesson. Timing takes into account the overall plan for the day's lesson, the preparation of other members of the class at that moment to engage with the idea being raised, and the context in which the opening emerges.

For example, consider again Student A's question related to why adding two equations together does not change the solution set of a system of equations. Although the question creates an opening, the timing of the question must still be considered to determine whether the instance creates a pedagogical opportunity. Again, we present two contrasting situations: (a) Student A asks the question in the midst of the teacher's initial description of the procedure and (b) Student A asks the question after the teacher has finished describing the procedure and engaged the students in applying it. In the first situation, Student A's question does create a potential intellectual need for students to make sense of the procedure. However, because only part of the procedure has been presented, the majority of the class likely is not yet ready to engage with the idea being raised, and thus the timing is not yet right for discussing why it works. By contrast, in the second situation, the initial instructional goal (describing the procedure) has been completed, and the class as a whole has been sufficiently exposed to the procedure, so it is reasonable to infer that they are ready to engage in making sense of the mathematics underlying the procedure. Hence the timing is right, and Student A's question in the second situation creates a pedagogical opportunity.

Similarly, consider the situation when students are working on a worthwhile mathematical task individually or in small groups and the teacher is *monitoring* (Smith & Stein, 2011) the student mathematical thinking. The teacher may observe student mathematical thinking that is clearly related to a mathematical point that is both appropriate and central. One or more students may also display evidence of an intellectual need to make sense of the mathematics, creating an opening for one student or a small group of students. It may be more productive, however, for other students in the class to think about the mathematics further or to have other contrasting thinking available before a class discussion of the student mathematical thinking begins. In this case, the instance creates an opening, but the timing is not right—more would likely be gained by waiting to discuss the student thinking than by addressing it in the moment—and thus the instance is not a pedagogical opportunity.[6] Pedagogical opportunities occur when student mathematical thinking creates potential intellectual need at times that are particularly compelling—times when an in-the-moment discussion about that thinking will likely move the class toward a better understanding of a mathematically significant point and when not discussing the thinking at that moment risks diminishing or losing the intellectual

[6] This example highlights the benefits of the MOST analysis process even when the instance is not a MOST; since the instance satisfied all the criteria except for timing, the analysis identifies this student mathematical thinking as a good candidate for the teacher to *select* (Smith & Stein, 2011) for sharing during whole-class discussion of the task.

need to do so. The importance of timing is one reason why such instances are often referred to as teachable moments.

Summary. An instance embodies the pedagogical opportunity characteristic when (a) the expression of a students' mathematics creates an opening to build on student thinking to help develop an understanding of the mathematically significant point of the instance and (b) the timing is right to take advantage of the opening. Similar to the analysis process for the prior characteristics, determining whether an instance embodies the pedagogical opportunity characteristic requires measuring the student mathematics against the two individual criteria—opening and timing—and is done linearly (see Figure 4). It is only when both the opening and timing criteria are met that a pedagogical opportunity has occurred. Such an instance embodies all three characteristics and is thus considered to be a MOST—a **M**athematically Significant Pedagogical **O**pportunity to Build on **S**tudent **T**hinking.

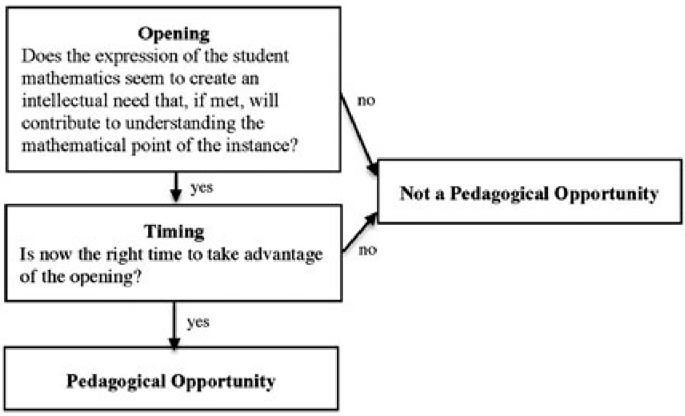

Figure 4. Analysis for the pedagogical opportunity characteristic.

MOST Analytic Framework

Determining whether an instance qualifies as a MOST involves a systematic analysis of whether the instance embodies the three MOST characteristics. The MOST Analytic Framework (Figure 5) summarizes the flow of analysis through the three characteristics and their associated criteria. As discussed previously, the characteristics and their associated criteria are analyzed linearly. It is important to note that the mathematically significant analysis takes place only when the student mathematical thinking characteristic is embodied in the instance and the pedagogical opportunity analysis takes place only when the instance embodies the mathematically significant characteristic. We have found that taking this flowchart approach to the analysis of an instance brings structure and order to what could be a chaotic and overly complex task.[7]

[7] We remind the reader that the MOST Analytic Framework is not intended to model the actual decision-making process a teacher might go through while teaching a lesson.

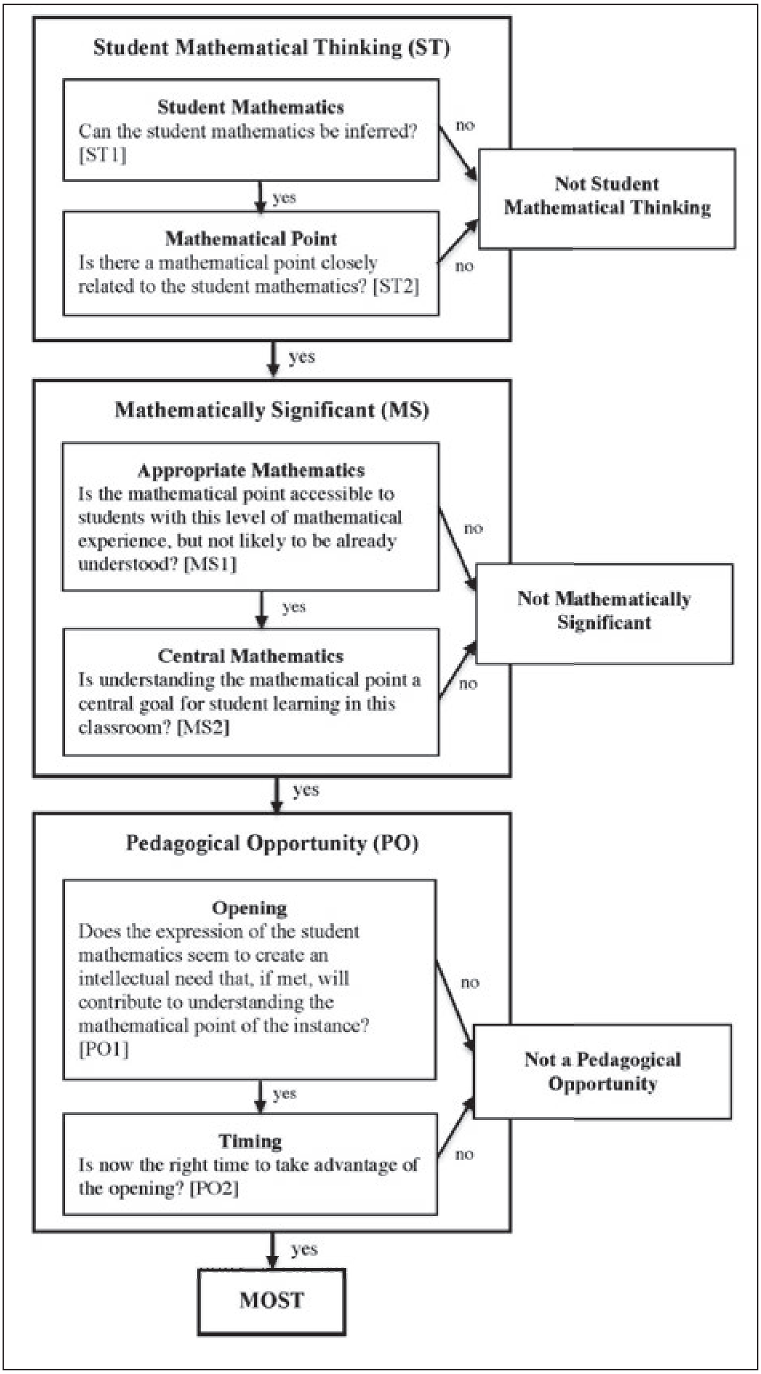

Figure 5. The MOST Analytic Framework.

We note that the presentation of this systematic analysis (as illustrated in the examples throughout the descriptions of the MOST characteristics and in the episode analysis that follows) may make it seem as though the analysis of a given instance is clear-cut and objective, but this is not the case. The characteristics and associated criteria provide a structure around which one can present a reasoned argument for the potential of building on a given instance, but this structure is not deterministic. In analyzing a given instance, individuals with substantial education and experience might draw different inferences about student mathematics; articulate different mathematical points based, in part, on the mathematics they feel is closest to the student mathematics; have different expectations of the mathematics students at a given age are capable of learning but typically do not yet understand; prioritize different aspects of mathematics as more or less central to a given area of study; differ in judging the likelihood of the creation of intellectual need; or differ in just where the cutoff is between not good timing and good timing. Thus, even with a reasoned argument, observers may come to different conclusions about an instance. The analysis we present in this article is the agreed upon consensus of the authors, not the "correct" answer.

Analysis variations of a different nature may occur because of differences in education and experience. For example, the more complex an understanding one has of mathematics and what it takes for students to learn it, the more likely one is to identify student mathematical thinking and a closely related mathematical point that thinking can be used to build toward. That is, variations in knowledge and experience make it possible for two observers of the same instance of instruction to arrive at different mathematical points. Consider, for example, the difference between an experienced observer familiar with mathematics education literature on rational numbers, and thus quite knowledgeable about multiplicative reasoning, and a novice observer with a limited knowledge of multiplicative reasoning both observing the same lesson. Suppose that, given a problem in which a horse is twice as tall as a dog, students are able to find the height of the horse by multiplying the given height of the dog by 2. Given the height of the horse, these same students divide by 1/2 to find the dog's height. The expert observer would likely recognize that this thinking could be used to address a common misconception and thus would see a closely related mathematical point such as "Neither multiplication nor division always 'makes smaller' or 'makes larger.' " The novice observer, however, might dismiss the student work here as a simple error and thus conclude that there is no mathematical point. In the latter case, however, the novice observer would likely not be in a position to provide a reasoned argument for their conclusion.

Despite the potential variation in analysis results, the process of making the reasoned arguments focuses attention on instances of student thinking that have potential to become the object of rich discussion about important mathematical ideas. If observers come to different conclusions based on informed, reasoned arguments, it is likely that either conclusion could lead to actions that improve student mathematical understanding. It is precisely because of the subjectivity inherent in the complex activity of teaching that the MOST construct has such potential. The very

act of constructing and defending a reasoned argument using a common framework and language provides an opportunity for learning to occur among observers of student thinking—teachers, teacher educators, and researchers.

Analysis of Examples from the Literature

In this section we analyze two brief teaching episodes from the mathematics education literature. Analyzing these episodes allows us to illustrate how the MOST Analytic Framework can be used to identify instances of student mathematical thinking that have the potential to be built upon and to compare our framework to those used in the articles from which the episodes are drawn. This analysis provides a snapshot[8] of how the research team has used the MOST Analytic Framework (see Figure 5) to determine whether classroom instances are MOSTs and, if not, the reasons they fall short. The first episode (see Figure 6 on pp. 106–109) is a fifth-grade classroom discussion of fractions led by an expert teacher. The second episode (see Figure 7 on pp. 110–111) is a seventh-grade classroom discussion about solving equations led by a student teacher. Figures 6 and 7 provide the dialogue being analyzed (with line numbers and italics from the original transcripts) as well as the student mathematics (SM) and mathematical points (MP) articulated as part of the analysis process. The ST, MS, and PO columns in the figures indicate, respectively, the three MOST characteristics: student mathematical thinking, mathematically significant, and pedagogical opportunity. A check mark in a column indicates that the characteristic has been met for an instance; a number in brackets indicates the last criterion of the characteristic the instance satisfied. Instances with three check marks are those that embody all three characteristics and are MOSTs.

Episode 1: Fractions

Episode 1 (see Figure 6) is taken from Leinhardt and Steele (2005) and comes from a fifth-grade class discussion about finding output values for the rule $3x + 1$ when given different input values. After the students had demonstrated fluency with large integer values in an input-output table, the teacher, Magdalene Lampert, added 1/4 as an input and asked for the output. Her initial intent (as expressed in the article) was to focus students on the scope of the rule in preparation for graphing linear functions. The dialogue in Episode 1 diverts from that plan in order to focus on the multiplication and addition of fractions. The transcript begins with a response to the teacher's request for the output value.

The first instance occurs in Line 7, when Soochow gives a correct answer. Having inferred the SM to be "The output of $3x + 1$ when x is 1/4 is 1 3/4," we articulated the MP to be "To evaluate a linear expression for a fractional input, one replaces the variable with the given fraction and then performs the appropriate operations." The instance therefore qualifies as student mathematical thinking. This MP is

[8] Our work typically focuses on video recordings of classroom mathematics lessons, which allow us to include in our analysis subtleties such as students' vocal inflections, facial expressions, gestures, and body language as well as the reactions of others in the classroom to students' actions.

			Student Mathematics (SM)	Mathematical Point (MP)	ST	MS	PO
7	Soochow:	One and three fourths.	The output of $3x + 1$ when x is 1/4 is 1 3/4.	To evaluate a linear expression for a fractional input, one replaces the variable with the given fraction and then performs the appropriate operations.	✓	✓	
8	T:	How would you explain it please?					
9 10	Soochow:	*Because one-fourth times three is three-fourths and then you just* *add o– add a one.*	The output of $3x + 1$ when x is 1/4 is 1 3/4 because 1/4 times 3 is 3/4 and adding the one gives 1 3/4.	To evaluate a linear expression for a fractional input, one replaces the variable with the given fraction and then performs the appropriate operations.	✓	✓	
11 12 13	T:	Okay, so first you times by three and then you add one. Who can explain why one fourth times three is three fourths? Sun Wu?					
14 15 16	Sun Wu:	*One fourth, like one fourth of a pie and then somebody brings two* *more and one times three is three—three pieces of pie that came* *out of four pieces of pie?*	Three times 1/4 is like having 3 pieces of pie that are each 1/4 of the pie (1/4 of a pie plus 2/4), giving you 3 out of the four pieces of the pie.	1) Fraction multiplication can be thought of as repeated addition.	✓	✓	✓
				2) The fraction $1/n$ represents one piece of a whole partitioned into n equal size pieces.	✓	✓	✓
17	T:	Okay, are they all the same size? Those three pieces of pie? Lisa?					

Lisa:	Yes	The three pieces of pie are the same size.	The fraction 1/*n* represents one piece of a whole partitioned into *n equal size* pieces.	✓	✓	
T:	How do you know?					
Lisa:	Because if you're adding one fourth times three you're going to [—] [—] equal parts	Cannot infer.				
T:	Okay. Cause I'm, I'm taking three things that are all the same size. They're all the size of one fourth. Ali?					
Ali:	*It could be one fourth* [—] *could be a whole one.*	Cannot infer.				
T:	Can you explain what you mean?					
Ali:	Can I come to the board?	None.				
T:	Yes, here take this [chalk], it's easier to see.					
Ali:	Here's like a big pie [draws circle and divides it into fourths]	1/4 of the circle is one of four equal pieces.	The fraction 1/*n* represents one piece of a whole partitioned into *n equal size* pieces.	✓	✓	
T:	Um-hum					
Ali:	And then you could divide it into fourths, four pieces. *And then one fourth could be one* [*points to one segment of circle*] *and then would be like this one* [*points to the 1 on the input side of the chart*].					

		Student Mathematics (SM)	Mathematical Point (MP)	ST	MS	PO
T:	I don't understand what you mean. Does anybody else understand what Ali means?					
Bridgette:	Me-, he means that if you ha-, if you have one fourth and you make say you color in three of the four pieces [—] equal one whole.	Cannot infer.				
T:	Is that what you meant?					
Ali:	Yeah.	Cannot infer.				
T:	Okay, what do you think about that? Ali is saying three times one fourth is one fourth [*sic*]. Add one fourth and you'd get four so it would be just like here [points to the 4 beside the 1 in the function chart]. But the input number here was one [writes faint 1 in input column beside the 4] and now the input number here is one fourth [points to the 1/4 in new chart]. What do you think Sun Wu?					

47 48	Sun Wu:	*He thinks the um, the one is like one fourth. But it's really one, another, four.*	Cannot infer.				
49 50	T:	What do you think about that Ali? [draws another circle]. How many fourths are there in one whole?					
51	Ali:	Four fourths [T draws new circle divided into fourths].	There are four fourths in one whole.	n pieces of size $1/n$ make one whole.	✓		
52 53	T:	Four fourths? So if I was going to put a number in here I could put one and a fourth [sic] [writes in column]					
54 55	T:	Is there anything I could put in there besides one and a fourth? Elsie?					
56	Elsie	Wouldn't it be one and three fourths?	Cannot infer.				
57 58	T:	Oh, I'm sorry. It should be one and three fourths like that anyway [changes chart]. Is that what you meant?					
59	Elsie:	Yeah.	Cannot infer.				

Figure 6. Analyzed transcript excerpt from G. Leinhardt and M. D. Steele (2005), "Seeing the Complexity of Standing to the Side: Instructional Dialogues," *Cognition and Instruction, 23*(1), pp.107–108. Copyright 2005 by Lawrence Erlbaum Associates, Inc. Reprinted by permission of the publisher (Taylor & Francis Ltd, http://www.tandf.co.uk/journals). A check mark in a column indicates that the characteristic has been met for an instance. Instances with three check marks are those that embody all three characteristics and are MOSTs.

			Student Mathematics (SM)	Mathematical Point (MP)	ST	MS	PO
16	Teacher:	OK, here we already have subtraction [indicating the symbol "–" in "*m* – 12 = 5"], so what's the opposite of subtraction?					
17	Students:	Addition	Addition is the opposite of subtraction.	Addition and subtraction are inverse operations.	✓		
18	Teacher:	So if I want to make a zero here, what can I do?					
	Once more, students offered various responses, including again "subtract twelve."						
19	Students:	Subtract twelve.	To make zero in the equation *m* – 12 = 5, you can subtract 12.	Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.	✓	✓	✓
20	Teacher:	If I subtract twelve, it's going to be a plus minus and a plus minus, so I didn't make a zero. So how am I going to get rid of the subtraction?					
21	Student:	Add	To get rid of subtraction, you need to add.	Addition and subtraction are inverse operations.	✓		
22	Teacher:	Addition. So what am I going to add?					
23	Student:	Twelve	I am going to add twelve.	Cannot articulate.	**[1]**		
24	Teacher:	OK, do I add it to one side or …?					

25	Student:	All sides.	You need to add 12 to both sides.	Addition property of equality.	✓	✓	
26	Teacher:	Both sides. OK, so I have a minus twelve plus twelve, and what is that?					
	Students again call out various (inaudible) responses, none of which were zero.						
27	Teacher:	Zero					
28	Karl:	It's 24. You're supposed to add twelve.	When you add 12 to 12 you get 24.	Subtracting an integer is equivalent to adding the additive inverse.	✓	✓	✓
29	Teacher:	OK, this is not an addition sign right here [indicating the symbol "–" in "m – 12 = 5"]. It's a subtraction sign.					
30	Karl:	I know.	I know that the symbol in front of the twelve is a subtraction sign.	+, –, ×, and ÷ are used to represent addition, subtraction, multiplication, and division respectively.	✓		

Figure 7. Analyzed transcript excerpt reprinted from T*eaching and Teacher Education, 17*(2), M. L. Blanton, S. B. Berenson, and K. S. Norwood, "Using Classroom Discourse to Understand a Prospective Mathematics Teacher's Developing Practice," p. 232, Copyright (2001), with permission from Elsevier Science Ltd. A check mark in a column indicates that the characteristic has been met for an instance; a number in brackets indicates the last criterion of the characteristic the instance satisfied. Instances with three check marks are those that embody all three characteristics and are MOSTs.

appropriate because it builds on knowledge available to most fifth-grade students, making it accessible, but using fractions in this way is not typically part of their past experience, so they are unlikely to already understand it. In addition, the MP is central because it is in direct service of a goal for the day's lesson, namely, understanding that a linear equation can be evaluated for various input values. Therefore, the instance is mathematically significant. In this instance, as is often the case with unjustified correct answers, Soochow's mere statement of the correct answer provides neither unique nor unexpected reasoning and thus is not likely to have created disequilibrium in the students. With no need to "reach equilibrium" (Harel, 2013, p. 122), Soochow's answer does not seem to create an intellectual need for the class to make sense of that answer. Thus, the instance fails to create an opening, the first criterion of a pedagogical opportunity, and is not a MOST.

In Lines 9 and 10, Soochow explains how she arrived at her answer. The SM now includes an explanation, but the MP remains the same, so the instance embodies, as before, the student mathematical thinking and mathematically significant characteristics. Because Soochow's explanation simply describes the basic operations she performed and not her rationale for using them, it still does not seem to create an intellectual need for the class to make sense of her thinking. Thus, the instance fails to create an opening and is not a MOST.

In Line 12, the teacher focuses the discussion on fraction multiplication. We infer that the SM associated with Sun Wu's response in Lines 14–16 is, "Three times 1/4 is like having 3 pieces of pie that are each 1/4 of the pie (1/4 of a pie plus 2/4), giving you 3 out of the four pieces of the pie." We articulated two related MPs:[9] MP1, "Fraction multiplication can be thought of as repeated addition," and MP2, "The fraction $1/n$ represents one piece of a whole partitioned into n equal size pieces." Therefore, the instance qualifies as student mathematical thinking. Typically, rational number multiplication has been taught by fifth grade, but because students commonly struggle with the idea throughout their school experience (e.g., Moss & Case, 1999; Smith, 1995), MP1 is appropriate for students at this mathematical level. Understanding rational number multiplication is certainly a core mathematical goal for elementary school students, so MP1 is also central and is therefore mathematically significant. Sun Wu's explanation with respect to fraction multiplication is compelling because he explains why the product is 3/4 by appealing to repeated addition, a connection that may create disequilibrium for students in the class. Thus, the instance creates an opening to make Sun Wu's thinking the object of discussion in order to build toward an understanding that multiplying a fraction by a whole number is similar to multiplying two whole numbers because both can be thought of as repeated addition. Given the teacher's prior move to focus the discussion on understanding fraction multiplication, the timing is right for the teacher to capitalize on the opening, and the instance is a pedagogical opportunity. The instance (with respect to MP1) is a MOST.

[9] The following analysis illustrates how one instance can lead to a MOST with two viable mathematical points that could be made from the student mathematics. The fact that one brief student utterance could be used productively in multiple ways highlights the complexity of working with student thinking.

MP2 is appropriate because it is common for students across a number of elementary grades to continue to struggle with the value and necessity of equipartitioning. Because equipartitioning is a fundamental concept in elementary mathematics (e.g., Confrey, Maloney, Nguyen, & Rupp, 2014), understanding MP2 is a central goal for students at this level; thus MP2 is mathematically significant. The imprecision in Sun Wu's statement about "three pieces of pie that came out of four pieces of pie" creates an opening to build toward an understanding of equipartitioning. Again, given the focus of the current discussion on understanding operations on fractions, the timing is ideal. The instance (with respect to MP2) thus creates a pedagogical opportunity and is a MOST.

In Line 18, we infer the SM of Lisa's statement to be "The three pieces of pie are the same size," and the MP is the same as MP2 in the previous instance. Thus, this instance qualifies as student mathematical thinking and, by the same argument as above, is mathematically significant. However, in this instance the bulk of the mathematical thinking originated in the teacher's question, not in Lisa's response. The routine nature of Lisa's response does not seem to create an intellectual need for students to make sense of her thinking; thus, it fails to create a pedagogical opening and is not a MOST.

The student comments by Lisa in Lines 20–21 and by Ali in Lines 24 and 26 all fail to qualify as student mathematical thinking, although the last instance for a different reason than the first two. In the first two instances the students appear to be thinking mathematically, but it is unclear what that mathematical thinking is, so we cannot infer the SM. Ali's request to come to the board in Line 26 gives insight into the fact that he wants to show what he is thinking, but the instance does not involve a mathematical idea and therefore does not qualify as SM. These instances fail to meet any of the characteristics of a MOST.

Ali's turn in Lines 28–33 contains two statements. The SM of the statement in Lines 28–30 is inferred to be "1/4 of a circle is one of four equal pieces." In Lines 30–33, Ali makes another statement that seems to be related to the first one, but we cannot with confidence infer in what way he thinks a segment of the circle might be "like" the one in the input chart. Thus, the SM of the instance is inferred to be "1/4 of a circle is one of four equal pieces." This SM is related yet again to the MP "The fraction $1/n$ represents one piece of a whole partitioned into *n equal size* pieces." So, this instance qualifies as student mathematical thinking and, as before, is mathematically significant. In this case, however, it appears from the transcript that Ali's pie is indeed correctly equipartitioned (divided into fourths). The correct, straightforward nature of the instance does not seem to create an intellectual need for students to make sense of the SM. Thus, the instance fails to create a pedagogical opening.

Bridgette's attempt in Lines 36–38 to interpret Ali's thinking is incomplete and not clear enough for us to infer the SM. Likewise we cannot infer the SM in Ali's confirmation that Bridgette's thought captured his meaning (Line 40) or in Sun Wu's unclear statement about Ali's thinking (Lines 47–48). In each of these instances, we cannot infer the SM, so the instance is not student mathematical thinking.

The SM in Ali's response in Line 51 is inferred to be "There are four fourths in one whole," which is connected to the MP "n pieces of size $1/n$ make one whole," so this instance qualifies as student mathematical thinking. Ali's response is similar to Soochow's response in Line 7, a brief correct response to a question, but it differs on the first mathematically significant criterion of appropriate mathematics. By fifth grade, students typically have mastered the mathematical idea that a whole is made up of the number of parts into which it has been equipartitioned. As a result, the MP is not appropriate, and the instance is not mathematically significant.

Elsie's suggestion that the teacher should be using 1 3/4 instead of 1 1/4 (Line 56) and her acknowledgment when the teacher makes the change (Line 59) provide insufficient evidence for us to infer the SM. Although Elsie might be reasoning mathematically with respect to the error, she could, in essence, be correcting a typographical error.

We note that this dialogue contains several student utterances that fall short of being observable evidence of student mathematical thinking (Lines 20–21, 24, 26, 36–38, 40, 47–48, 56, and 59). Although these instances suggest that the students are engaged, they do not provide sufficient insight into their mathematical thinking to enable us to infer what the student is thinking or to determine an MP. Building on such instances would be risky because it is not clear what the object of discussion would be. They are, however, instances in which a teacher might make a move to elicit further information from the students, whose responses, in turn, might generate a MOST. Recall that teachers cannot generate a MOST; they can only create fertile circumstances for students to do so. A MOST can occur only when there is observable evidence of student thinking—when one can confidently infer student mathematics.

This brief segment of classroom dialogue contains one MOST with two mathematical points (Lines 14–16 of Figure 6). The teacher productively used the MOST in her instruction by engaging the class in making the SM an object of discussion, capitalizing on the opportunity to improve students' understanding of MP2.

Episode 2: Solving Equations

The second episode (Figure 7) comes from Blanton, Berenson, and Norwood (2001). The dialogue occurred during a discussion in a seventh-grade general mathematics class. The teacher, a student teacher just beginning her field experience, is leading a discussion about this problem:

> Alex had $5 left in his wallet after he spent $12 on snacks and souvenirs at the Jubilee. How much money did he take to the Jubilee? (p. 231)

At the point where the transcript begins, the equation $m - 12 = 5$ was on the overhead projector, and the teacher had asked the students to recall how they isolated the variable in the previous day's lesson. The students provided various responses, including the suggestion to subtract 12 from both sides of the equation, apparently following a procedure they had used the prior day to solve similar equations that had a plus sign instead of a minus sign.

In their analysis, Blanton et al. (2001) identified the teacher's "incremental questioning routine" (p. 232) and the clear alignment with the oft-observed Initiate-Respond-Evaluate (Mehan, 1979) discourse pattern. We chose this transcript excerpt to illustrate the analytic value of the MOST construct even when the discourse pattern *funnels* (Wood, 1998) students toward particular responses. As with the previous episode, we analyze each instance to illustrate how we use the MOST Analytic Framework (see Figure 5).

The lesson excerpt begins with students responding "addition" (Line 17) to the question "What's the opposite of subtraction?" (Line 16). We thus infer the SM to be "Addition is the opposite of subtraction." This SM is related to the MP "Addition and subtraction are inverse operations," so this instance qualifies as student mathematical thinking. Recognizing these inverse pairs, however, is likely to have been mastered by seventh-grade students, so the instance fails the first mathematically significant criterion of appropriate mathematics and is not a MOST.

In Line 19, in response to the teacher's question about making zero in the equation, students respond with various answers, including "subtract twelve." The SM "To make zero in the equation $m - 12 = 5$, you can subtract 12" is associated with the MP "Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation," so this instance qualifies as student mathematical thinking. This MP is accessible to students at this level but is something that beginning algebra students struggle to understand, so it is appropriate. Also, given the current discussion and line of reasoning, this mathematical idea is related to immediate mathematical goals for these students and thus is central, making this instance mathematically significant. Clearly, the teacher expected students to respond "add twelve," given that they had just been reminded of the fact that addition is the opposite of subtraction. That some students say "subtract twelve" in this context provides evidence of an intellectual need for students to make sense of the SM, creating a pedagogical opening. The timing of this instance is also quite compelling, both because of the previously "established" fact that addition is the opposite of subtraction and because "subtract 12" will not actually make zero in this equation. This instance thus creates a pedagogical opportunity and qualifies as a MOST. We have here a mathematically significant pedagogical opportunity to build on student thinking—to make the students' "subtract twelve" statement the object of discussion in order to explore how students are thinking about "making zero" and how that goal might be related to considering inverse operations.

In Line 21, students again respond "add," only this time they respond not to a question of what is the opposite of subtraction but instead to a question about how they can "get rid of the subtraction" (Line 20). We infer the SM of this instance to be "To get rid of subtraction, you need to add." This mathematics is again related to the MP "Addition and subtraction are inverse operations," so the instance qualifies as student mathematical thinking but is not mathematically significant. As previously argued, recognizing these inverse pairs is likely to have been mastered by the students in a seventh-grade general mathematics classroom, so the instance is not appropriate.

In Line 23, a student responds “twelve” to the question “What am I going to add?” (Line 22). This response begs the question “Add twelve to what?” The vagueness of the teacher’s question interferes with our ability to infer SM beyond “I am going to add twelve.” Thus, although we can infer this SM, we cannot articulate an MP because the SM is so disconnected from the implicit logic of the teacher’s questioning. There is no mathematical point one could build toward by making “twelve” the object of discussion. Thus, the instance does not qualify as student mathematical thinking.

The teacher then asks whether 12 should be added “to one side or [to both]” (Line 24) to which a student responds “all sides” (Line 25). The SM of this instance, “You need to add 12 to both sides,” is related to the MP “Addition property of equality,” so this instance qualifies as student mathematical thinking.

Although students at this stage of their mathematical development have been introduced to the addition property of equality, it is common for them to struggle with the concept on into their high school years, so the MP is appropriate. The MP is central because it is directly connected to the lesson goal of understanding how to solve linear equations, so the instance is mathematically significant. The statement, however, is routinely correct; it does not seem to create an intellectual need to make sense of the SM and thus does not create an opening.

In Line 26, the teacher asks students for the sum “minus twelve plus twelve.” Various incorrect answers are apparently called out, but the transcript provides no clear evidence of student mathematical thinking at this point. The teacher states the correct answer is zero, and then in Line 28, Karl speaks up: “It’s 24. You’re supposed to add twelve.” The SM of this instance is inferred to be “When you add 12 to 12 you get 24.” This SM is associated with the MP “Subtracting an integer is equivalent to adding the additive inverse,” so the instance qualifies as student mathematical thinking. This MP is accessible but something seventh-grade general mathematics students still struggle to understand, so it is also appropriate mathematics. Because this MP is clearly related to the lesson goal of solving linear equations, the MP is central, and the instance is therefore mathematically significant. Karl’s emphatic statement that the answer should be 24, despite the teacher telling him and the rest of the class that the correct answer should be zero, likely creates an intellectual need for the students to resolve the contradiction and thus creates an opening. The timing of the instance is ideal, given that the class is currently attempting to apply knowledge of additive inverses in the context of solving linear equations. The instance creates a pedagogical opportunity and is a MOST.

One final instance of observable evidence of student thinking is when Karl responds with “I know” (Line 30) to the teacher’s reminder that the sign in front of the 12 in the equation is a subtraction sign not an addition sign. We infer the SM to be “I know that the symbol in front of the twelve is a subtraction sign,” and the MP is “+, –, ×, and ÷ are used to represent addition, subtraction, multiplication and division respectively.” Although we certainly do want students to understand this MP, it is not appropriate because we would expect students at this level to already know the meaning of these symbols.

Episode 2 has two instances that contained MOSTs (Lines 19 and 28 of Figure 7). These instances occur in an episode of dialogue that is only minutes long and in a classroom primarily focused only on correct answers. In each case, the teacher responded to the MOST by lowering the cognitive demand of the task. That is, rather than engaging the class in making the SM an object of discussion and capitalizing on the opportunity to improve students' understanding of the MP, she focused the students on mathematical facts that had already been mastered. Not only did this failure to capitalize on MOSTs lead to missed opportunities to improve learning, but responding inappropriately also appeared to decrease students' opportunities to learn.

Discussion

Our analysis of two contrasting episodes suggests that MOSTs can occur across grade levels, in different mathematical contexts, in different types of classrooms, and with teachers of varied experience. In addition, using examples from existing literature allows us to examine the value added by using the MOST construct as a lens for viewing classroom mathematics discourse. In the remainder of this section, we carry out such an examination by contrasting the emphasis of the MOST construct with the emphasis of the theoretical lenses of the original studies from which the episodes were taken. Our intent is not to critique or minimize this related research in any way. Rather, we seek to build on work such as theirs and to explicate just what the MOST construct contributes to the discussion.

Episode 1 was taken from Leinhardt and Steele (2005), who analyzed classroom discourse from the perspective of critical features of instructional explanations, using that perspective "to help analyze and systematize the complexity of the classroom discourse" (p. 87). We see their analysis leading to a better understanding of instructional explanations that incorporate student thinking and of the moves that teachers can make both to create opportunities for MOSTs to occur and to act on them when they do. Specifically, Leinhardt and Steele described some of the options (or paths) that the teacher had available to her in response to the students' comments about fractions in Episode 1, and they framed her decisions about what comments to pursue in terms of managing dilemmas of teaching (Ball, 1993; Lampert, 2001). The MOST construct further contributes to understanding the dilemmas of teaching by unpacking the characteristics of student mathematical thinking that make it potentially productive to pursue.

To illustrate what the MOST construct adds to the discussion, consider the first few lines of the dialogue in Figure 6. Leinhardt and Steele (2005) described Soochow's response in Line 7 as a correct answer with "no apparent need" for follow-up (p. 109). Although their analysis did not provide an explanation for why there was no apparent need, analysis with the MOST construct does. Our analysis points out that, although Soochow's comment involves significant mathematics, her statement of a correct answer without unique or unexpected reasoning is unlikely to have created disequilibrium in the students and thus does not seem to create an intellectual need for the class to understand the mathematics at hand. Without the

intellectual need, Soochow's comment does not create an opening (and thus a pedagogical opportunity) to build on her thinking.

Although there was no apparent need to pursue Soochow's response (Line 7), Leinhardt and Steele's (2005) analysis noted that the teacher "chose this point to start a dialogue" about fraction multiplication (p. 109). Their analysis of this episode focused on the mathematical paths the teacher did and did not follow; the MOST construct allows our analysis to focus on how the content of the students' thinking might compel one to take a particular path. In response to Line 7 as well as to Lines 9–10, the teacher's choice to pursue the path she did appears to be because of the topic (which was on the table because the teacher asked the students to perform fraction multiplication) rather than because of the nature of the student's responses, which were correct and fairly routine. The teacher could have chosen not to go down the paths she did without any adverse consequences. By contrast, in response to Lines 14–16 the teacher focuses on an underlying idea related to Sun Wu's response—the size of the pieces and whether they are equal. This path is created by the nature of Sun Wu's response; as Leinhardt and Steele state, Sun Wu appears to have "blurred the unit" (p. 109). Two possible paths are highlighted by the two MPs identified in our analysis. MP1 (Fraction multiplication can be thought of as repeated addition) is in line with the teacher's original path, but MP2 (The fraction 1/n represents one piece of a whole partitioned into n equal size pieces) is the one the teacher chooses to pursue at this time. Each of these paths provides the opportunity to build on student thinking to further understanding in ways that other paths do not. Soochow's responses (Lines 7 and 9–10) do not compel one to choose any particular path; Sun Wu's response, on the other hand, compels one to choose between two productive paths. Teachers constantly make decisions about which paths to pursue. There are often multiple equally valid paths to take, but this is not always the case. The MOST construct highlights instances in which the nature of student mathematical thinking might compel one to take a particular path because of the opportunity it provides at that moment to build on that thinking to further student mathematical understanding.

As seen in this analysis, the MOST construct provides a mechanism for unpacking aspects of the complex practice of building on student mathematical thinking. It elaborates on how mathematically productive student thinking can be identified in ways that provide a roadmap for teachers less expert than Lampert to resolve "the dilemmas of teaching" (Leinhardt & Steele, 2005, p. 109; see also Ball, 1993; Lampert, 2001). In particular, the MOST construct provides a means for grappling with the dilemma of which path to pursue by explaining why the nature of student mathematical thinking might make certain paths more compelling than others.

The MOST construct centers attention on students' thinking and provides a concrete way to identify which student thinking can be productively used to further student mathematical understanding. This is not to say that we are not interested in the teacher or that Leinhardt and Steele (2005) were not interested in the students but rather to highlight the difference in what is in the foreground and background of the two approaches. They focus on the fact that "particular types of questions and

teacher moves prove more fruitful than others in understanding how students are thinking and making sense of the content" (Leinhard & Steele, 2005, p. 99). In contrast, we focus on the fact that particular instances of student thinking prove more fruitful than others in creating opportunities to build on that thinking to develop significant mathematics. As mentioned in the introduction, one reason we began the MOST work was because existing research on teachers' use of student thinking tended to focus on the teachers' responses to student thinking rather than on identifying what it was about the thinking that teachers needed to understand to make effective decisions about which thinking was most productive to pursue.

Episode 2 was taken from Blanton et al. (2001). The focus of their study was classroom discourse and the associated emerging practice of a student teacher. Their stated purpose was to look for "linkages between classroom discourse and learning to teach mathematics" (Blanton, Berenson, & Norwood, 2001, p. 227). In a sense, they studied learning to teach through the lens of classroom discourse, looking at episodes over time to try to characterize the evolution of the teacher's basic discourse routines. The MOST construct provides a lens to conduct similar research related to the specific teaching practice of recognizing and building on mathematically significant student thinking. This construct allows researchers to examine classroom discourse for evidence of both mathematically productive instances of student thinking and teachers' developing abilities to identify them. It might also serve as a reflective tool for teachers who are learning to productively incorporate student thinking into their instruction. Rather than analyzing an episode for the type of discourse that is evident, the MOST Analytic Framework provides a means for examining discourse to determine whether it provides opportunities to enact a particular practice—the practice of building on students' mathematical thinking.

Blanton et al. (2001) found that the early discourse of the novice teacher they studied (represented by the dialogue in Episode 2) was quite "univocal" in nature. Although they do discuss a few of the students' comments, the main focus of their analysis is on the teacher's discourse moves. For example, they identified Lines 16, 18, 20, 22, and 24 as instances of "instructional questions" often associated with univocal discourse (p. 233). Through analyzing the transcript with the MOST Analytic Framework, we see that students' answers to these similar questions provide varying opportunities for the teacher to build on student thinking. In fact, the instances in Lines 17, 19, 23, and 25 are each coded differently in our analysis. The instance in Line 23 does not have an MP; the instance in Line 17 (as well as in 21) is not appropriate mathematics; and the instance in Line 25 does not create a pedagogical opportunity. It would not likely be productive for the teacher to try to build on these instances. Line 19, however, is a MOST, providing a compelling opportunity for the teacher to break out of her univocal discourse pattern and build on students' mathematical thinking. The application of the MOST Analytic Framework reveals that, although the nature of the teacher's questions was fairly consistent across the episode, the nature of students' responses was not. Our analysis allows us to characterize those responses and the various affordances they provide.

Blanton et al. (2001) also identified specific instances in Episode 2 when the

teacher did not seem to be focused on students' mathematics. With respect to the instance in Line 19, they noted that the teacher's response (Line 20) was "supplying students with explanations of her own methods" (p. 232), and with respect to Line 28, they noted that the teacher "interpreted his [Karl's] utterance as a breakdown in communication" (p. 233) and responded by "explaining her own perspective (29, 31), rather than questioning his utterance" (p. 233). Blanton et al. identified key instances in which the teacher pushed her own mathematical thinking rather than pursuing students' thinking. It is possible that the student teacher shared her mathematics with the students because she perceived that something about these instances was mathematically significant. Our analysis sheds further light on the situation, identifying why the teacher's moves feel so disappointing. Yes, the teacher supplanted the students' mathematics with her own, but in these two instances, she does so not just in the presence of significant mathematics but in the presence of MOSTs. The MOST construct illuminates why such occurrences feel like missed opportunities and provides a rationale for using the students' mathematical thinking in these instances to advance students' understanding rather than imposing the teacher's mathematical thinking.

Two main points emerge from comparing our analysis with that of Blanton et al. (2001) in this way. First, the MOST construct illuminates the variety of student mathematical thinking that can be elicited despite a univocal discourse pattern. Even though these types of teacher questions are not ideal, they still provide opportunities—although limited—for students to share their mathematical thinking. Second, the MOST Analytic Framework supports identification of the instances this teacher might best pursue if she decided to try to focus on "using student thinking." In seeking to change from univocal to dialogic discourse, a teacher may overcompensate by attempting to respond to every student comment with statements like "Does everyone agree?" or "Say more." As a more effective alternative, the MOST Analytic Framework provides a means for analyzing which instances have the most potential to be used productively.

Like the focus on discourse in Blanton et al. (2001), the MOST construct is intended to illuminate important aspects of teacher practice. It does so, however, by focusing primarily on students and their mathematics rather than teachers and theirs. This focus allows for detailed analysis of students' mathematics and of whether observable evidence of student thinking provides opportunities to build on that thinking. Although we desire the construct to eventually be applied to better understand whether and how teachers have capitalized on MOSTs, part of the power of the construct lies in its focus on evaluating instances of student thinking rather than evaluating the teacher. Focusing on students and their mathematics allows one to unpack the building blocks of a teaching practice without necessarily critiquing any particular attempt to enact that practice (cf. Hiebert, 2013).

Comparing the analyses undertaken by Leinhardt and Steele (2005) and Blanton et al. (2001) with ours illustrates the usefulness of the MOST construct. The construct zooms in on instances of student thinking that are likely to be productive for a teacher to pursue. This allows a teacher to move beyond knowing that he or she

should build on student thinking to knowing when building on student thinking is likely to be fruitful in terms of supporting student understanding of mathematics. The MOST Analytic Framework also provides researchers and practitioners with a mechanism to analyze subtle differences in students' responses to questions that might be similar in nature. Recognizing such differences helps to explain why some instances of student thinking feel like missed opportunities if they are not pursued, why other instances that a teacher tries to pursue do not result in productive discussion, and why still others appear to significantly enhance student understanding of mathematics when made an object of discussion. Thus, the MOST construct provides a lens through which teachers, teacher educators, and researchers can better understand instances of student thinking that can be built on to increase student understanding of significant mathematics.

Conclusion

Researchers and practitioners in mathematics teacher education advocate the use of student thinking as a means of improving mathematics instruction (e.g., NCTM, 2000, 2014). Many teachers we have observed, particularly novices, seem to interpret this call to mean that all student thinking is equally valuable and, consequently, should all be pursued in similar ways. We argue, however, that this interpretation is fundamentally flawed. Although teachers certainly need to carefully listen to all student ideas, this listening must be accompanied by thoughtful consideration of whether a particular idea or comment is worth pursuing in the limited amount of instructional time that is available. By clearly defining three critical characteristics that distinguish instances that provide opportunities to productively build on students' mathematical thinking from those that do not, the MOST construct has the potential to become a tool to make sense of student mathematical thinking. In particular, it provides both a means of systematically analyzing instances of classroom discourse and a vocabulary for discussing the mathematical and pedagogical importance of student thinking that arises within such discourse. Thus, the MOST construct can be used to *frame* (Levin, Hammer, & Coffey, 2009) mathematics teaching practice in productive ways by allowing teacher educators and researchers to focus on available student mathematical thinking. This framing shifts the focus of the work from whether a teacher is using student thinking to what student thinking a teacher could incorporate into a lesson and why that incorporation might be valuable.

The main purpose of this article was to articulate the details of the MOST construct. We have shown how the construct provides a lens to examine—and a vocabulary to talk about—instances of student mathematical thinking. This lens also allows researchers to determine whether and to what degree instruction attends to student thinking, significant mathematics, and pedagogical opportunities. With this lens in place, we believe that subsequent research could be conducted to answer questions such as "How might MOSTs vary across classrooms with diverse populations of students?" and "How can teachers be supported in learning to recognize MOSTs?" Although these questions focus on identifying MOSTs, the intent is that such identification lead to learning to productively use the student mathematical

thinking found in MOSTs. After the MOST Analytic Framework is used to identify mathematically productive instances of student thinking that occur in classroom dialogue, researchers could study the teacher actions and utterances surrounding these instances. Further research questions include "What teacher moves tend to facilitate or inhibit students' generation of MOSTs?" and "What are effective ways to capitalize on MOSTs?" This vein of research could contribute to increased understanding of the nature of instruction that uses student mathematical thinking to support student learning of mathematics.

The MOST construct has the potential to contribute to the important work of researching and facilitating teachers' mathematically productive use of student thinking. Teachers' abilities to provide high-quality mathematics education for all students can be enhanced by helping teachers to identify instances of student thinking about significant mathematics that occur in their classrooms and to recognize when this thinking creates a pedagogical opportunity to build on that thinking to support student learning. Such a lens is important because MOSTs are instances of student thinking that could significantly enhance mathematics instruction. Our conceptualization of MOSTs provides a tool for analyzing practice that can help make more tangible the often abstract but fundamental goal of building on students' mathematical thinking.

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