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Relationships Between Students' Fractional Knowledge and Equation Writing

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To understand relationships between students' fractional knowledge and algebraic reasoning in the domain of equation writing, an interview study was conducted with 12 secondary school students, 6 students operating with each of 2 different multiplicative concepts. These concepts are based on how students coordinate composite units. Students participated in two 45-minute interviews and completed a written fractions assessment. Students operating with the second multiplicative concept had not constructed fractional numbers, but students operating with the third multiplicative concept had; students operating with the second multiplicative concept represented multiplicatively related unknowns in qualitatively different ways than students operating with the third multiplicative concept. A facilitative link is proposed between the construction of fractional numbers and how students represent multiplicatively related unknowns.

Key words: Algebraic reasoning; Fractional knowledge; Iterative fraction scheme; Middle school students; Multiplicative concept; Quantitative reasoning; Unknowns

More students are taking algebra courses (Nord et al., 2011; Rampey, Dion, & Donahue, 2009), in part because doing so has been linked to increased educational and economic opportunities (Gamoran & Hannigan, 2000; Moses, 2001). Another reason for increased algebra course taking is that in the last decade, passing algebra courses has become an expectation or state requirement for many students

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(e.g., Council of Chief State School Officers [CCSSO], 2009; National Center for Educational Statistics [NCES], 2006; Seeley, 2005). However, many students continue to struggle to pass algebra courses (e.g., Helfand, 2006; Stein, Kaufman, Sherman, & Hillen, 2011). So, understanding how students learn algebra and facilitating that learning continue to be critical concerns in the field of mathematics education.

To help students see algebra as a fundamental way of sense making rather than as one or two, sometimes dreaded, middle or high school courses, researchers have studied and advocated for the integration of algebraic ways of thinking across the K–12 curriculum (e.g., Davis, 1985; Kaput, 1998; National Council of Teachers of Mathematics [NCTM], 1989, 2000; Wagner & Kieran, 1989). A subfield of research has emerged on *early algebra*, which refers to developing algebraic ideas in elementary and middle grades, rather than taking formal algebra courses at earlier ages (Carraher & Schliemann, 2007; Kaput, Carraher, & Blanton, 2008; Russell, Schifter, & Bastable, 2011b). Those who study early algebra hold that arithmetical and quantitative ways of thinking have an inherently algebraic character that can be explored and articulated (Carraher, Schliemann, Brizuela, & Earnest, 2006). For example, researchers can treat multiplication by three as a function by proposing that students apply it to unknown quantities and represent it symbolically (Carraher, Schliemann, & Schwartz, 2008). Doing so means that students may experience ideas of algebra, such as generalizing and reasoning with unknowns and variables, far earlier than is typical, thereby potentially easing their transition into more formal study of algebra.

Much of the research on early algebra focuses on how students may transform their whole-number knowledge to develop algebraic ideas (e.g., Bastable & Schifter, 2008; Carpenter, Franke, & Levi, 2003; Carraher et al., 2006). To this point, researchers have not extensively studied connections between students' fractional knowledge and algebraic reasoning (Kilpatrick & Izsák, 2008; Lamon, 2007). However, the basic argument for an algebraic character to fractions is not different from the argument with respect to whole numbers. For example, reasoning with fractions can elicit the use of fundamental mathematical properties (Empson, Levi, & Carpenter, 2011), and arithmetical operations with fractions can be thought of as functions (Steffe, 2001). Indeed, in order to fully understand how students' arithmetical and quantitative reasoning can be a basis for algebraic thinking—that is, to fulfill the intent of early algebra—researchers need to investigate relationships between students' fractional knowledge and their algebraic reasoning. This research could flesh out policy recommendations that proficiency with fractions is essential for achievement in algebra (Fennell et al., 2008). It could also illuminate how students' algebraic reasoning might support learning fractions, a direction that, to our knowledge, has not been explored.

To understand relationships between students' quantitative reasoning with fractions and their algebraic reasoning in the area of equation writing, we conducted a clinical interview study with 18 middle and high school students. The study was designed to include six students operating with each of three different multiplicative

concepts (Hackenberg & Tillema, 2009; Steffe, 1992, 1994). These concepts are based on how students create and coordinate composite units (units of units), and they have been found to significantly influence students' fractional knowledge (e.g., Hackenberg, 2010; Norton & Wilkins, 2012; Steffe & Olive, 2010). Initial evidence indicates that these concepts also influence students' algebraic reasoning (Olive & Çağlayan, 2008), specifically how students conceive of quantitative relationships and represent them with algebraic notation. Because the six students operating with the most basic multiplicative concept did not use letters to represent unknown quantities without significant guidance from the interviewer and wrote very few equations, we address their work elsewhere (Hackenberg, 2013). In this article, we focus on the other 12 students.

The first purpose of this article is to show that, in this study, students' multiplicative concepts influenced how students represented quantitative situations involving two multiplicatively related unknowns. The second purpose is to propose a facilitative relationship between students' fractional knowledge and how students represent multiplicatively related unknowns. These research questions are addressed:

1. How do students reason with fractions as quantities? What fraction schemes and operations are attributable to the students?
2. How do students solve algebra problems that involve using unknowns to write equations?
3. How are students' solutions to algebra problems related to students' fractional knowledge?

Students' Fractional and Algebraic Knowledge

Perspectives on Connections between Arithmetical and Algebraic Knowledge

According to Linchevski and Livneh (1999), researchers have developed at least two perspectives on the relationship between arithmetical and algebraic knowledge. In one perspective, students' difficulties with algebra reflect challenges in negotiating a new system and new language (algebraic notation), rather than difficulties that are traceable to students' understanding of arithmetic or numerical structure (e.g., Balacheff, 2001; Filloy & Rojano, 1989; Heffernan & Koedinger, 1998; Lee & Wheeler, 1989; Matz, 1980). In another perspective, algebra is viewed as arising from students' generalizations and abstractions of their arithmetical and quantitative reasoning (e.g., Booth, 1984; Carpenter et al., 2003; Carraher & Schliemann, 2007; Empson et al., 2011; Kaput, 2008; Olive & Çağlayan, 2008; Stacey, 1989; Steffe, 2001). Much of the research on early algebra, and on developing algebraic reasoning across the curriculum, takes this second perspective (Carraher & Schliemann, 2007), as do we.

Based on this research, we view students' beginning algebraic reasoning to be about (a) generalizing and abstracting arithmetical and quantitative relationships, and systematically representing those generalizations and abstractions, not

necessarily with standard algebraic notation and (b) learning to reason with algebraic notation in lieu of quantities. These two points correspond to Kaput's (2008) two core aspects of algebra (i.e., algebra as systematically symbolizing generalizations of regularities and constraints, algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems), with a focus on the first of his three strands—algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic and quantitative reasoning. We take *generalizing* to be

> an activity in which people in specific sociomathematical contexts engage in at least one of three actions: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases. (Ellis, 2011, p. 311)

We take *abstracting* to be a process that underlies making general statements, involving a stripping away of particulars and an ability to view particulars as representing a phenomenon broader than the particulars (Kaput, Blanton, & Moreno, 2008; von Glasersfeld, 1991, 1995).

Generalizations and Abstractions Involving Fractions

Some researchers study students' generalizations and abstractions of numerical and quantitative relationships involving fractions (e.g., Ellis, 2007; Empson et al., 2011; Russell, Schifter, & Bastable, 2011a; Steffe, 2001). These researchers view students' use and awareness of patterns and structures in their ways of thinking with fractions to be algebraic. For example, Empson and colleagues (Empson & Levi, 2011; Empson et al., 2011) called strategies in which students explicitly or implicitly use fundamental properties of arithmetical operations and equality, such as the Distributive Property, *relational thinking*. A student's *implicit* use of a fundamental property suggests that an adult can see an aspect of his or her own mathematical knowledge in a student's way of thinking that is not within the student's awareness (cf. Steffe, 2001). For example, in a quest to determine 20 times 3/4, a sixth-grade student reasoned as follows: 20 times 3/4 is the same as 10 times 1 1/2, which is the same as 5 times 3, which is 15 (Empson & Levi, 2011, p. 85). This reasoning is an implicit use of the Associative Property of Multiplication because the student's reasoning could be modeled as follows: $20 \times 3/4 = 1/2 \times (20 \times 3/4) \times 2 = (1/2 \times 20) \times (3/4 \times 2) = 10 \times 1\ 1/2$, etc. Empson and colleagues hold that students' relational thinking with fractions is itself algebraic and can be a foundation for sensible, explicit use of standard algebraic notation to represent generalizations and abstractions of numerical structure (cf. Kaput, Blanton, & Moreno, 2008).

Another example of research that speaks to connections between fractional knowledge and algebraic reasoning is Ellis's (2007) teaching experiment with seven middle school students. In this study, students created emergent ratios as quantities, which meant that they used two quantities that were directly measured, such as distance and time, to create a third, intensive quantity, such as speed (Schwartz, 1988). For example, if a character walking at a constant speed went

15 cm in 12 s, students reasoned that for every 5 cm he walked, 4 s would elapse, and ultimately students reasoned that 1 cm corresponded to 4/5 s. Students used these emergent ratios to make explicit statements about slopes of lines in cycles of generalizing and justifying activity. Ellis's research indicates that creating equivalent ratios with fractional quantities supports constructing ideas of slope that are rooted in quantitative reasoning. However, to our knowledge, researchers have not addressed how students' fractional knowledge may be involved in equation writing and solving, another aspect of algebraic reasoning.

Development and Use of Algebraic Notation

Many researchers describe the process of active symbolization as nested and cyclical: Students' representations (oral, written, drawn) of their experiences mediate a refined conception of those experiences, which in turn produces a refined set of representations, and the nested cycle continues (e.g., Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Kaput, Blanton, & Moreno, 2008; Sfard, 2000). For Kaput, Blanton, and Moreno (2008) this process of symbolization becomes algebraic if it occurs "in the service of expressing generalizations or in the systematic reasoning with symbolized generalizations using conventional algebraic symbols systems (including more recent graphical and dynamic systems)" (p. 49); in contrast, symbolization is "quasi-algebraic" (p. 49) if it follows the purposes above but uses any symbols, not just conventional algebraic ones. Clearly, this process of algebraic symbolization is far from simple. As Kieran (2007) observed, "Generating equations to represent the relationships found in typical word problems is well known to be an area of difficulty for algebra students" (p. 721).

Numerous research studies with students in Grades 6 through 10 have shown that secondary school students have difficulty using letters to represent unknowns and tend to ignore letters, substitute specific values for letters, treat letters as labels of objects, use letters as an alphabetical code, or treat each letter as having the value 1 (e.g., Booth, 1984; Küchemann, 1981; MacGregor & Stacey, 1997). Often students do not generate algebraic equations, or solve them, if it is possible to solve a problem using other methods, such as guessing and checking numerical values or unwinding arithmetical operations (e.g., Bednarz & Janvier, 1996; Johanning, 2004; Nathan & Koedinger, 2000; Swafford & Langrall, 2000). When students do generate equations to represent quantitative situations, they may use strategies such as word-order matching, static comparisons (Clement, 1982), and the comparison of unequal quantities (MacGregor & Stacey, 1993), which have been found to underlie incorrect responses, such as the famous reversal error (Clement, 1982).[1] Overall, it seems clear that treating letters as representations of unknowns and then arithmetically operating on these unknowns are both

[1] The reversal error refers to the finding that even freshmen undergraduate engineering majors wrote equations like $6S = P$ for a problem in which the number of students, S, was six times the number of professors, P, reversing the way the equation should appear, $S = 6P$, if it is to represent an equality between two quantities.

significant challenges for students (Bednarz & Janvier, 1996; Clement, 1982; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994).

Some researchers have distinguished between arithmetic, prealgebraic, and algebraic uses of letters in equations (e.g., Küchemann, 1981; Slavit, 1999; Vlassis, 2002). An arithmetic use of letters means a student may use them but only because she thinks of them as temporary placeholders for numbers. In a prealgebraic use of letters (Vlassis, 2002), students operate arithmetically on letters but their meaning for the operations comes from concrete models such as weights on a balance. In contrast, an algebraic use of letters means that a student "is able to refrain from immediately attributing a concrete meaning to the letter" (Vlassis, 2002, p. 354). Similar to Vlassis (2002), Slavit (1999) viewed equation writing with the algebraic use of letters as "acts of generalization" (p. 257) because the meaning of the arithmetic operations used on the letters is adjusted to "a focus on the operation itself" (p. 257) without a need to focus on concrete referents.[2]

In a study with four eighth-grade students who worked on a problem involving values and amounts of coins, Olive and Çağlayan (2008) found that only students who had interiorized three levels of units (a particular multiplicative concept that is discussed in the Theoretical Framework) could generate a complete quantitative structure for the situation and represent it with standard algebraic notation. This finding suggests that writing equations, and in particular using letters algebraically, could be linked to particular numerical ways of operating.

Some researchers have gained traction on the difficulty of equation writing and solving with young students in Grades 1 through 6 by helping them engage in many specific instances of relationships among quantities prior to representing generalizations with equations (e.g., Carraher et al., 2006; Carraher et al., 2008; Gravemeijer et al., 2000; Russell et al., 2011b).[3] For example, Carraher, Schliemann, and Schwartz (2008) used a variable number line and indeterminate quantities to develop concepts of unknowns and variables with third- through fifth-grade students. In a fourth-grade lesson, the researchers introduced a problem in which Mike had $8 in his hand and the rest in his wallet, while Robin had three times the amount of money Mike had in his wallet. Instruction involved developing a table and graph for different possible values for the two amounts of money, discussing when Mike and Robin had the same amount of money, and developing solutions to that question using equations. Following the work with tables and graphs, 39 of the 63 students used letters to represent the indeterminate amounts of money, and they operated on these letters additively or multiplicatively. However, it is not known whether the students had developed arithmetic, prealgebraic, or algebraic meanings for these letters.

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[2] This act of generalization might be viewed as extending one's reasoning beyond the range in which it originated, category (b) of Ellis's (2007) definition of generalizing.

[3] This approach could be seen as encompassing all three parts of Ellis's (2007) definition of generalizing.

Algebraic Notation with Fractions: Fractions as Multipliers and Reciprocal Reasoning

Researchers have investigated middle school students' use of algebraic notation in problem solving, but few, if any, studies have addressed how students generate and understand fractions as multipliers of unknowns or variables—for instance, conceiving of and symbolizing one fifth of an unknown length as $(1/5)x$.[4] For example, Swafford and Langrall (2000) asked 10 sixth-grade students who had not received formal algebra instruction to work on six algebraic problems that involved whole numbers as multipliers of unknowns. Similarly, Johanning (2004) engaged 31 seventh- and eighth-grade students who had not taken a formal algebra course in two algebraic problems in which whole numbers were the only multipliers of unknowns. In Olive and Çağlayan's (2008) study, the four eighth-grade students used decimals (values of coins) as multipliers of unknowns but not fractions. Because multiplying unknowns and variables by fractions is critical for reciprocal reasoning in solving equations and for representing functional linear relationships, more needs to be known about how students conceive of fractions as multipliers on unknowns and variables.

Researchers have found that conceiving of fractions as multipliers on known quantities is a significant challenge for students (e.g., Behr et al., 1993; Hackenberg, 2010; Lamon, 2007). In Hackenberg's (2010) study, only students who had interiorized three levels of units (discussed in the Theoretical Framework) and had abstracted a fraction as a concept constructed fractions as multipliers of knowns. In our work, we say that students *operate on unknowns multiplicatively* if they routinely use fractions and whole numbers as multipliers of unknowns and if we have evidence that this use is in reference to a quantitative situation for the students. For example, if a student wrote $(3/5)x$ but then was unable to explain what that meant in the quantitative situation of the problem, or was unable to determine 3/5 of a known quantity, we would assess that the student was not using 3/5 as a multiplier of an unknown despite a seemingly correct expression.

Operating multiplicatively on unknowns is necessary for *reciprocal reasoning* in equation solving. We attribute reciprocal reasoning to a student if, given that y is 3/5 of x, she can reason that x must be 5/3 of y because each 1/5 of x is equal to 1/3 of y, and so 5/5 of x is equal to 5/3 of y (Hackenberg, 2010, 2014; Thompson & Saldanha, 2003). Reasoning reciprocally is one version of *reasoning reversibly,* which we define as being able to start with an outcome of reasoning and use it to produce the starting point for that outcome (Hackenberg, 2010). For example, if a student can start with an unmarked bar (rectangle) that is, say, three fifths of another bar and she can produce that other bar, we would attribute reversible reasoning with fractions to that student. We explain this idea further in the next section.

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[4] This meaning of fractions has often been referred to as its *operator meaning* (Behr, Harel, Post, & Lesh, 1993), but we use the term *multiplier* to distinguish between operating multiplicatively versus additively with fractions.

Theoretical Framework

Based on research that demonstrates that students' quantitative reasoning is a fruitful foundation for their algebraic reasoning, we took a quantitative approach in this study. In this section we discuss this approach, tools we use to model students' mathematical thinking, and our view of students' multiplicative concepts.

A Quantitative Approach

We take a *quantity* to be a property of one's concept of an object or phenomenon "that can be subjected to comparison" (Steffe & Olive, 2010, p. 49). To conceive of an extensive quantity[5] requires a person to conceive of a measurement unit, of the property as subdivided into some number of these measurement units, and of a way to enumerate the number of these units to find a value, or measurement, of the quantity (Smith & Thompson, 2008; Thompson, 1993, 1995, 2011). However, extensive quantities can be thought about even if they are not yet measured. In our work, approaching fractions from a quantitative perspective means we pose problems to students in which fractions have the potential to be values of lengths; these lengths may represent other quantities as well, such as weight. This approach requires identifying a length that is defined as the unit of measurement, often depicted as a rectangle. For example, three fifths as an extensive quantity is the result of partitioning the length of the rectangle into five equal parts, disembedding one of those parts, and iterating it three times (Figure 1).

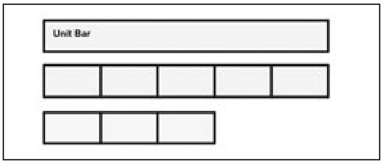

Figure 1. A representation of three fifths as an extensive quantity.

Approaching algebraic reasoning from a quantitative perspective means that unknowns can be thought of as extensive quantities for which a value is not known but for which a value could be determined. So, extensive quantitative unknowns are potential results of measuring fixed extensive quantities before actually measuring them, whereas extensive quantitative knowns can be thought of as extensive quantities for which a value has been determined. For example, the length of the rectangle in Figure 2 represents a known because the length is subdivided into some number of equal units, and those units can be counted—there are 11. In contrast, the length of the rectangle in Figure 3 represents an unknown. We can imagine subdividing the length into some number of equal units, but we do not yet know how many will span the length.

[5] Generally speaking, an extensive quantity can be directly counted or measured (Schwartz, 1988).

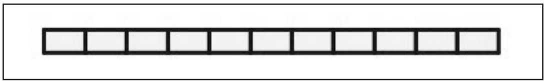

Figure 2. A representation of a known quantity with value 11 units.

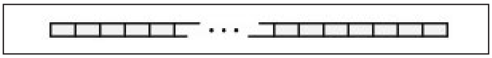

Figure 3. A representation of an unknown.

Tools for Modeling Students' Mathematical Thinking: Operations and Schemes

From our perspective, to imagine subdividing a length into equal parts involves engaging in mental actions, or *operations* (Piaget, 1970; von Glasersfeld, 1995). Operations that are critical for fractional knowledge include *partitioning*, marking a part or whole into some number of equal pieces; *disembedding*, removing a part from a whole without mentally destroying the whole; and *iterating*, repeatedly instantiating a part to make a larger amount (Kieren, 1980; Lamon, 1996; Mack, 2001; Pothier & Sawada, 1983; Steffe & Olive, 2010; Tzur, 1999, 2004). Operations are the components of *schemes*, goal-directed regularities in a person's functioning that consist of three parts: a situation, an activity triggered by how the person perceives the situation, and a result of the activity that a person assimilates to her or his expectations (von Glasersfeld, 1995). We take students' *reasoning* to be the functioning of their schemes and operations in ongoing interaction in their experiential worlds.

The fraction scheme highlighted in this paper is an *iterative fraction scheme* in which fractions are conceived of as multiples of unit fractions (Hackenberg, 2007; Steffe, 2002; Steffe & Olive, 2010; Tzur, 1999). For example, students who have constructed an iterative fraction scheme view 3/7 as 3 times 1/7, and 7/5 as 7 times 1/5, which means these students have constructed *fractional numbers* (Steffe & Olive, 2010). One situation of an iterative fraction scheme is a request to make a length that is seven fifths of a given length. For this request to be sensible, students have to be able to posit a length that stands in relation to the given length yet is freed from relying on being part of a whole for meaning. Students who have constructed an iterative fraction scheme can make the posited length by partitioning the given length into five equal parts, disembedding one part, and iterating it seven times. For these students, the result is a multiple of a unit fraction (7/5 is 7 times 1/5) as well as one whole (5/5) and 2/5.

Students who have constructed iterative fraction schemes can usually solve a problem in which they are given an improper fractional amount, say a candy bar (a rectangle) that is 14/9 of another bar, and they are to make the other bar. Solving this problem requires that students reverse their iterative fraction schemes—that is, start with an improper fraction and create the whole to which it refers. To make the whole, students need to view 14/9 as 14 times the amount they need to iterate nine times to make the whole, which involves operations that students with iterative

fraction schemes have constructed. So, in general, an iterative fraction scheme and a reversible iterative fraction scheme tend to emerge close together in a student's construction of fractional knowledge (Hackenberg, 2010; Steffe & Olive, 2010).

Students who have not yet constructed iterative fraction schemes may construct a *partitive fraction scheme* (Steffe & Olive, 2010) in which students create proper fractions by iterating a unit fraction so many times. For example, if these students are asked to make a length that is 3/5 of a given length, they partition the given length into five equal parts, disembed one part, and iterate that part three times. However, their meaning for the result is based on parts out of wholes rather than on a multiplicative relationship. That is, for them the result is three parts out of five, even though their behavior may appear similar to students who think of 3/5 as 1/5 times 3. So, students who have constructed only partitive fraction schemes are limited to the whole in their work with fractions; making fractions like 7/5 does not make sense to them because it is puzzling to take seven parts out of five (Hackenberg, 2007; Steffe & Olive, 2010).

Students' Multiplicative Concepts

To construct fractions as lengths requires lengths to be constituted at least as units of units, or *composite units*. How students generate and coordinate composite units is the foundation of how we understand students' multiplicative concepts. These multiplicative concepts are the interiorized results of students' units-coordinating schemes (Steffe, 1992); to progress from one concept to the next requires a significant reorganization of schemes and is a protracted process that can take 2 years (e.g., Steffe & Cobb, 1988). Steffe (2007) has estimated that 50–70% of incoming sixth-grade students have interiorized at least two levels of units. If we take the optimistic estimate of 70% and assume that it represents equal numbers of students who have interiorized two and three levels of units, this implies that roughly one third of sixth-grade students are operating with each of the two concepts outlined below (cf. Norton & Wilkins, 2012).

MC2 students. A *units coordination* involves two composite units, such as 5 and 7, and it entails distributing the units of one composite unit across the elements of another composite unit (Steffe, 1992, 1994)—for example, distributing 5 units of 1 across each of the units of the 7 to get a unit of 35 that may be structured in various ways by students. Some students view the result of seven 5s as a composite unit in which the elemental units of 1 are *iterable*, which means that for these students, there is a multiplicative relationship between elemental units of 1 and composite units. So, for these students, 35 is a number that can be created by iterating 1 35 times. Students who view the results of a units coordination as a composite unit consisting of iterable units of 1 have interiorized units of units, or two levels of units. *Interiorizing* refers to reprocessing the result of a scheme so that it is available in further operating; having interiorized the result of a scheme means a student can anticipate that result prior to activity (Tzur & Simon, 2004). These students can anticipate the coordination of two levels of units prior to operating (Steffe, 1992),

and we refer to them as MC2 students (Hackenberg & Tillema, 2009).

MC2 students have the potential to treat a length as a unit containing some number of equal units prior to operating in a situation (Steffe & Olive, 2010). For example, these students can treat a length that represents one foot as a unit containing seven units—a unit of units structure—without having to actually make the partitions. These students can also make three levels of units in activity, which means they can insert units into each unit in solving a problem. For example, they can insert five parts into each of the seven parts in the 7/7-foot segment and determine that they have made 35 parts in all (Figure 4). However, in further operating the 35/35-foot becomes "only" a unit of 35 units for these students: MC2 students do not continue to view the 35/35-foot as a three-levels-of-units structure, such as a unit of seven units, each of which contains five units (Hackenberg, 2010; Hackenberg & Tillema, 2009).

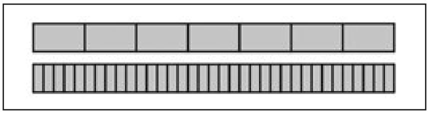

Figure 4. Seven sevenths of a foot, with five parts inserted into each seventh, resulting in 35/35.

MC3 students. Students operating with the third multiplicative concept, whom we refer to as MC3 students, can take three levels of units as given and flexibly switch between three-levels-of-units structures (Hackenberg, 2010; Steffe & Olive, 2010). So, in the whole-number coordination of seven 5s, MC3 students can take as given both the distribution of seven 5s and the structure of the result, 35, as a unit of seven units, each of which contains five units. In short, MC3 students can operate strategically on different organizations of the 5s as if they were units of 1 without losing track of the 5s as composite units. Prior to operating, MC3 students can also treat a length as a unit containing some number of units, each of which contains some number of units. So, in the insertion of five parts into each of the seven parts of the 7/7-foot segment, MC3 students can do what MC2 students do, but they can also maintain views of the 35/35-foot segment as a unit of seven units, each of which contains five units, and they can switch to viewing the segment as a unit of five units, each of which contains seven units (Figure 5). Being able to flexibly switch between such unit structures is critical for constructing many fraction schemes (e.g., Hackenberg & Tillema, 2009; Steffe & Olive, 2010).

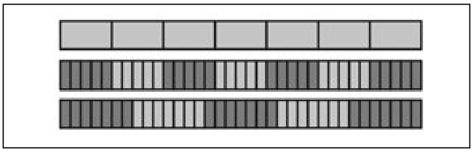

Figure 5. One foot as a unit of seven units of five units, and as a unit of five units of seven units.

In particular, the third multiplicative concept has been found to be necessary for the construction of an iterative fraction scheme (Hackenberg, 2007; Norton & Wilkins, 2012; Steffe & Olive, 2010). For example, to conceive of seven fifths as a number that is usable in further operating means to take it as a unit of seven units (fifths), any of which could be iterated five times to create the "whole," with respect to which seven fifths is named (Figure 6). So, seven fifths is a composite unit of seven units, but any one of those units refers to a composite unit of five of those units. This three-levels-of-units structure seems somewhat different from the three levels of units involved in conceiving of a partitioned length as a unit of seven units, each of which contains five units (contrast Figures 5 and 6), and we return to this point later. In contrast, the second multiplicative concept has been found to be necessary for the construction of a partitive fraction scheme (Steffe & Olive, 2010).

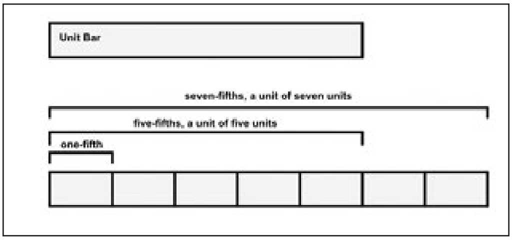

Figure 6. The three-levels-of-units structure in seven fifths.

Methods

To address our research questions, we conducted a clinical interview study (Clement, 2000). In this section we discuss participant selection, data collection, and data analysis.

Participant Selection

Four seventh-grade students, seven eighth-grade students, and one tenth-grade student, all from a small Midwestern town, participated in the study; six of these students were operating with the second multiplicative concept, and six were operating with the third multiplicative concept.[6] Participant selection occurred via classroom observations, consultation with students' teachers, and one-on-one, task-based selection interviews to assess students' multiplicative concepts (see Appendix A for selection interview protocol). These assessments were based on student responses to selection interview questions that involved embedded units, such as asking students to determine the number of cans of juice in a crate if six cans of juice were in a package, eight packages were in a box, and a crate contained

[6] The tenth-grade student was selected because the original intent of the study was to include both middle and high school students. However, due to scheduling constraints it was not possible to engage other high school students in the study. So more middle school students were interviewed to achieve the targeted total number of participants, 18.

five boxes (see S1 in Appendix A); whole-number partitive division (see S2); and multiplicative comparisons between two quantities (see S3, S4, and S5). Additional questions in the selection interview allowed researchers to develop initial ideas about the students' fraction schemes (see S6–S9). Course enrollment for the 12 MC2 and MC3 students who are the focus of this article is listed in Table 1. All students had experienced some instruction in their mathematics classes on unknowns and equation solving. We chose to work with students who were enrolled in prealgebra or algebra classes because nearly all middle and early high school students in our area are enrolled in these classes. So, in order to study secondary school students' algebraic reasoning, we had to take students from these classes.[7] We realize that early algebra frameworks have been used with students who have not yet studied algebra formally. However, we did not have a choice of students in our desired grade bands who had not received some formal algebra instruction. We viewed these students as beginning algebraic reasoners because they were taking prealgebra or a first algebra course.

Table 1

Course Enrollment of the MC2 and MC3 Students

Course	MC2	MC3
Two-year basic mathematics[a] (Seventh-grade student)	1	
Advanced seventh-grade mathematics	1	2
Eighth-grade prealgebra	4	
Eighth-grade algebra		3
High school algebra (Tenth-grade student)		1

[a]This course was described by school personnel as "a class for students who struggle."

Data Collection

Interviews. Students participated in two 45-minute, task-based interviews: a fractions interview and an algebra interview (Goldin, 2000). All interviews were conducted by the two authors. When not acting as interviewers, we assisted with and observed the interviews in order to provide another perspective (cf. Steffe & Thompson, 2000). All interviews occurred in a conference room at the schools during lunch periods or study halls. Each interview was video-recorded with two cameras: one focused on the interaction between the researcher and student and one focused on the student's written work. These videos were digitally mixed into one file for analysis with the work video inset into the interaction video.

All students completed the fractions interview prior to the algebra interview,

[7] A small number of middle school students in our area are enrolled in geometry classes; we did not include them in the study.

but the time between interviews varied from 3 weeks to 4 months with an average time of just less than 2 months.[8] A main reason for the length of time between interviews was that interviews with middle school students could occur only on one day during the school week, on the day with scheduled study halls around lunch periods. Interviewing at this time was recommended by the principal in order to lessen class disruptions, not interfere with sports practices after school, and not burden parents with picking up children who rode the bus.

The interview protocols were refined in a prior pilot study (Hackenberg, 2009) and were designed so that the reasoning involved in the fractions interview could be drawn upon for solving problems in the algebra interview. For example, in the fractions interview, students encountered this situation: "A 65-cm stack of CDs is five times the height of another stack." Students were asked to make a drawing of the situation and determine the height of the other stack. In the algebra interview, students were given a similar problem but both heights were unknown. Students were asked to make a drawing and write equations to represent the situation. (See Appendices B and C for interview protocols.)

During the interviews we followed the interview protocol but were free to depart from it when necessary to investigate conjectures about students' thinking (Clement, 2000). Overall, we sought to harmonize with student thinking (Steffe & Thompson, 2000) to the extent possible in the short time that we interacted with the students, and we aimed for our questions to be open-ended rather than funneling (Wood, 1998). For example, we rephrased questions to students as requested or as we deemed necessary, but we tried to keep rephrasing in the spirit of using language or ideas that might elicit further mathematical activity from the student, and we tried to avoid guiding students to particular answers. Above all, we aimed to communicate respect for and genuine interest in the students' ways of thinking (Ginsburg, 1997).

Written fractions assessment. Students also completed a written fractions assessment (Norton & Wilkins, 2009) to triangulate claims about their fractional knowledge. At the time of data collection, two forms of this assessment were available at each of Grades 7 and 8. The two forms at a particular grade level were made by rotation of 22 items that were designed to assess students' construction of particular fractions operations and schemes, such as a splitting operation and an iterative fraction scheme. All seventh-grade students took a Grade 7 form, while the eighth- and tenth-grade students took a Grade 8 form. To score the written assessments, we followed the methods of Norton and Wilkins (2009, 2012). Because the results of these assessments were compatible with our findings from the interviews but do not illuminate them further, we include information about the assessments in Appendix D.

[8] Four months between interviews was unusual and occurred for only one student who switched schools in the middle of the year. It took some time to locate this student and schedule a second interview. The median length of time between interviews was also a little under 2 months, and the modes were 3 weeks and 1 month.

Data Analysis of Interviews

Analysis occurred in two phases. The aim of the first phase was to formulate a second-order model of each student's fraction schemes and equation writing and solving activity (Clement, 2000). A second-order model is a researcher's constellation of constructs to describe and account for another person's ways and means of operating (Steffe, von Glasersfeld, Richards, & Cobb, 1983). Researchers generate second-order models out of their repertoire of tools, which for this study included theoretical constructs (operations, schemes, concepts), models from prior research (e.g., Hackenberg, 2007, 2010; Steffe & Olive, 2010; Tzur, 2004), and a commitment to use theoretical constructs in an orienting but not deterministic way (Clement, 2000). That is, we aimed to construct second-order models that were consistent with models from prior research, which lends explanatory power and believability to the findings, while expecting students to surprise us with novel ways of operating.

To accomplish this first phase of analysis, we engaged in repeated viewing of video files and took detailed analytic notes for each student (Cobb & Gravemeijer, 2008), which included transcriptions of major portions of each interview, summaries of each student's work on each problem in each interview, and memos (Corbin & Strauss, 2008) consisting of interpretations of and conjectures about a student's work on a particular problem. To write the memos, we asked probing questions about each student's data and, when possible, made theoretical comparisons to second-order models from prior research. Then, we wrote a narrative summary of the second-order model for each student and shared these summaries with each other. We each read all summaries and discussed questions about the summaries together, viewing segments of video to clarify interpretations and refine conjectures. Final decisions about schemes or other ways of operating attributed to students were recorded in a summary grid. The resulting models from this phase provided a portrait of students' fraction schemes and concepts in response to the first research question as well as their ways of writing equations to solve algebra problems in response to the second research question.

In the second phase of analysis, we assessed differences in how students with different multiplicative concepts solved the problems in the algebra interview and articulated relationships between students' fractional knowledge and their use of unknowns to write and solve equations. To accomplish this second phase, we looked across students to identify differences in students' conceptions of unknowns, use of letters to represent unknowns, use of whole numbers and fractions as multipliers on unknowns, and reciprocal reasoning. The first author wrote syntheses across students with each multiplicative concept, drawn from the narrative summaries, which she discussed with the second author. Based on these syntheses, we developed written accounts of how students' multiplicative concepts and fractional knowledge were related to their use of unknowns to write and solve equations. These accounts were, in essence, an integration (Corbin & Strauss, 2008) of the models and syntheses, which provided the basis for responding to the third research question.

Findings

In the next two sections we present findings about our models of students' fractional knowledge (first research question) and how students used unknowns to write equations to represent quantitative situations (second research question). In the third section we propose a facilitative link between the construction of fractional numbers and how students represent multiplicatively related unknowns (third research question).

The Fractional Numbers Findings

Regarding the first research question, we focus on whether students had constructed an iterative fraction scheme because this scheme indicates that students have constructed fractional numbers (Steffe & Olive, 2010). Consistent with prior research (e.g., Hackenberg, 2007; Steffe & Olive, 2010), none of the six MC2 students had constructed an iterative fraction scheme, but all six MC3 students had done so. The third multiplicative concept has been found to be necessary to construct an iterative fraction scheme because improper fractions involve coordinating two different, embedded composite units, as we have discussed. We present this finding to confirm prior research and to give a sense of the students' work with fractions. Central evidence for this finding comes from students' solutions to the fifth question in the fractions interview (see Appendix B for the interview protocol):

> F5. *Improper Fraction Problem*. This rectangle is a picture of a candy bar. Draw a separate candy bar that is nine sevenths of that bar.

In addition, we discuss student work on another interview question intended to elicit a reversible iterative fraction scheme (F6).

MC2 students. When working on F5, two MC2 students articulated verbally that drawing an improper fraction was strange. For example, Lisa_8[9] said, "That's weird . . . *can* there be nine sevenths?" She said there could not be because "you can't take nine out of something that's seven." Five MC2 students drew bars longer than the given bar, but Lisa_8 did not. She partitioned a copy of the given bar into seven parts with the last two parts smaller than the first five parts. Then, she partitioned each of the first two parts into two approximately equal parts to make a total of nine parts (lower bar in Figure 7). She concluded that nine sevenths was smaller than seven sevenths because at least some of the parts for nine sevenths were smaller than the sevenths. So, we do not attribute an iterative fraction scheme to Lisa_8.

In response to F5, the other five MC2 students drew bars longer than the given bar. However, in two of the students' drawings, the parts in the original and nine-seventh bars were not obviously the same size. For example, in Sheila_8's drawing,

[9] We use a subscript to show the student's grade level at the time of the study.

the extra parts were each about half the size of a part in the original bar. In addition, Sheila$_8$ identified only the first seven parts as her answer (Figure 8). Thus, we cannot attribute an iterative fraction scheme to Sheila$_8$ or to the other student (Charlie$_7$) who exhibited similar reasoning.

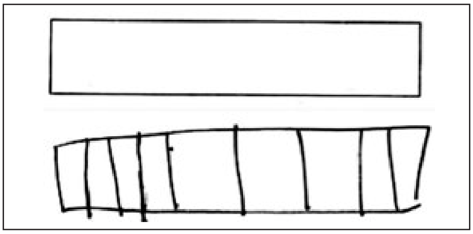

Figure 7. Lisa$_8$'s drawing of nine sevenths (lower bar) of the given bar (upper bar).

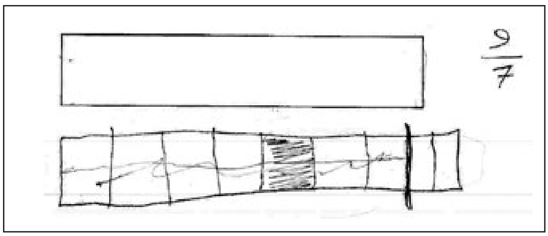

Figure 8. Sheila$_8$'s drawing of nine sevenths.

The work of two other students on F5 featured parts that changed size. For example, Samantha$_7$ drew two copies of the original bar, and she partitioned each into sevenths (middle two bars in Figure 9). At first she identified nine sevenths as two whole bars (the original and first copy) and two more parts (sevenths) in the next bar. Twenty seconds later, she identified nine sevenths as the first copy of the original and two parts of the next copy. When asked to draw nine sevenths as one single bar, she drew a bar that looked about two parts more than a whole bar (lowest bar in Figure 9). However, when the interviewer asked her to show all the parts, Samantha$_7$ partitioned her long bar into seven, not nine, equal parts. This work is counterindicative of an iterative fraction scheme for Samantha$_7$ and the other student (Matt$_8$) who exhibited similar reasoning.

The sixth MC2 student, Martin$_8$, initially demonstrated promising evidence of an iterative fraction scheme. Martin$_8$ began F5 by marking the original bar into nine equal parts and then stopped himself. “I should have cut that (original) into seven pieces and made the other one nine,” he said. He did so on a new copy of the original bar in which each part in the original and the new bar were the same width. Furthermore, he identified the parts in the 9-part bar he made as one seventh of the original. However, when drawing a whole bar given a fraction of it, Martin$_8$ came to different conclusions about how the parts of the new bar related to the original. For example, he worked on this problem:

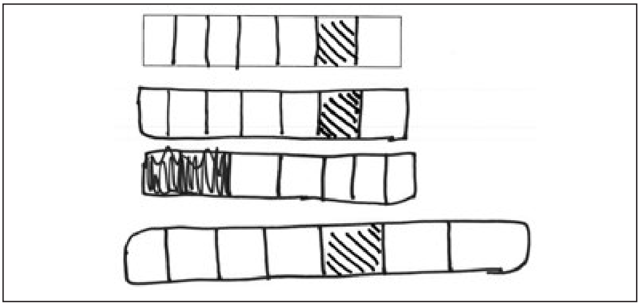

Figure 9. Samantha$_7$'s drawings of nine sevenths.

F6b. *Making the Whole Problem.* This rectangle represents a candy bar. It's four thirds the size of another candy bar. Make a drawing of the other candy bar.

Martin$_8$ drew a bar longer than the original and said that bar had to be cut into nine pieces "because we were talking about that the other day in math" (Figure 10). Martin$_8$'s work on F6b indicates that he had not constructed a reversible iterative fraction scheme.

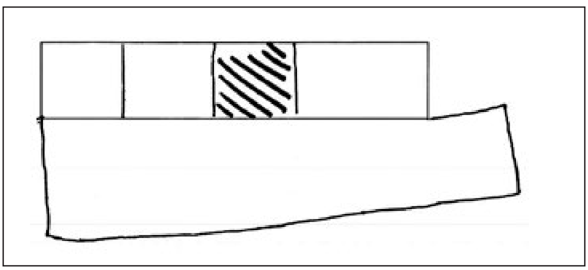

Figure 10. Martin$_8$'s drawing of the whole bar (bottom), given a bar (top) that was 4/3 of it.

As a follow-up, in the algebra interview, the interviewer asked Martin$_8$ to draw four thirds of a given bar. Martin$_8$ partitioned the given bar into fourths and added on a part the size of one of these fourths. He seemed to know he had not solved the problem, stating, "This is hard." The interviewer suggested he draw a bar that was two thirds of the given bar followed by a bar that was three thirds of it. He did. Then, she asked him if he could draw a bar that was four thirds of the given bar. He said no because, if he drew a 4-part bar, it would become fourths. Thus, despite some confirming evidence in the fractions interview, we cannot attribute an iterative fraction scheme to Martin$_8$.

MC3 students. All MC3 students solved F5 by drawing a bar that was two parts longer than the original bar in which each of the parts was the size of one seventh of the original. Suzanne$_8$'s drawing was representative of the responses of MC3 students (Figure 11). In her drawing, she aligned the parts in the two bars. She

explained her drawing as follows: "I divided this [original bar] up into sevenths, and then I drew out the whole bar, and then I, here [pointing to the right end of the copy of the whole bar] I kind of measured how much two sevenths would be, and added it to the end of the bar." When asked about the size of one of the parts in her new bar, Suzanne_8 called it "one seventh of this bar [pointing to the original bar] or one ninth of this bar [pointing to the new bar]." We note that all MC3 students also solved F6, demonstrating that they had constructed reversible iterative fraction schemes.

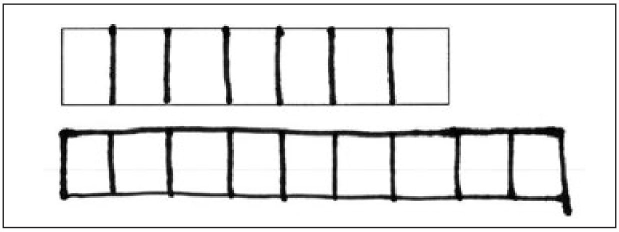

Figure 11. Suzanne_8's drawing of nine sevenths.

Multiplicatively Related Unknown Findings: Reversible and Reciprocal Reasoning

In discussing students' fractional knowledge, we focused on two problems from the fractions interview (F5 and F6) and on one issue: students' construction of an iterative fraction scheme. Similarly, in discussing the students' equation writing, we focus on two problems from the algebra interview (A1 and A5, see Appendix C) and on a central issue: how students represented multiplicatively related unknowns. We found that all students except one MC2 student (Charlie_7) created accurate drawings for both problems. However, based on the equations that the students wrote, we claim that the drawings stood for different generalizations of quantitative structure for the MC2 students compared with the MC3 students. In particular, the MC2 students demonstrated a lack of reversibility in equation writing as well as a lack of reciprocal reasoning, which we explain. In addition, the ways in which the MC2 students used numbers in checking their equations reflected different levels of abstraction consistent with distinctions between arithmetic and algebraic uses of letters (Slavit, 1999; Vlassis, 2002).

MC2 students' work on A1: Effortful equation writing and lack of reversibility. All MC2 students worked first on A1:

> A1. *Cord Length Problem.* Stephen has a cord for his iPod that is some number of feet long. His cord is five times the length of Rebecca's cord. Could you draw a picture of this situation? Can you write an equation for this situation? Can you write another equation?

All MC2 students drew a long segment or bar and a smaller one that they indicated to be one of five equal parts of the longer length, and they justified their drawing

in relation to the problem (e.g., Figure 12). Four of the six MC2 students made their drawing in under 2 minutes, while Charlie_7 and Matt_8 spent 5 minutes and nearly 10 minutes, respectively, producing their drawings. All MC2 students also used letters to represent unknown quantities in A1, but in all cases, they did so at the interviewer's suggestion. That is, when the interviewer asked students to write an equation to represent the situation in their drawing, MC2 students responded with statements like "What do you mean?," "I don't know," and "I don't think so."[10]

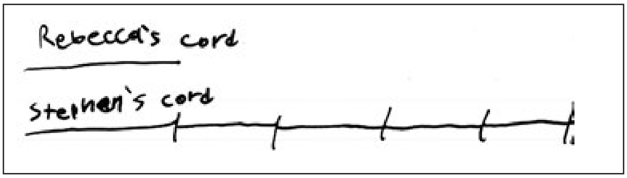

Figure 12. Matt_8's final picture for A1.

One MC2 student (Charlie_7) wrote only additive equations for A1 that, from our perspective, were incorrect (e.g., $y + x + x + x + x + x = 15$, where x represented Rebecca's cord length and y represented Stephen's cord length). Another student (Sheila_8) wrote multiplicative equations that featured conjoining letters without reference to quantitative meaning from her picture (e.g., $5x \cdot 5y = a$, where x and y are defined as above, and a represented the "answer"); from our perspective, these equations were incorrect. Two other students (Matt_8 and Martin_8), with significant prompting from the interviewer, eventually wrote multiplicative equations that we viewed as correct (e.g., $x \cdot 5 = y$, where x and y are defined as above). The final two MC2 students (Lisa_8 and Samantha_7) wrote correct multiplicative equations more promptly, but Lisa_8 did not appear to have a meaning for division (or for fractions as multipliers of unknowns) that she could use to produce a second equation. We focus on Matt_8 as an example of someone who produced one correct multiplicative equation with considerable support and Samantha_7 because she produced two correct multiplicative equations.

Matt_8's work on A1. When the interviewer posed A1, Matt_8 sat in silence for 30 seconds and then said, "I don't really know how to solve this." The interviewer encouraged him to make a drawing, and Matt_8 spent nearly 10 minutes doing so. During this time, he deliberated about whether Stephen's length should be Rebecca's cord length and five more of her lengths or Rebecca's cord length and just four more of her lengths. Because of his uncertainty, after 5 minutes the interviewer chose to introduce an example where Stephen's cord length was 15 feet. In 16 seconds Matt_8 determined that Rebecca's cord length was 3 feet because "15 divided by 5 is 3." The interviewer asked, "So, if Rebecca's cord is 3 feet, how

[10] Usually the interviewer then asked students about whether in their math class they used letters to represent quantities for which they did not know a value. All MC2 students except one (Charlie_7) said yes.

many of those 3-foot sections do you need to make Stephen's?" Matt wrote, "Rebecca would need five more pieces." When asked whether 3 feet and five more 3-foot pieces would make 15, Matt_8 nodded yes. The interviewer asked if he could show how he knew, and Matt_8 said, "I don't think so." The interviewer pointed at each of the six segments that made up Stephen's length, naming each 3 feet, and asked Matt_8 to check. He said, "It wouldn't, would it! It would actually give you 18, and so it should be . . . I don't really know." The interviewer asked him how he knew to divide 15 by 5 and what that looked like in the picture. Matt_8 talked about cutting Stephen's length into five pieces, and he crossed off the sixth (last) part of the length representing Stephen's cord length. On his paper, he wrote, "It [Stephen's cord length] would actually be 4 more pieces." Then he and the interviewer addressed writing equations, as the following data excerpt shows.

Data Excerpt 1: **Matt_8 works on equations for A1.**

I: Great. So now we have the picture worked out. So here's my next question about this problem. Do you think you could write an equation too, to show the relationship between the lengths of the cords?

M: I don't think so. I didn't really understand the question. [Both smile.]

I: Okay. Do you think you understand it better now?

[Matt_8 shakes his head no.]

I: No? Um, well let me ask one more question about that.

M: I just pretty much winged that one [gesturing to the drawing of the situation].

I: You winged that one? Oh, that's okay. You did a nice job of thinking through it. [Then, for 1.5 minutes the interviewer helps Matt_8 set up letters to represent the unknown lengths in his drawing: x represented "Rebecca's cord" and y represented "Stephen's cord."[11]]

I: Okay. Now the question is can you think of a way to write an equation to relate x and y. And we know that Stephen's cord length is five times the length of Rebecca's.

[After a 15-second pause, Matt_8 shakes his head no.]

I: Not sure? [pause] You've also observed that it would be four more pieces [pointing to what he wrote on his drawing]. Like you would need four pieces, I guess added on to Rebecca's, to make Stephen's. Do you think you could show that with an equation?

Matt_8 wrote $x + 4 = y$ and then $x + x + x + x + x = y$. When asked how he saw his second equation in his picture, he wrote, "if you add these to Rebecca's cord you

[11] Articulating the unknowns specifically as lengths, or as a number of feet, was introduced by the interviewer for 10 out of the 12 students. Students always agreed with this idea, but they did not always record it on paper. For example, students sometimes wrote something like "y = Stephen's cord." We put their exact writings in quotations.

will get Stephen's cord." The interviewer asked Matt_8 to check the equations with the example they had developed while working on the drawing: When Stephen's cord length was 15 feet, Rebecca's was 3 feet. Upon checking the values in the equations, Matt_8 changed $x + 4 = y$ to $x + 12 = y$ but reported that $x + x + x + x + x = y$ worked. The interviewer then asked, "Do you think there's a way to write this equation [$x + x + x + x + x = y$] but use multiplication?" Matt_8 wrote "$x \times 5 = 15$." Upon suggestion from the interviewer that 15 was the example but y was "more general," Matt_8 changed the equation to "$x \bullet 5 = y$." Then, the interviewer questioned Matt about writing another equation.

Data Excerpt 1, continuation: **Matt_8 works on equations for A1.**

I: Could you write another equation for this situation?
M: [25-second pause] I don't know; I don't think so.
I: No? Okay. I'm just wondering because here we have, if we know Rebecca's, we can take it times five, we get Stephen's. Is there a way to write an equation so that if you know Stephen's, you can do something to it to find Rebecca's?
M: [6-second pause] Mm-mm [no].

Matt_8's drawing, for him. Given that Matt_8 did not write equations swiftly or easily, one has to ask: What did Matt_8's drawing mean to him? Based on his deliberation over whether the longer cord length consisted of five versus four parts added to the shorter cord length, one issue for Matt_8 was how he viewed the relationship between the two lengths. He seemed certain that Stephen's cord was longer than Rebecca's and that it consisted of some number of parts equal to Rebecca's. However, his lack of certainty about the number of those parts may mean that the multiplicative relationship between two quantities was not fixed for him. In other words, Stephen's length was bigger, and one could keep adding some amount of Rebecca's lengths to make it, around five of them total, or five more of them beyond Rebecca's. This view could mean that Stephen's length was somewhat uncertain in relation to Rebecca's length. Support for this interpretation includes Matt_8's comment that he had "winged" his drawing—that he was uncertain of it or how it represented the relationship. Although not all MC2 students deliberated as Matt_8 did, two others (Martin_8 and $\mathrm{Charlie}_7$) were similar to Matt_8 in not producing multiplicative equations swiftly or at all.

Matt_8's reticence and use of numerical examples in equation writing. If Stephen's length were somewhat uncertain in relation to Rebecca's, then writing an equation to represent that relationship would indeed be difficult. Because Matt_8 did not write a multiplicative equation after the interviewer restated the given multiplicative relationship, the interviewer chose to remind Matt_8 of the additive relationship that he had articulated. Matt_8 then represented that relationship in two ways, as $x + 4 = y$ and $x + x + x + x + x = y$. He did not produce an equation using multiplication until explicitly prompted by the interviewer. When he did produce it, he used a value from the example, 15 feet, in the equation. Doing so is

evidence that $Matt_8$'s ideas about multiplying were tied to the numerical example rather than to the relationship between the unknowns. So, rather than use the numbers to help him see the structure of the relationships between the two unknowns, $Matt_8$ seemed to rely on the numbers to write his multiplicative equation, only changing 15 to y at the prompting of the interviewer. Thus, for $Matt_8$, the equation $x \cdot 5 = y$ may not have represented a general case beyond the specific example. All other MC2 students demonstrated similar uses of numerical examples in the algebra interview.

$Matt_8$'s lack of reversibility in equation writing. This reliance on numbers is also seen in the lack of reversibility in equation writing for $Matt_8$. Note that Matt did not produce a second equation, such as $y \div 5 = x$, despite the fact that he stated clearly in the numerical example that he had divided 15 by 5 to get Rebecca's cord length, 3 feet. So, his ability to operate with the numerical example to produce the smaller length given the larger one did not translate into an ability to view the structure of the relationships between two unknowns as involving division. In fact, three other MC2 students ($Charlie_7$, $Sheila_8$, and $Lisa_8$) manifested a similar lack of reversibility in equation writing for A1. Another student, $Martin_8$, did write an equation involving division, but like $Matt_8$, he required considerable time and interviewer support to do so.

We explain this lack of reversibility by appealing to the MC2 students' multiplicative concepts. Recall that with whole numbers, MC2 students can produce three levels of units in activity, but in further operating, they do not maintain the three levels. So, for example, $Matt_8$ could generate 15 as a unit of five units, each of which contains three units. However, in further operating, 15 would be a unit of units for $Matt_8$—that structure is what he could take as given. So, consider that working quantitatively with x as a length means x would need to be a unit of an unspecified number of units. Matt could take x five times to make y because he could make three levels of units in activity. However, to divide y to produce x would require continuing to view y as a three levels of units structure, one that could consist of so many equal parts (five), each of which were a unit of units (i.e., x). This structure is precisely not one that $Matt_8$ would be likely to make. So, for $Matt_8$ and other students like him, we interpret the longer unknown length in A1 as an end but not as an entity that could be taken as a new beginning—it could not easily be used to feed back into the situation and produce the starting point. We propose that this way of viewing the longer unknown length helps account for $Matt_8$'s uncertainty in making his drawing and his reticence to write any equations about the situation at all.

$Samantha_7$'s work on A1. Unlike $Matt_8$, $Samantha_7$ produced a picture for A1 in less than a minute (Figure 13). Like $Matt_8$, she indicated uncertainty when asked for an equation: "What do you mean?" she asked. However, in 40 seconds she wrote the equation $S = R(5)$, where S represented "Stephen's length" and R represented "Rebecca's length." She said that this equation showed "Stephen's cord is

Rebecca's times five" and that Stephen's cord would equal Rebecca's if Rebecca's cord were five times its size. When asked for another equation, she sat silently for 25 seconds. Then, she wrote $S/5 = R$, saying, "I don't know if this is right; I'll just guess." Her explanation for this equation was that to "solve it," a person would multiply five on both sides, which would result in her first equation. The interviewer then pressed Samantha_7 on the meaning of her second equation in relation to her picture of the situation. Samantha_7 said that the second equation meant Stephen's cord divided by five would be Rebecca's cord, but she also said she was not sure if that was true.

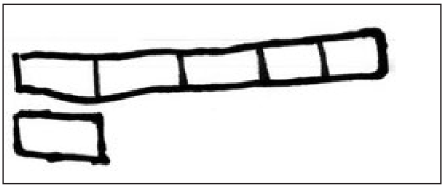

Figure 13. Samantha_7's picture for A1.

Using 5 as both a multiplier and divider of an unknown with quantitative meaning is initial evidence that Samantha_7 had constructed reversibility in her equation writing, at least with a whole-number multiplicative relationship between two unknowns. This evidence could mean that, in contrast with the other five MC2 students, Samantha_7 could take S as an entity that could be operated upon further. One possible account is that Samantha could maintain S as three-levels-of-units structure and so was likely becoming an MC3 student. We explain this possibility in the next subsection on the MC3 students' work. However, because Samantha_7 expressed quite a bit of uncertainty about her equations for A1, we believe we should be conservative in attributing this possibility to her. We entertain a different account later in the paper, when we present our hypothesis about a facilitative link between students' construction of fractional numbers and how students represent multiplicatively related unknowns.

MC3 students' work on A1: Swift equation writing and reversibility. In their work on A1, all six MC3 students spent no more than 1.5 minutes drawing and justifying their pictures in relation to the situation. Typically, they drew a long segment or bar partitioned into five equal parts and a smaller segment or bar that spanned one fifth of the longer length (cf. Figure 13). When the interviewer asked whether students could write an equation to represent the situation they had drawn, all six MC3 students wrote an initial equation with 5 as a multiplier of an unknown representing Rebecca's cord length, setting that equal to another unknown representing Stephen's cord length. No MC3 student spent more than 1.75 minutes doing so. For example, Gloria_8 wrote $y = 5x$, where x represented "Rebecca's cord length" and y represented "Stephen's cord length." She articulated this equation as "Stephen's cord is 5 times Rebecca's cord." When asked to write another equation, Gloria_8 wrote $x = (1/5)y$, explaining that

"Rebecca's cord is equal to one fifth of Stephen's."

The other MC3 students did similar work, although four of them initially wrote their second equation using a whole number as a divider (e.g., $y/5 = x$). When these four students were asked whether they could write this equation using multiplication, they used 1/5 as a multiplier on their unknown. Three did so within 30 seconds of being asked, and one did so within 1 minute. Overall, the equations the MC3 students produced relatively swiftly are evidence that they had constructed reversibility in equation writing; indeed, their work is initial evidence of reciprocal reasoning. All MC3 students also used a numerical example to check their work in which they appeared to "see through" the numbers to a quantitative structure (Thompson, 1993), verifying that they had produced correct equations.

An account of MC3 students' work. The swiftness of the equation writing of MC3 students can be partly explained by familiarity with algebraic notation. The eighth- and tenth-grade MC3 students were in algebra classes in which they routinely used letters to represent unknowns and variables; the other students were in prealgebra classes where they did so as well, but perhaps not as regularly. However, we don't think that familiarity explains everything, particularly because the two seventh-grade MC3 students were also in a prealgebra class.[12] We offer an account of the MC3 students' work based on their multiplicative concept.

Because MC3 students take the coordination of three levels of units as a given, they can view 5 times an unknown as a three-levels-of-units structure: Stephen's cord length can be seen as a unit of five units, each of which could contain some number of units that is unspecified. The students can take this three-levels-of-units structure as a given in further operating, which would facilitate writing an equation like $y = 5x$, where the students operate further on both y and $5x$ by equating them. In addition, maintaining this three-levels-of-units structure in further operating means the students can view Rebecca's cord length as one of these five units of units or as one fifth of Stephen's cord length. Doing so means that the equation $y/5 = x$ is a natural consequence of $y = 5x$. For the MC3 students in this study, dividing a quantity by 5 appeared to be the same as multiplying that quantity by 1/5. So even though four students did not initially produce $(1/5)y = x$ as a second equation, they produced it relatively promptly when asked about whether they could write their division equation using multiplication. Their activity provides initial evidence that they had constructed a unit fraction as a multiplier of an unknown, although some may have used 1/5 as a multiplier because they had been told to write algebraic expressions involving division in a certain way. Table 2 summarizes the equation writing for A1 for all students.

[12] In our selection interviews we did not find any MC2 students in the algebra classes.

Table 2
Equations Written for A1

	Name (pseudonym), grade, class	$y = 5x$	$x = y/5$	$x = (1/5)y$
MC2	Charlie, 7, basic mathematics	No	No	No
	Sheila, 8, prealgebra	No	No	No
	Matt, 8, prealgebra	Eventually	No	No
	Martin, 8, prealgebra	Eventually	Eventually	No
	Lisa, 8, prealgebra	Yes	No	No
	Samantha, 7, advanced mathematics	Yes	Yes	No
MC3	Willa, 7, advanced mathematics	Yes	No	Yes
	Gloria, 8, algebra	Yes	No	Yes
	Peter, 7, advanced mathematics	Yes	Yes	Yes
	Suzanne, 8, algebra	Yes	Yes	Yes
	Liam, 8, algebra	Yes	Yes	Yes
	Hector, 10, algebra	Yes	Yes	Yes

Note. y represents Stephen's cord length, and x represents Rebecca's cord length. "Yes" means the student wrote the equation with little prompting and explained it with reference to quantities in the problem. "Eventually" means the student wrote the equation after more than 10 minutes of discussion and interviewer questioning.

MC2 students' work on A5: No reciprocal reasoning. Based on these interviews, none of the MC2 students had constructed reciprocal reasoning. For example, despite explicit questioning during work on A1, no MC2 student multiplied an unknown by 1/5 to write an equation to find Rebecca's cord length given Stephen's cord length (see Table 2). After spending nearly 2.5 minutes on this problem, Samantha_7 concluded, "Yeah, it is kind of a hard question."

Confirming evidence of this claim occurred in the students' work on A5:

> A5. *Theo-Sam CD Stack Height Problem*. Theo has a stack of CDs some number of cm tall. Sam's stack is two fifths of that height. Draw a picture to represent this situation. Can you write an expression for how tall the height of Sam's stack is? Can you write an equation based on your expression? Can you write another equation for the situation?

All MC2 students except Charlie_7 drew a segment or rectangle partitioned into five equal parts to show Theo's stack height, and they identified Sam's stack height as spanning two of those parts (Figure 14). The five students spent 20–35 seconds drawing their pictures. These five students also wrote at least one equation for A5,

taking from 70 seconds to 3 minutes. Two students, $Lisa_8$ and $Matt_8$, wrote only additive equations with fractions. For example, $Lisa_8$ wrote $S + 3/5 = T$, where S represented "Sam's stack" and T represented "Theo's stack." A third student, $Martin_8$, used division in his equation: $25 \div x = 2/5$, where x was not defined. He wrote no other equations. Thus, we could not attribute reciprocal reasoning, or fractions as multipliers on unknowns, to these students. The other two students, $Sheila_8$ and $Samantha_7$, used multiplication in their equations for A5. Because $Sheila_8$ did not show evidence of quantitative meaning in explaining her equation, we focus on $Samantha_7$'s work below.

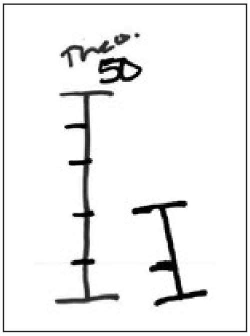

Figure 14. $Samantha_7$'s picture for A5.

$Samantha_7$'s work on A5. After drawing a picture (Figure 14), $Samantha_7$ wrote an additive equation, $T - 2/5 = S$, which she then changed to $T - 3/5 = S$. For her, T represented "Theo's," and S represented "Sam's." The interviewer proposed an example, 10 cm as Sam's stack height, to check this equation. $Samantha_7$ determined with her picture that Theo's stack height was 25 cm. She checked her equation with this pair of numbers and quickly saw that the equation did not work. The interviewer confirmed her conclusion by suggesting that three fifths "by itself" did not work and then asked what the three fifths was "of." $Samantha_7$ indicated that it was three fifths of Theo's stack of CDs. When asked how to represent that, she said it would either be $3/5 - T$ or $T - 3/5$. Then, based on earlier evidence that she had used whole numbers as multipliers and dividers of unknowns, the interviewer made the following suggestion.

Data Excerpt 2: **$Samantha_7$'s work to write an equation for A5.**

I: So, so far you've been using subtraction to write the equations. We can see it's not quite working out even though there's some things that make sense about it. Do you think you could use multiplication to write equations?

S: Yeah [uncaps her pen].

I: What would it look like?

S: Should I try this? [She computes 25/1 • 3/5 using her standard computational algorithm, getting 15. She considers subtracting this from 25 in the equation.]

I: I wonder, what would you multiply 25 by to directly get Sam's stack height?

S: [5-second pause, softly, slowly] You could multiply . . . two fifths maybe.
I: Would that work?
S: [Computes $25/1 \times 2/5$, getting 12 and then 10] Yeah, it's right.
I: Oh, okay. All right. Does that make sense in the problem?
S: [Shaking her head no] Well, not in the drawing. It did if I left this [changes 3/5 back to 2/5 in her equation, writing $T - 2/5 = S$]. This is what I was saying when I had it as two fifths, I was saying that T minus 2/5 is the same as Sam's.
I: [Nodding] Now you wrote T minus two fifths here, but I'm not sure you did T minus two fifths here [pointing to her computation].
S: I did T times 2/5.
I: Hmm.
S: So I should change that to T times 2/5 [writes "$T \bullet 2/5 = S$"].

Following this exchange, the interviewer proposed another example, letting Theo's stack height be 50 cm. Samantha$_7$ was not sure how to determine Sam's height using reasoning and her picture. She spent nearly 4 minutes doing so with questioning support from the interviewer. When Samantha$_7$ arrived at 20 cm for her answer, she used the standard computational algorithm to check that her equation $T \bullet 2/5 = S$ was correct. In the final work on A5, the interviewer asked Samantha$_7$ whether she could write an equation to determine Theo's stack height given Sam's stack height. Samantha$_7$'s final equation for this part of the problem was $S \bullet 2 + 10 = T$, which she developed over the course of 3.5 minutes.

Samantha$_7$'s picture, for her. Given that Samantha$_7$ did not initially produce multiplicative equations and that her work was the most advanced of all MC2 students, we must consider what Samantha$_7$'s picture meant to her. Like other MC2 students, she clearly seemed to see that 2/5 combined with 3/5 made Theo's length as a referent unit. Because she persisted with equations involving subtraction even after she had made some multiplicative computations, it is likely that this was an enduring view for her: 2/5 of a quantity was made by taking away 3/5 of it. It seemed to be quite a change, influenced by fairly explicit interviewer prompting with a numerical example, to view 2/5 of a quantity to be made from multiplying by 2/5. We link this view to the fraction scheme we attribute to Samantha$_7$ in which fractions are not multiples of unit fractions, a partitive fraction scheme. For Samantha$_7$, fractions such as 2/5 can be made by iterating one fifth two times, but the result is still based on part-whole meanings, two parts out of five parts. Without an explicit multiplicative view of fractions, Samantha$_7$'s main way of operating with fractional parts of an unknown appeared to be additive (cf. Steffe & Olive, 2010).

Samantha$_7$'s use of numerical examples in equation writing. In addition, like Matt$_8$ in solving A1, Samantha$_7$'s work demonstrates some aspects of being tied to numerical examples rather than seeing through them to a general, quantitative structure. With interviewer prompting, she did produce a correct initial multiplicative equation for a general relationship between the two unknowns.

However, her second equation, $S \cdot 2 + 10 = T$, was tied to the example of Theo's stack height being 50 cm.

Samantha$_7$'s nascent reversible reasoning but lack of reciprocal reasoning. Still, with that second equation, Samantha$_7$ demonstrated that she was trying to use Sam's stack height as a unit with which to measure Theo's: Sam's stack height times two produced most of Theo's stack height. We can conclude that Samantha$_7$ was beginning to reason reversibly in trying to produce Sam's stack given Theo's. However, Samantha$_7$ did not create the fifths of Theo's stack height as halves of Sam's, and so to deal with the extra part necessary to make Theo's, she added on 10 cm, which came from the numerical example she had been using. The only other MC2 student to work somewhat in this way was Lisa$_8$ (Hackenberg, 2014), but she did so solely with a numerical example without any use of unknowns.

We use Samantha$_7$'s multiplicative concept and fraction scheme to help account for the beginnings of reversible reasoning but lack of reciprocal reasoning. Because Samantha$_7$ could make three levels of units in activity, she could create Theo's stack height as a unit of five units, each of which contains some number of units. Based on her partitive fraction scheme, she disembedded two of those units of five units, each containing some number of units, to represent Sam's stack height. To then use Sam's stack height to make Theo's requires taking Sam's stack height as given in further operating. We infer that Samantha accomplished this goal because she determined that Theo's stack height was two of Sam's and some more. However, we infer that Samantha$_7$ did not maintain Sam's height as a unit of two units, each of which was equal to one of the five units that comprised Theo's height. If she had done so, she would have seen Theo's stack as a unit of five units, any of which could be iterated twice to make Sam's. This way of thinking is precisely that of the iterative fraction scheme, which Samantha$_7$ had yet to construct.

MC3 students' work on A5: Elicitation of reciprocal reasoning. In their work on A5, all MC3 students drew a long segment or bar marked into five equal parts and then a smaller one that spanned two of those parts, and they justified their pictures in relation to the situation (cf. Figure 14). They spent from 42 to 90 seconds on their drawings. All MC3 students wrote a correct initial equation in which they used a fraction as multiplier on an unknown, taking from 18 to 78 seconds. Five students used 2/5 as a multiplier. For example, Suzanne$_8$ wrote $S = 2/5T$, where S represented "height of Sam's" and T represented "height of Theo's." She stated that this equation meant "Sam's is two fifths the height of Theo's." In contrast, one MC3 student (Gloria$_8$) wrote an initial equation using 5/2 as a multiplier ($T = 5/2S$).

Although their initial equations on A5 were similar in structure, the MC3 students' work to write a second equation for this situation varied. One student, Willa$_7$, never used a fraction as a multiplier in her second equation but instead wrote $(5S)/2 = T$, using whole numbers as multipliers and dividers. We could not conclude that Willa was reasoning reciprocally in her work on A5 even though

she did produce a correct second equation (Lee & Hackenberg, 2014). However, the other five students developed reciprocal reasoning in the course of the interview. Two students (Gloria$_8$ and Liam$_8$) developed a correct second equation with a fraction as a multiplier, and they used only multiplicative equations in their solution process. The three remaining students, Suzanne$_8$, Hector$_{10}$, and Peter$_7$, began by writing additive equations and then produced multiplicative equations—and reciprocal reasoning. We give an example of Suzanne$_8$'s work to show this development in contrast with Samantha$_7$'s work on A5.

Suzanne$_8$'s work on A5. When asked for a second equation, Suzanne$_8$ wrote $S = T - 3/5$. Upon determining that Sam's stack height was 4 cm if Theo's was 10 cm and using that to check her equation, Suzanne$_8$ changed her equation to $S = T - 3/5T$. She explained, "You need the variable so you can make sure that you have three fifths of Theo's height, not just three fifths in general" (as a number). Then, the interviewer asked for an equation to determine Theo's height if she knew Sam's. Over a duration of 30 seconds, Suzanne$_8$ wrote $T = S + 3/5S$. In the following excerpt, the interviewer asked her to explain that equation.

***Data Excerpt 3:* Suzanne$_8$'s work on A5.**

I: Okay. So explain that equation.

S: Well, um [4-second pause]. [She crosses out the equation. Then she writes "$T = 2.5S$".]

I: Oh, okay, you have a new idea there. So tell me how you came up with that.

S: Well, because if S is two fifths of T, then if you times it by 2.5 [softly] you'll get it.

I: You'll get it—it will work out?

S: Mm-hmm. [pause] Like I have these numbers in my head.

I: I see. Did you use 10 and 4 or did you—

S: [Nods] Mm-hmm [yes].

I: Okay. So when you did this one [pointing to $T = S + 3/5S$], did you test it with numbers? [Suzanne shakes her head no.] How did you know that you wanted to cross that one out?

S: [Smiling] Because I went to explain it and I couldn't.

I: Fair enough. Do you see in your picture this idea of two and a half [pointing at the $T = 2.5S$ equation]? [Suzanne$_8$ nods yes.] Can you explain how you see it in your picture?

S: 'Cause this is 2/5 [pointing at 2/5 of Theo's height in her drawing], so if you doubled Sam's, then it would be 4/5, and then if you added another half of it, it would be 5/5.

I: I see. Could you write this [pointing to the $T = 2.5S$ equation] as a fraction?

[Suzanne$_8$ writes "$T = 2\ 1/2\ S$".]

I: Does this multiplier of S [circling the 2 1/2 with finger] have any relationship to the multiplier of T in your original equation?

S: [4-second pause] Well, this [pointing to the 2 1/2] is—could be the reciprocal.
I: Oh, okay. If you wrote it differently?
S: Mm-hmm. If you wrote it [writes "$T = (5/2)S$"].

Suzanne$_8$'s use of numerical examples in equation writing. First we note that Suzanne$_8$ appeared to be using her letters algebraically. She likely tried out her equation $T = 2.5S$ with a numerical example mentally, because she said "I have these numbers in my head." However, she did not then remain tied to that specific example but returned to the letters in her equation. So, she used the numerical example to inform the more general statement. Furthermore, she did not always accept or reject an equation based on the use of a numerical example: When asked if she had tested $T = S + 3/5S$ with numbers, she said no; she rejected that equation because she could not explain it, presumably in relation to how she had portrayed the quantities in her picture.

Suzanne$_8$'s reciprocal reasoning. This work also demonstrates that determining how to measure T in terms of S was not automatic for Suzanne$_8$, but in the process of solving the problem, she began to view T as two-and-one-half lengths of S. That is, Suzanne$_8$ switched from viewing Theo's stack height as a referent unit by which to measure Sam's, and instead viewed Sam's stack height as a referent unit by which to measure Theo's. However, in contrast with Samantha$_7$, Suzanne$_8$ went further because she proposed 2.5 as a multiplier of S. By doing so, Suzanne$_8$ demonstrated that she was conceiving of fifths of Theo's stack height as halves of Sam's. In summary, to develop reciprocal reasoning, Suzanne$_8$ had to learn to take 2/5 of one referent unit as a new referent unit and then use it to measure the original referent unit. Making this switch required that Suzanne$_8$ see Theo's height as consisting of wholes and parts of Sam's height, specifically as two wholes and one half of Sam's or as five halves of Sam's. Thus, developing reciprocal reasoning to solve A5 required that 5/2 was a fractional number for Suzanne$_8$—that is, it required the construction of an iterative fraction scheme.

Indeed, we suggest that constructing reciprocal reasoning requires a bit more because any unknown composite unit can be taken as a referent unit to measure any other. We propose that doing so involves an abstraction of a student's iterative fraction scheme so that it can be used in relation to any two composite units. Although we cannot claim this level of generality for Suzanne$_8$, she did use her reasoning from A5 in the next interview problem, A6, which indicates that it held meaning for her. All other MC3 students except Willa$_7$ also used their reciprocal reasoning on A6. A summary of the students' equations for A5 is shown in Table 3.

A Facilitative Link Between Fractional Numbers and Multiplicatively Related Unknowns

Based on data and analysis presented in the prior sections, the results from this study suggest that operating with the third multiplicative concept facilitates representing multiplicatively related unknowns with algebraic notation. It also suggests

that an iterative fraction scheme is a constructive resource in reciprocal reasoning. We develop both points further by proposing a facilitative link between the construction of fractional numbers and how students represent multiplicatively related unknowns. This account addresses our third research question about how students' quantitative reasoning with fractions may be related to their use of unknowns to write equations in solving algebra problems.

Table 3
Equations Written for A5

	Name (pseudonym), grade, class	Initial additive equation	$y = (2/5)x$	Second additive equation	$x = 2.5y$ or $x = (2\frac{1}{2})y$	$x = (5/2)y$
MC2	Charlie, 7, basic math	--	--	--	--	--
	Martin, 8, prealgebra	No	No	--	--	--
	Sheila, 8, prealgebra	No	Yes[a]	--	--	--
	Matt, 8, prealgebra	Yes	No	--	--	--
	Lisa, 8, prealgebra	Yes	No	--	--	--
	Samantha, 7, adv. math	Yes	Yes	Yes	No	No
MC3	Willa, 7, adv. math	No	Yes	No	No	No[b]
	Gloria, 8, algebra	No	Yes	No	No	Yes
	Liam, 8, algebra	No	Yes	No	Yes	Yes
	Hector, 10, algebra	No	Yes	Yes	Yes	Yes
	Suzanne, 8, algebra	No	Yes	Yes	Yes	Yes
	Peter, 7, adv. math	No	Yes	Yes	Yes	Yes

Note. y represents Sam's stack height and x represents Theo's stack height. Two dashes (--) means that the student did not produce equations.
[a]Sheila did not appear to have quantitative meaning for this equation.
[b]Willa's second equation was $x = (5y)/2$.

Implicit units coordinations in fractional numbers and multiples of unknowns. We invite readers to recall the three-levels-of-units structure involved in conceiving of a number like seven fifths as the result of an iterative fraction scheme (Figure 15, top). A student who has constructed an iterative fraction scheme views seven fifths as a unit of seven units (fifths), any of which imply a unit of five units (five fifths). So the student coordinates two composite units that are nested, although they are not nested in quite the same way as the units in the equal partitioning of each part of a partitioned length. That is, consider structuring a length as a unit of seven units, each of which contains five units. Doing so involves inserting composite units of five units into each unit of the composite

unit of seven units (Figure 15, bottom). Explicit insertion is instrumental in initially making this structure. In contrast, with seven fifths, the units coordination is not so explicit. Instead, each unit fraction (fifth) in the unit of seven units (seven fifths) *implies* the composite unit from which it was made (five fifths). Here the word *implies* means that a person with an iterative fraction scheme does not have to run through all of the coordinations involved—that is, does not have to take each fifth from the seven fifths and iterate it to make five fifths. Instead, making the coordination one time, as in creating seven fifths with one of the fifths from the given whole, stands in for making it any of the other times. Thus, the construction of an iterative fraction scheme involves an abstraction of one's units coordinations.

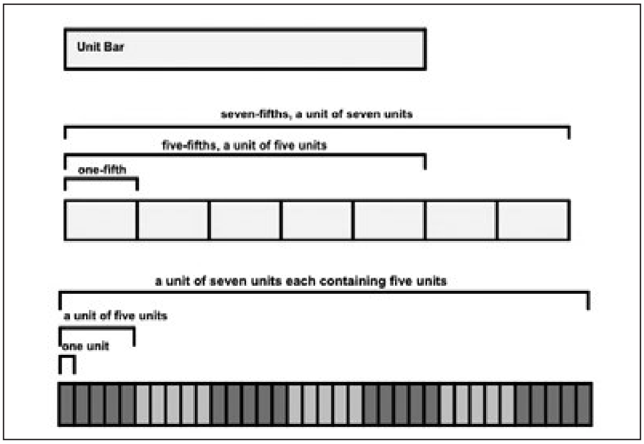

Figure 15. The three-levels-of-units structures of seven fifths (top) and a partitioned seven-unit length (bottom).

In whole-number units coordinations, the insertion of units within units may become implied for students (i.e., when units coordinations are interiorized); however, in the fractional number case, the nesting of units is implicit from the start. We propose that the construction of an iterative fraction scheme may allow students to adapt their multiplicative concepts so that implicit, abstracted units coordinations are featured. In other words, constructing an iterative fraction scheme may foster implicit, abstracted units coordinations as a norm in a student's mathematical thinking.

To us, this analysis suggests a facilitative link between the construction of fractional numbers and the construction of whole-number multiples of unknowns[13] that are usable in further operating, such as equation writing. Both constructions involve implicit, abstracted units coordinations in similar, though not identical, ways. In making a multiple of an unknown that is usable in further operating, students may iterate a partitioned composite unit (the unknown) the requisite number of times and take that amount as a unit. Students who do so treat the result

[13] The word *multiple* usually means the product of a quantity and an integer. We restrict the meaning to positive multiples here.

as a unit consisting of the number of “containing” units (number of times the unknown is iterated), each of which contain a number of units that is unspecified (Figure 16). So, this aspect of the coordination is implicit. Here the word *implicit* means that a person cannot actually run through all of the coordinations: A person can imagine making the coordination once, or several times, but cannot actually make all of them because the number of units making up the unknown is unspecified. Thus, constructing multiples of unknowns appears to require an abstraction of one’s units coordinations.

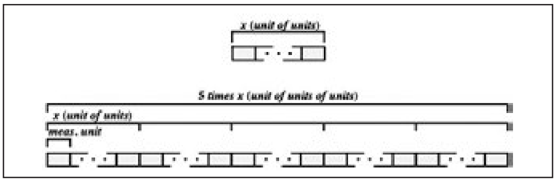

Figure 16. Units coordinations involved in x (top bar) and $5x$ (bottom bar).

We suggest that it could be useful to experience the notion of one coordination as representative of all (even though a person could actually make all), as in the case of fractional numbers, in order to be able to use one coordination as representative of all when a person cannot actually make them all, as in the case of whole-number multiples of unknowns.[14] In short, abstracting units coordinations in the construction of an iterative fraction scheme may be quite useful in abstracting units coordinations to construct and operate with multiples of unknowns. Furthermore, the link could proceed in the other direction. That is, constructing and operating with multiples of unknowns could be one component in facilitating the construction of an iterative fraction scheme. Thus, the link may be mutually supportive. This proposal is consistent with researchers who suggest such links between students’ fractional and algebraic knowledge (e.g., Kilpatrick & Izsák, 2008; Steffe, 2001).

The case of MC2 student Samantha$_7$. A mutually supportive link could shed light on the case of Samantha$_7$, the MC2 student who made more progress than other MC2 students in representing multiplicatively related unknowns. Samantha$_7$ was an MC2 student, similar to some other students in prior research, who had not interiorized three levels of units but had constructed schemes and operations beyond what might be expected of a “typical” MC2 student (Hackenberg, 2007, 2010). Our model of her fractional knowledge that extends beyond what is presented in this paper is consistent with models of these other students. Together these models suggest either that there may be an intermediary multiplicative concept in between the second and third concepts or that some MC2 students can engage in a great deal of learning that allows them to construct some schemes and operations that are unexpected for their multiplicative level.

[14] Note that we are not claiming that the construction of an iterative fraction scheme is the *only* way to experience this kind of implicit units coordination.

One account that we offer for further investigation is whether MC2 students like Samantha$_7$ can learn to use whole numbers as multipliers on unknowns but not hold the three-levels-of-units structure in mind in the way that we have described it. For example, perhaps these students can make the three-levels-of-units structure in their activity and then “drop out” the units of units within each of the five units, while still continuing to act on the unit of five units—at least within some boundaries. In this sense, their concepts of unknowns may not be quantitative in the way that we have described because while they are operating, the unknown ceases to be a unit of an unspecified number of units. We conjecture that MC2 students like Samantha$_7$ will still not be able to operate multiplicatively with unknowns in the way that MC3 students do, and this was borne out in our study because we could not attribute fractions as multipliers of unknowns, or reciprocal reasoning, to Samantha$_7$.

Discussion and Concluding Remarks

Representing and operating on multiplicatively related unknowns are necessary for writing and solving equations as students progress in learning algebra. In this study, we found that students with different multiplicative concepts represented multiplicatively related unknowns in qualitatively different ways. In particular, MC2 students did not use fractions as multipliers of unknowns, and many of them were also challenged to use algebraic notation to represent whole-number multiplicative relationships between unknowns. These students also had not constructed iterative fraction schemes. In contrast, MC3 students used both whole numbers and fractions as multipliers of unknowns, and they had constructed iterative fraction schemes. Our analysis indicates that the students’ multiplicative concepts play an important role in these differences. Thus, this study provides an operationally based account for why representing and operating on multiplicatively related unknowns is challenging for many students.

The complexity of representing and operating on multiplicatively related unknowns may help explain several findings from prior research. First, it may help account for why students do not write and solve equations when they can use strategies like guessing and checking values or unwinding arithmetical operations (Bednarz & Janvier, 1996; Johanning, 2004; Nathan & Koedinger, 2000; Swafford & Langrall, 2000). Second, it may be one reason why researchers have posited a gap between students’ arithmetical and algebraic knowledge that hinges on students’ difficulties in operating on unknowns when solving equations (Bednarz & Janvier, 1996; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). Third, it may help illuminate why some early algebra researchers have made progress with students by focusing initially on sustained work with additive changes to unknowns (e.g., Carraher et al., 2006; Carraher et al., 2008).

Fourth, this study may help explain the distinction between arithmetic and algebraic uses of letters (e.g., Küchemann, 1981; Slavit, 1999; Vlassis, 2002). Recall that students who use letters arithmetically are temporarily waiting to evaluate with numbers, whereas students who use letters algebraically can refrain from attributing numerical values to the letters. In this study, we saw related

distinctions in how MC2 versus MC3 students used letters to represent unknowns. Often the MC2 students seemed to be tied to numerical examples, forming some of their equations around specific instances of the unknowns. In contrast, the MC3 students tended to use numerical values to inform their more general statements rather than remaining tied to them. We propose that being able to refrain from using specific values, or using values but not being tied to them, may be linked to being able to hold units coordinations implicit. MC3 students do not have to implement all of the units coordinations involved in an equation in order for the equation to be meaningful to them: The units coordinations are meaningful without implementation, so the students can refrain from evaluating them. In contrast, MC2 students do have to implement units coordinations at three levels for them to be meaningful; so, refraining from evaluation may not seem natural or sensible to these students. To us, this suggests that MC2 and MC3 students are operating at different levels of abstraction with respect to multiplicatively related unknowns.

Taken together, the findings about fractional numbers and multiplicatively related unknowns in this study led to our proposal that the construction of improper fractions may facilitate representing multiplicatively related unknowns, based on an analysis of the nature of the units coordinations involved in each case. The proposal of this facilitative link supports the stance of Empson and colleagues (Empson & Levi, 2011; Empson et al., 2011) that relational thinking with fractions is fundamentally algebraic. Specifically, it suggests that time spent abstracting units coordinations so that they become implicit in one's thinking is time well spent in developing both robust fractional knowledge and algebraic reasoning. More broadly, the proposed facilitative link may help develop claims that learning fractions is important for learning algebra (Fennell et al., 2008). Certainly, this proposed link requires further study. If it proves robust, it implies that curricular pathways should be designed to support both MC2 and MC3 students in constructing iterative fraction schemes and in cultivating ideas about multiplicative relationships between unknowns as potential measurements of quantities. We anticipate the need for more than one curricular pathway because MC2 students appear to encounter a rather resilient constraint in constructing iterative fraction schemes (Hackenberg, 2007). Such differentiation of instruction seems essential if we are to support students in learning algebra rather than use algebra as an "engine of inequity" (Kaput, 1998, p. 25).

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APPENDIX A

Selection Interview Questions

S1. There are 6 cans of juice in a package and 8 packages in a box. A crate contains 5 boxes. How can you figure out how many cans of juice are in a crate? Can you draw a picture to show how you know?

S2. There are 96 students going on the field trip and 6 buses. The same number of students is to ride on each bus. Draw a picture to show how you will determine how many students will ride on each bus. How many students will ride on each bus?

a. *If this problem is hard, try*: In a classroom, there are 6 rows with 4 seats in each row. The teacher adds 12 more seats to the classroom. How many total rows can be made with four seats in each row? How many seats will be in the classroom?

S3. This segment represents a picture of a piece of licorice. Theo's piece of licorice is five times longer than this one. Can you imagine Theo's piece? Can you draw it? Show me how you know that this piece of licorice is five times longer than Theo's. What fraction of this piece of licorice is Theo's piece? (If this problem is hard, pose the problem with two times or three times.)

S4. The Giant Soda at the convenience store is 24 ounces. That's 8 times the amount of soda that Stephanie drank. What would you do to find out how much Stephanie drank? Draw a picture to show.

S5. Camika has $21. That's 1/7 the size of the amount of money that Rickard has. What would you do to find out how much money Rickard has? Draw a picture to show.

S6. At a small party, five friends split the submarine sandwich shown (a rectangle) fairly. Can you draw that? You take one of these pieces and share it fairly among four people, you and three friends. How much of the whole sandwich do three of these people get? (If this problem is hard, ask about the share for just one of the three people.)

S7. The drawing below (a rectangle) is a picture of a candy bar. Draw a candy bar that is seven fifths of that bar.

S8. The two rectangles represent two identical candy bars. Show how you'd share them equally among three people. How much of one candy bar would one person get?

S9. The picture (a rectangle partitioned into three equal parts) represents a sub sandwich that is 3 feet long. That's five times the amount of sandwich Sara wants to eat. Draw how much she will get. How much of a foot does she eat?

APPENDIX B

Fractions Interview Questions

F1. The drawing (a line segment) below shows my piece of string. Think of your piece of string so that mine is five times longer than yours. Can you draw what you're thinking of? Can you show for sure that mine is five times longer than yours?

F2. Sara has a stack of CDs that is 65 centimeters tall. That's five times the height of Roberto's stack. Draw a picture of this situation. How tall is Roberto's stack of CDs? How did you solve this problem?

F3. Tyrone has \$21. That's 1/7 the size of Cammy's amount of money. Draw a picture of this situation. How much money does Cammy have? How did you solve this problem?

F4. Share three identical sub sandwiches (shown by three identical rectangles) equally among five people. Show the share for one person. Probe their picture and how they made it. How much of a sandwich does each person get?
  a. If difficult, try two sandwiches among three people.
  b. If they solve the problem, try five sandwiches shared fairly among seven people.

F5. *Improper Fraction Problem*. The drawing (a rectangle) is a picture of a candy bar. Draw a separate candy bar that is 9/7 of that bar. If the student completes this problem, ask her or him to shade one piece and tell how much it is of the original bar.

F6. *Making the Whole Problem*. This candy bar (a rectangle) is 3/5 the size of another candy bar. Make a separate drawing of the other candy bar. Ask how they made their drawing.
  a. If difficult, try: This candy bar (a rectangle) is 1/5 of another candy bar. Make a separate drawing of the other candy bar.
  b. If they solve the problem, try: This candy bar (a rectangle) is 4/3 of another candy bar. Make a separate drawing of the other candy bar.

F7. Tanya has \$84, which is 4/7 the size of David's amount of money. Draw a picture of this situation. How much does David have?
  a. If difficult, try: Tanya has \$15, which is 1/5 the size of David's amount of money. Draw a picture of this situation. How much does David have?
  b. If they solve the problem, try: Cassie earned \$48 babysitting. That's 4/3 the amount of money Serena earned. Draw a picture of this situation. How much money did Serena earn?

F8. This candy bar (a rectangle) is partitioned into three equal parts, so it's a 3/3-bar. Make that bar into a 5/5-bar but don't erase the half mark. Probe their reasoning.

F9. Draw a picture of 1/3 of 1/7 of this rectangular cake (a skinny rectangle). How much is that piece of the whole cake? How do you know?

a. If they solve the problem, try 3/5 of 1/7 of the cake.

b. If they solve (a), try 1/7 of 3/5 of the cake.

APPENDIX C

Algebra Interview Questions

A1. *Cord Length Problem*. Do you have a cord with earplugs for listening to music? How long do you think it is? It's not a value that we know exactly, right? But we could measure it to find the exact value. Stephen has a cord for his iPod that is some number of feet long. His cord is five times the length of Rebecca's cord.

a. Could you draw a picture of this situation? Describe what your picture represents.
b. Can you write an equation for this situation? Can you tell me in words what your equation means?
c. *As necessary*. Can you check your equation with your picture?
d. *As necessary*. Check your equation using this question: Who has a longer cord, Stephen or Rebecca?
e. Can you write more than one equation? *As necessary* (if they have only written something like $t = 5*q$, where t represents Stephen's cord length and q represents Rebecca's cord length): Can you write an equation to express Rebecca's cord length in terms of Stephen's?
f. *As necessary* (if they have written something like $t = q \div 5$): Can you write this equation using multiplication?
g. Let's say that Stephen's cord is 15 feet long. Explain how to find the length of Rebecca's cord.

A2. *If A1 is difficult, try this*: How tall do you think your math teacher (or the principal) is? It's not a value that we know exactly, right? But we could ask him or have him stand by a measuring tape and find out the value. Let's say we know that he is three times the height of a little boy who's one year old. We don't know the one-year-old's height.

a. Could you draw a picture of this situation? Describe to me what your picture represents.
b. Can you write an equation for this situation? What does your equation mean in words?
c. *As necessary*. Can you check your equation with your picture?
d. Can you write more than one equation? *As necessary* (if they have written something like $t = 3*b$, where t represents the height of the teacher and b represents the height of the baby): Can you write an equation to determine the height of the little boy in terms of the height of the teacher?
e. *As necessary* (if they have written something like $b = t \div 3$): Can you write this equation using multiplication?
f. Let's say that the teacher/principal is 6 feet tall. How tall is the little boy?
g. Let's say that the teacher/principal is 7 feet tall. How tall is the little boy?
h. Let's say that the boy is 2 and a half feet tall. How tall is the principal/teacher?

A3. There are five identical candy bars (rectangles) and each candy bar weighs some number of ounces. Let's say that h = the weight of one bar. How much does 1/7 of all the candy weigh?

a. If this question is hard, start with two or three bars and ask about 1/3 or 1/5.
b. If still hard, use sharing language to find out about whether the student can make equal shares.
c. Could you draw a picture of 1/7 of all the candy?
d. Can you write down an expression for the weight of 1/7 of all the candy?

A4. *Optional, depending on time*. There are three candy bars on the table, each of different weight. The first weighs a ounces, the second weighs b ounces, and the third weighs c ounces.

a. How much does 1/5 of all the candy weigh?
b. Could you draw a picture of 1/5 of all the candy?
c. Can you write down an expression for the weight of 1/5 of all the candy?

A5. T*heo-Sam CD Stack Height Problem.* Theo has a stack of CDs some number of cm tall. Sam's stack is two fifths of that height.

a. Draw a picture to represent this situation.
b. Can you write an expression for how tall the height of Sam's stack is? Can you tell me in words what your expression means?
c. Can you write an equation based on your expression in (b)? Can you tell me in words what your equation means?
d. Can you write another equation for the situation? Can you tell me in words what your equation means?
e. Ask them to test their equations with particular numbers, such as 50 cm as the height of Theo's stack.

A6. *Optional, depending on progress on A5.* Christina earned some money babysitting. That's 4/3 of what Serena earned.

a. Draw a picture to represent this situation.
b. Can you write an equation that relates the amount of money Christina earned to the amount of money Serena earned? Can you tell me in words what your equation means?
c. Can you write another equation? Can you tell me in words what your equation means?
d. Ask them to test their equations with particular numbers, such as $36 for Christina's money.

A7. Here is a picture of a 10-by-10 grid with the squares on the border highlighted. Without counting one by one, and without writing anything down, can you find a way to determine how many squares are on the border?

a. Elicit and probe reasoning.
b. Can you find another method?

c. Now, let's say your square was 6 by 6. Could you use your first method to determine the number of squares on the border? What about your second method?

d. Could your first method apply to a square of another size? Can you give an example of that?

e. How would you describe in words how to use your first method on any square?

f. How would you use algebra to write an expression to communicate your first method to someone? Probe the meaning of letters and the expression.

g. Do parts (d) through (f) for the second method, if time permits.

APPENDIX D

Written Assessments

To score the written assessments, we followed the methods of Norton and Wilkins (2009, 2012). In most cases, four or five items targeted a particular operation or scheme. For each item we assigned a "1" if the student's markings and responses seemed strongly compatible with demonstrating the targeted scheme or operation. We assigned a "0" if the student's markings and responses counterindicated that the student could operate in a manner compatible with the targeted scheme or operation. In ambiguous cases, we used 0.4 or 0.6 to indicate inconclusive evidence. We discussed ambiguous cases to resolve them. We then summed the student's scores for each targeted operation or scheme. Students needed to earn a sum of 3 (out of 4 or 5) for the operation or scheme to be attributed to them. Our inter-rater reliability on scoring the written assessments was 93%. On the Grade 7 forms, it turned out that there were only two items that targeted the iterative fraction scheme, an issue we did not discover until we were in the midst of data collection. So, for seventh-grade students we did not have enough evidence to make assessments about their iterative fraction schemes from the written assessment alone; we did make assessments about this scheme from the interviews.

The written assessment results confirmed our conclusions from analysis of the interviews. Based on this assessment, to all MC3 students we could attribute an iterative fraction scheme and a reversible iterative fraction scheme. We could not attribute an iterative fraction scheme to any MC2 student, and we could not attribute a reversible iterative fraction scheme to any MC2 student except eighth-grade student Matt. Matt earned 1 out of 5 points on the five items that targeted an iterative fraction scheme, and 3 out of 5 points on the five items that targeted a reversible iterative fraction scheme. These scores could indicate he had constructed a reversible scheme but not the scheme itself. Because doing so is a logical impossibility, we decided to look across the 10 items. Matt's score, 4 out of 10, is less than half and not enough to attribute these closely paired schemes to him. In fact, the maximum score on the 10 items for MC2 students was 4. In contrast, all MC3 students earned a minimum of 7. One explanation for Matt's unusual scores is that he may have been on the verge of constructing these closely paired schemes. Still, the written assessment evidence is not strong enough to conclude that Matt had constructed either scheme at this time.