The Content-Focused Methods Course: A Model for Integrating Pedagogy and Mathematics Content*

Michael D. Steele  
*Michigan State University*

Amy F. Hillen  
*Kennesaw State University*

In the majority of secondary mathematics teacher preparation programs, the work of learning mathematics and the work of learning to teach mathematics are separated, leaving open the question of when and how teachers integrate their knowledge of content and pedagogy. We present a model for a content-focused methods course, which systematically develops a slice of mathematics content in the context of typical methods course activities. Three design principles are posited that undergird the design of such a course, addressing the nature of the mathematics content, the sequencing and design of activities, and the ways in which the course addresses the needs of diverse learners. Data from an instantiation of one such course is presented to illustrate the ways in which the course design framed teachers’ opportunities to learn about both content and pedagogy.

**Key words:** Teacher education; Mathematical knowledge for teaching; Mathematics methods.

As Linda Darling-Hammond (2010) points out, teacher education in the United States finds itself in a Dickensian conundrum. On one side, a great deal of political attention has been paid to improving the quality of teaching and learning, particularly in the fields of science, technology, engineering, and mathematics. On the other, pointed questions are being raised about the specific value of formal teacher education. On one side, a great deal of political attention has been paid to improving the quality of teaching and learning, particularly in the fields of science, technology, engineering, and mathematics. On the other, pointed questions are being raised about the specific value of formal teacher education. In light of studies criticizing the mathematical training that US teachers receive (Schmidt et al., 2007; Schmidt, Houang, & Cogan, 2011), there has been a return to favoring mathematical preparation over education coursework for teacher certification in many states. This press has largely focused on the preparation of secondary teachers, with the notion that disciplinary specialists with some basic pedagogical instruction might be well equipped to teach. While the research community generally agrees that teachers certified through formal teacher preparation programs effect stronger learning outcomes in students (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2008; Darling-Hammond, 2006; Darling-Hammond & Bransford, 2005), little consensus exists regarding the features of mathematics teacher preparation that promote teacher and student learning. The call for common school mathematics standards has cascaded into calls for common mathematics teacher education standards (Simon, 2000; Wilson, 2011) consistent with student-centered instruction, and frameworks that support the development of such a knowledge base. Several researchers have taken up this call, describing the knowledge base for teaching mathematics (e.g., Ball, Thames, & Phelps, 2008; Stein, Engle, Smith, & Hughes, 2008), creating instruments for measuring teacher knowledge (e.g., Brown, Bush, & McGatha, 2006; Hill & Ball, 2004; Izsák, 2008), and linking those measures to student outcomes (Hill, Rowan, & Ball, 2005). While the field has made substantial progress in describing mathematical knowledge for teaching and in linking that knowledge to student outcomes, little work has been done to describe features of mathematics teacher education that support the development of this knowledge. In this paper, we describe design principles that undergird a model for a mathematics methods course for secondary teachers that systematically integrates mathematics content in ways that provide opportunities to learn mathematical knowledge for teaching.

Researchers have conceptualized the complex knowledge base for teaching in ways that incorporate content, pedagogy, and several conceptualizations of the intersections between the two across the K–16 spectrum (e.g., Ball, Thames & Phelps, 2008; Shulman, 1986; Speer & Wagner, 2009; Steele, 2005). A common thread across this work is that pedagogical knowledge is neither discrete nor conceptually separable from the knowledge of the mathematics content being taught. Knowledge of how to teach a particular slice of mathematics rests on one’s knowledge of the mathematics in question; however, research that has investigated the development of mathematical knowledge for teaching has shown this process to be less additive (e.g., learn the content, then learn to teach it) and more...

* In accordance with the policy of the MTE Editorial Panel regarding potential conflicts of interest involving the editor, the review of and publication decision for this manuscript were handled by Denise A. Spangler.
iterative. For example, Steele (2008) demonstrated the ways in which engaging in mathematical and pedagogical tasks can enhance different aspects of both knowledge bases; Speer and Wagner (2009) identified pedagogical dilemmas that arise during teaching that spark re-examination of the content and the further development of pedagogical capacity. Yet in both policy and practical circles, the work of learning mathematics content and learning to teach mathematics are bifurcated. Prospective teachers receive content and pedagogical instruction in different courses, often separated both temporally and organizationally within teacher education systems. While many elementary preparation programs feature mathematics for teachers courses that sometimes attempt to integrate these learning experiences, few such opportunities exist for secondary teacher candidates (for exceptions, see Hill, 2006; Senk, Keller, & Ferrini-Mundy, 2000).

One model integrating the study of content and methods for secondary teachers is what Markovits & Smith (2008) term a content-focused methods course. Content-focused methods courses (CFMC) situate the systematic development of mathematical knowledge for teaching in the context of the typical activities in a methods course (Markovits & Smith, 2008). Whereas a methods course might treat content opportunistically through isolated tasks or lesson plans that teachers prepare, and a content course might provide plausible connections to pedagogical practice, the content-focused methods course features discernible mathematical and pedagogical storylines that are tightly connected. In this article, we look back at a content-focused methods course intended to enhance teachers’ mathematical knowledge for teaching and articulate a set of design principles common to the work. These principles can serve as a framework for the design of teacher education experiences that target mathematical knowledge for teaching across a wide range of mathematical content and in a variety of contexts: both preservice and practicing, both elementary and secondary teachers.

**Content-Focused Methods Course Design Principles**

1. Focuses on a narrow slice of mathematical content or process central to developing mathematical proficiency in secondary school.
2. Uses a guiding inquiry to frame and motivate the course and provide a unifying thread.
3. Organizes content and pedagogical activities into sequences that engage teachers across the continuum from learner to teacher.

**Illustrating the Model With a Specific Example: A Content-Focused Methods Course on Function**

A content-focused methods course centered on function (herein referred to as the functions course for simplicity) was designed using the three principles. We begin with a description of the course and context, followed by a discussion of the ways in which each of the principles influenced course design. We then describe in general the learning evident from the teachers who participated in the course and relate those data to the design principles.

**Description of the Functions Course**

The course was intended to enhance teachers’ mathematical knowledge for teaching functions and to develop their capacity for enacting meaningful student-centered learning experiences around these ideas for secondary students. It was taught as a graduate-level course at a large urban university in the Midwestern United States. Course development and implementation were part of a larger research project whose goals were to design and study courses around case-based mathematics education materials. The goals of the course are shown in Table 1.

The course targeted preservice and practicing secondary teachers and was promoted as an “advanced methods” course. It was a required course for preservice secondary teachers at the end of a yearlong master of arts in teaching program and was offered as an elective for practicing teachers pursuing master’s-level study. In addition, a number of elementary preservice teachers and in-service special educators with particular interests in mathematics took the course as an elective. (We reflect on the impact of the diverse teacher population later in this article.) The background of the 21 teachers enrolled in the course is shown in Table 2.

The principal investigator of the research project served as the lead designer and course instructor with support from a research team made up of teacher education researchers and graduate students. The authors of this article were graduate students on the research team and have subsequently refined and enacted the course as faculty members at other institutions. The research team (RT) began by selecting sets of mathematical tasks and narrative or video cases (featuring the same or a similar task) that represented rich learning opportunities related to functions, drawing primarily from Smith, Silver, and Stein’s (2005a) set of algebra tasks and cases. The RT then created or adapted additional activities related to the mathematical tasks and assembled activity sequences
Table 1
Goals for the Functions Course

<table>
<thead>
<tr>
<th>Mathematical goals</th>
<th>Pedagogical goals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Develop a mathematically accurate definition of function and use it to distinguish examples and nonexamples of function</td>
<td>Support the development of students’ understanding of functions by encouraging and facilitating rich mathematical discussions</td>
</tr>
<tr>
<td>Distinguish linear and nonlinear and proportional and nonproportional functions</td>
<td>Identify and enact cognitively challenging mathematical tasks</td>
</tr>
<tr>
<td>Solve a variety of problems involving functions, using recursive or closed form terminology and notation</td>
<td>Identify factors that impact the maintenance and decline of cognitive demands during implementation</td>
</tr>
<tr>
<td>Create and make connections among multiple representations of functions</td>
<td></td>
</tr>
</tbody>
</table>

Design Principle #1: A narrow focus: Algebra as the study of patterns and functions. The first principle for the content-focused methods course prescribes a narrow focus on an aspect of mathematical content central to the secondary mathematics curriculum. This principle establishes relevance for the mathematical content to be explored with respect to the work of teachers in their classrooms and affords an in-depth exploration of the content rather than a surface-level treatment of a variety of mathematical ideas. The content focus should cut across grade levels in some important way, be identified in standards documents as important to secondary mathematics, and be complex and challenging for both teachers and their students.

Therefore, the RT selected algebra as the study of patterns and functions as the focus for this course because it met the preceding criteria well. Function is an important cornerstone of secondary mathematics, which has become even more prominent with the rise of secondary mathematics curricula that explicitly use function as the grounding concept for the development of algebraic thinking (Alper, Fendel, Fraser, & Resek, 1997; Center for Mathematics Education, 2009; Cooney, 1996; Coxford et al., 1997). The Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010) underscore this importance by positioning functions as a high school content strand alongside algebra, geometry, modeling, and statistics and probability.

Table 2
Demographic Data on Course Participants

<table>
<thead>
<tr>
<th></th>
<th>Preservice: postbacc MAT program</th>
<th>In-service: masters of education</th>
<th>Secondary education doctorate</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elementary (K–6, all subjects)</td>
<td>3</td>
<td>1</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>Secondary (7–12, mathematics)</td>
<td>10</td>
<td>5</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Deaf Education</td>
<td>1</td>
<td>1</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>TOTAL</td>
<td>13</td>
<td>7</td>
<td>1</td>
<td>21</td>
</tr>
</tbody>
</table>

(called constellations) centered on a particular aspect of the mathematics of function. Figure 1 shows this collection of activities, with the colors representing the constellations, the shapes representing different activity types, and grey borders representing activities closely related to the guiding inquiry. (Figure 1 shows an enactment of the course during a 6-week summer term meeting 3 hours twice a week. The course has also been enacted during a typical 16-week semester.) Activities above the horizontal bar were enacted in class, with those below the bar representing homework assignments. We next describe the ways in which the design in Figure 1 reflected the three principles, and discuss how the team anticipated those principles and supported teachers’ opportunities to learn.
Figure 1. Activities in the functions CFMC.
Researchers investigating function have consistently found that students and teachers alike are frequently able to identify and generate examples at the heart of the function concept, such as continuous linear and quadratic functions, but struggle to identify more exotic functions, cannot always provide a mathematically complete definition of function, and are limited in their representational fluency (e.g., Even, 1998; Pitts, 2003; Sánchez & Llinares, 2003; Stein, Baxter, & Leinhardt, 1990). These characteristics together suggested that a content-focused methods course designed around functions would be a rich site for sustained mathematical inquiry.

Specifically, the RT conceived of the course as focusing on families of functional relationships that were familiar—largely linear and quadratic—to provide a diverse group of teachers with entry to the topic. At the same time, the team selected tasks that were mathematically extensible, allowing teachers to explore noncontinuous linear functions and rational functions built by transforming a simple linear function. The RT also recognized the importance of representational fluency—generating multiple representations of functions, moving flexibly between them, and describing the ways in which different representations are useful for noticing and analyzing specific features of a function. This set of ideas is important for developing both content knowledge and pedagogical practice.

**Design Principle #2: A guiding inquiry: What is a function, and what are examples and nonexamples?**

A guiding inquiry is a question (similar to an essential question in K–12) designed to frame the course-long content focus. This inquiry establishes the importance of the in-depth study of a particular slice of content within mathematics and mathematics teaching. This sort of big-picture view is often lacking in curricula and standards documents and is an important aspect of teachers’ curricular vision, which guides their decision making about what content is taught in the classroom (Drake & Sherin, 2008). The guiding inquiry also represents an opportunity to generalize from the set of particular mathematical tasks in the course to a larger mathematical structure and concept. The guiding inquiry should be about a topic for which most teachers will have initial ideas but one for which it is reasonable to believe teachers and students might have a limited understanding or misconceptions. In this spirit, the guiding inquiry should also be posed early in the course so as to reveal teachers’ initial conceptions and revisited at key moments in the course to provide opportunities to refine and elaborate those initial understandings.

The RT selected the guiding inquiry of *What is a function, and what are examples and nonexamples of functions?* for the reasons outlined above: Teachers would likely have some fluency with functions, yet a rich understanding of the concept can elude teachers and students. A significant body of research has demonstrated that even if teachers and students can work with examples of functions, they may not have a clear definition of a function and the specific criteria that distinguish functions from nonfunctions (Pitts, 2003; Vinner & Dreyfus, 1989). Lack of a clear definition of function can lead to the over- or under-generalization of the function concept and can engender a limited view of function and obscure its mathematical utility. For example, teachers who conceive of a function as something that can always be represented graphically potentially miss important function examples such as the Dirichlet function or nonnumeric functions such as the relationship between letters and mailboxes (Sand, 1996). A reliance on a graphing requirement on the Cartesian plane also obscures geometrically-based functions such as transformations.

Motivating a course-long inquiry into a topic for which teachers may feel as if they already have a great deal of knowledge can be a challenge. To motivate deep consideration of the definition of function, the RT positioned the definition of function as something to be constructed and revisited over time rather than simply stated and taken as shared. The language of function was used in the discussion of the first mathematical task, and teachers were then asked to define function individually and in small groups. They were able to state their initial ideas about functions, providing the instructional team with a baseline gauge of what the teachers knew. The course instructor assembled a list that captured all publicly shared ideas, including incomplete or vague conceptions, and this list was posted for all subsequent class sessions. This list was then used both as a resource when considering future examples and as a living document to be modified over the course. These recurring discussions helped to problematize the work on function.

**Design Principle #3: Engage teachers across the continuum from learner to teacher.** The notion of systematically developing content knowledge in a mathematics methods course is an important feature of the content-focused methods course model. In addition, the methods course must also develop pedagogical knowledge and link content and pedagogy in ways that are useful to the work of teaching. The use of authentic artifacts of practice (e.g., mathematical tasks, narrative and video cases of

---

1 While it did not happen in this particular instantiation, we have had teachers spontaneously request to edit an idea recorded on the list in subsequent iterations of the course.
teaching, student work, and lesson plans) is an important design consideration that supports the integration of the mathematical and pedagogical knowledge bases and connections to practice (Ball & Cohen, 1999). The content-focused methods course takes this connection a step further, using specific mathematical tasks as a grounding experience and starting point in an exploration of the mathematics as learner and teacher. Engaging in a mathematical task provides common ground to discuss the nuances of making sense of the mathematics. From this place, teachers can move back and forth between positions of learner and teacher, first in a protected way that may include analyzing third-party teaching artifacts such as narrative or video cases, sets of student work, or related mathematical tasks. As teachers develop deeper and more nuanced thinking about the content, they can move further on the continuum to consider the implications of taking different perspectives on the mathematics content on their teaching practice.

This principle led the RT to use particular activity structures in the content-focused methods course. For each mathematical task solved, teachers were asked to analyze the teaching of that task in some way (either through narrative or video cases), to consider students’ thinking about that mathematics in some way, and to make connections to their classroom practice. Beginning with solving the mathematical task as a learner is a critical element; in grappling with the mathematics themselves, teachers are better positioned to analyze students’ mathematical thinking and to consider how to support that thinking (Steele, 2008). The cases of teaching considered do not necessarily have to be exemplary cases but should raise important dilemmas about the teaching and learning of the content in question.2

Figure 2 shows the activities from the first constellation in the functions course placed along the learning-teaching continuum, with the numbering representing the order of activities. The constellation began with comparing the square and hexagon tasks and solving the hexagon task from a learner’s perspective, followed by reading and discussing the teaching of the tasks in The Case of Catherine Evans & David Young (Smith, Silver, & Stein, 2005a). Activities 4 and 5, both homework, pushed teachers to consider the implications of the use of patterning tasks in the classroom. The next class session looped back to talking about the mathematics by considering the mathematical standards in the hexagon task and posing the guiding inquiry (what is a function?) for the first time. The next three activities, analyzing student work, reading a practitioner article on teaching algebra, and interviewing a student around one of the tasks, represented a strong push toward teaching practice.

Activity sequences that keep the mathematics constant and traverse the continuum between learner and teacher provide teachers with a range of different opportunities to learn. First, the work begins in a relatively comfortable space for discussion—doing mathematics—and gradually moves to more sensitive spaces of a teacher’s classroom practice. Along the way, teacher participants build under-

---

2 The use of at least one exemplary case as a comparative measure can be particularly helpful. For a more extensive discussion of the selection and use of cases, see Smith and Friel, 2008.
standings of how other teachers and their students make sense of the mathematics content, and these understandings can then be applied to participants’ own classrooms. Like turning a gemstone in the light to see its different facets, considering multiple perspectives on the mathematics creates a more nuanced and robust sense of the intertwined package of teaching and learning. Moreover, the sequence also provides teachers with a model of what student-centered pedagogy might look like around a particular mathematical topic. While not every aspect of the mathematical work will transfer directly to practice, a strong socially constructed mathematical conversation is likely to include useful features adaptable to the classroom (Hillen & Hughes, 2008).

These three design principles together frame the opportunities teachers have to learn about content and pedagogy in a content-focused methods course. In the section that follows, we briefly describe teacher learning in the course with respect to both content and pedagogy. We then use data from the course to illustrate the ways in which the design principles may have afforded teachers particular sorts of opportunities to learn.

**Teacher Learning in the Functions Course**

The research team collected data to assess teacher learning about content and pedagogy in the functions course in several ways. Through written assessments and semi-structured interviews at the start and end of the course teachers were asked to solve mathematical tasks, analyze cases of teaching and student work artifacts, and plan lessons. The postcourse interview used a course map similar to Figure 1 and asked teachers to reflect on their learning of (a) mathematics; (b) students as learners of mathematics; and (c) teaching mathematics, and to identify activities that contributed to their learning. Course meetings were videotaped and transcribed, and all instructional artifacts were retained, which provided data related to opportunities to learn. All written assessment items were coded by both authors, with an inter-rater reliability of at least 92%.

In general, our analysis of the data suggests that teachers added to both their knowledge of content and of pedagogy. Prior to the course, many of the teachers struggled to produce a correct definition of function as well as an example and nonexample. Performance in generating the definition, example, and nonexample improved significantly on the postcourse assessment. Teachers were also asked to solve a number of mathematical tasks, both on written assessments and during course meetings, that involved functions. The use of representations and the ways in which teachers made connections between them improved from the start to the end of the course as well. From a pedagogical standpoint, teachers were better able to select high-cognitive demand tasks related to functions and plan for them in ways that supported the maintenance of the cognitive demand. They came to understand the ways in which one might systematically plan for and support work on multiple representations of functions with students, with a particular focus on meaningful questions that supported conceptual understanding. Teachers also considered the utility of having and supporting multiple mathematically correct definitions for function rather than a single canonical definition.

In the section that follows, we explore this data set in greater detail. Our goal is to use the three design principles as lenses through which to consider data on teacher learning and ways in which the course provided teachers with opportunities to learn about both content and pedagogy.

**Making connections among multiple representations: Using the lens of Principle 1.** One of the mathematical goals of the course was for teachers to make connections among multiple representations of functions. In this section, we use the lens of Principle 1—the focus on the content of function—to consider the ways in which course design using this principle offered teachers opportunities to learn related to connections between representations. Teachers had numerous opportunities to make connections between visual geometric patterns, symbolic equations, tables, graphs, mathematical language, and real-world contexts. The choice of specific tasks related to function and the design of a specific progression through those tasks contributed to these opportunities to learn.

Table 3 lists the mathematical tasks related to function that were used in the course and describes both the family of function (e.g., linear, quadratic, rational) and the starting representation used in each task. By holding the
content of function constant, teachers were able to experience the ways in which different representations made salient different features of the function relationship. As the course progressed, the teachers were using different representations spontaneously in their mathematical work and talk to make sense of the underlying mathematical constructs. For example, in a discussion of the Cal’s Dinner Card Deals task in Class 7 in which teachers were asked to make sense of the slope and y-intercept, teachers made 13 different connections between the 5 core representations of function (symbol, language, context, graph, table). Figure 3 shows the connections between the representations made by teachers in a 20-minute discussion.

Changes in teachers’ abilities to make connections between mathematical representations were assessed in part through their performance on the visual pattern

<table>
<thead>
<tr>
<th>Class</th>
<th>Linear proportional</th>
<th>Linear non-proportional</th>
<th>Quadratic</th>
<th>Piecewise</th>
<th>Rational</th>
<th>Non-numeric</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>Hexagon task (context)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>S-pattern task (context)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>Square/pool border task (context)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>Paul’s hair growth (context/table)</td>
<td>Sonya’s hair growth (context)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>Supermarket carts (context)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>Car wash (context)</td>
<td>Cal’s Dinner Cards: Plans A and B (graph)</td>
<td></td>
<td></td>
<td>Mail carrier (context) Students &amp; test scores (graph)</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>Cal’s Cost Per Meal: Regular Plan (table)</td>
<td>Cal’s Cost Per Meal: Plans A and B (table)</td>
<td></td>
<td></td>
<td>Weight and height (graph)</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>Graphs of functions: Functions 1 &amp; 2 (symbolic)</td>
<td>Graphs of functions: Functions 3 &amp; 4 (symbolic)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>Calling Plans (context)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td>S-pattern task (context)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note. Italics indicate situations that are or could be considered continuous; nonitalics indicate discrete situations.

Table 3
Range of Examples Used in Course Tasks
task shown in Figure 4. Results from the pre/post written assessment show evidence of changes in teacher capacity to make connections between a visual geometric pattern and a symbolic equation. Teachers’ responses to the task were coded using the rubric shown in Table 4. Initially, a majority of teachers determined an equation that generalized the pattern (rubric scores of 1 or higher), but the extent to which their explanations related their equation to the visual pattern varied. About 25% fully related their explanation to the pattern (score 4), with another 25% making no connections (score 1), and the remaining teachers making some connections (scores 2 and 3). By contrast, on the posttest, the majority (80%) completely explained how their equation related to the visual pattern, a significant difference (Wilcoxon sign-rank test; $W = -73$, $ns/r = 12$, $p = 0.0045$).

In addition to the changes in performance on the written assessment, our analysis of the postcourse interview data showed that teachers saw an important pedagogical use for connecting representations. When asked what they learned in the course, 13 of 21 teachers discussed connections between representations as a topic that helped them better understand students as learners of mathematics, and 11 of 21 named connections between representations as an important priority in the teaching of mathematics.

In following Design Principle 1, the RT held the content of function as a consistent thread throughout the course. Doing so allowed the instructor to focus on the ways in which representations helped to illuminate different aspects of functions. Teachers developed both a stronger fluency with the mathematical representations and a clearer sense of why and how one might use different representations as pedagogical tools.

**Defining function: Using the lens of Principle 2.**

Another goal of the course was for teachers to develop a mathematically accurate definition of function and use it to distinguish examples and nonexamples of function. Teachers considered the guiding inquiry, *What is a function, and what are examples and nonexamples of functions?*, which provided opportunities to meet this goal, at several points throughout the course, as shown in Figure 1. The prominence of the guiding inquiry, combined with the repeated opportunities to revisit and revise thinking about the definition of function and the nature of examples and nonexamples, provided teachers with opportunities to deepen their content knowledge related to function. The pre/post written assessment measured that learning, asking teachers, *What is a function? Give an example of a function and a nonfunction.*

Teachers’ responses to the first part of this task (*What is a function?) were coded as correct, incorrect, or inconclusive. A correct definition included the idea of univalence (i.e., the mapping of each element of the domain to exactly one element of the range) and did not explicitly rule out arbitrariness (i.e., elements of the domain and range do not need to be numeric). Correct definitions could use different terminology for $x$ and $y$ (e.g., input and output; domain and range; independent variable and dependent variable). Definitions that did not include univalence or made erroneous statements (e.g., functions must be linear relationships) were coded as incorrect. Definitions were coded as inconclusive if there was not enough information present to suggest the univalence criterion. For example, several definitions included correct statements (e.g., functions pass the vertical line test) but provided no further explanation regarding why the statement(s) implied that a relationship was a function. Teachers’ responses to the second part of the task (*Give an example of a function and a nonfunction*) were also coded as correct, incorrect, or inconclusive. Examples and nonexamples were also coded by family (e.g., linear; quadratic) and representation(s) used.

In general, teachers’ initial definitions were problematic, although their examples were not. Fewer than half of the 21 teachers in the course provided a correct definition.
on the pretest, with incorrect definitions cutting across all grade levels taught and experience. Most of these incorrect definitions did not include mention of univalence and included features that suggested a narrow conception of function (e.g., functions are linear, all functions can be graphed). Nearly all teachers (20 of 21)—including all of the teachers who provided an incorrect definition—provided a correct example of function, with most being linear or quadratic relationships presented as equations or graphs. Over half the teachers (13 of 21) provided a correct nonexample of function on the pretest; however, 6 of these teachers provided an incorrect definition of function.

By contrast, nearly all teachers (20 of 21) provided a correct definition on the posttest, a significant difference (Fisher’s exact test, p < 0.001). It is also important to note that all 12 teachers who did not provide a correct definition on the pretest improved in some way: 7 moved from incorrect to correct, 4 moved from inconclusive to correct, and 1 moved from incorrect to inconclusive. All teachers provided a correct example of function on the posttest, mostly linear or quadratic in nature. However, there was a significant increase in the number of correct nonexamples of function (Fisher’s exact test, p < 0.01). Given the background of these teachers, it is no surprise that the majority could easily produce and identify examples of functions. The types of examples provided by teachers were relatively straightforward relationships central to secondary mathematics. However, it is interesting to note that prior to the course, the majority of teachers were able to provide correct examples of functions, but not all teachers were able to correctly produce a definition or a nonexample of a function.

The second design principle that specifies a guiding inquiry, problematized early in the course and revisited throughout, provided teachers with repeated opportunities.

---

4 One teacher did not respond to this item; all 20 teachers who did respond to this item provided a correct example of function.
to learn the definition of function. By considering multiple examples, examining narrative cases of teachers seeking to support their students in understanding the construct of function, and thinking about tasks to use in their classrooms related to function, teachers had the opportunity to rise above simply learning a correct definition of function. By thinking through the definition and examples as learners and teachers, teachers developed knowledge of both content and pedagogy through this sustained inquiry.

The case of Olivia. Olivia was a practicing elementary teacher completing her sixth year of teaching who enrolled in the course as an elective. Olivia was known as a thoughtful teacher-learner who had taken part in many high-quality professional development experiences, including a similar content-focused methods course on proportional reasoning in her masters of education program.

At the beginning of the course, Olivia's knowledge of function was limited (see Figure 5). The definition of function she provided on her pretest did not include univalence and implicitly ruled out arbitrariness. Although she provided a correct example of function on the pretest, her work during the first interview (conducted after Class 3) revealed that she struggled to explain why her example was a function, even though a correct definition of function had been made public in class by the time of her interview:

Interviewer: I have the example of a function that you gave on the pretest. So you gave \( y = 2x \). And I wanted to ask you why this is an example of a function?

Olivia: Well, I think that looking at that, there would be one \( y \) for every \( x \), and one \( x \) for every \( y \), so I think that that's why it's a function.

Interviewer: OK. And that's based on the discussion in (the third class)?

Olivia: Yes, I mean, doesn't—I don't think it would have to be one value for each of them, I mean, every time you have \( y \), or \( x \), um, \( y \) is going to be two times that.

Interviewer: OK. What could you do to make it not a function?

Olivia: Make it \( x \) squared, or something. If I made it \( x \) squared, then, I think you'd have more than one value for \( x \). And [long pause], I don't really know. I think that because it's \( x \) squared, I think you'd have more than one value. But I'm not really sure what I'm doing.

In this excerpt, Olivia attempted to use univalence to explain why her example is a function and to create a nonexample of a function but grappled with its meaning and determining the variable (\( x \) or \( y \)) to which she should attend.

By the end of the course, however, Olivia had a more robust understanding of function and its definition. Her posttest function definition (Figure 5) satisfied both conditions for a correct definition. During the postcourse interview, Olivia successfully classified a set of relationships as functions and nonfunctions and explained her classifications drawing on the definition. In the excerpt below, she explained why the graphs of \( x = 2 \) and \( y = \pm x^{1/2} \) are not functions, using univalence as the justification:

Olivia: \( x = 2 \) is a nonfunction because \( x \) would be 2, but on that line, you could have any value for \( y \). And also because of the multiple values of \( y \), and also because if you think of drawing a vertical line through it, it is a vertical line, it'd hit the whole line. So it wouldn't be just one spot. And for \( y = \pm x^{1/2} \), if you do the vertical line test, it goes through the graph twice.

Interviewer: What is the vertical line test?
Tracing Olivia’s Learning

Precourse Assessments

A function is a relationship that can exist for a variety of numbers. In other words, different numbers can be used in the place of a variable, and the relationship can be maintained.

Example of function: 
\[ y = 2x \]
y is different, depending on the value of \( x \)

Example of nonfunction: 
\[ x + 4 = 20 \]
This is not a function because \( x \) can only have one value → 16

Course activities identified as contributing to learning:

Hair Growth Task (Class 5)

Cal’s Dinner Card Deals (Class 7)

A Function Is a Mail Carrier (Class 9)

Creating a definition of function (several classes)

Postcourse Assessments

A function occurs when 2 variables vary together. One variable is dependent on the other variable. For each value of the independent variable, there must be only one value of the dependent variable.

Example of function: 
\[ f(x) = 4x + 2 \]
The initiation fee for getting into a club is $2. At each meeting, the dues are $4. How much money will be spent after any given meeting?

Example of nonfunction:

includes univalence?

Doesn’t rule out arbitrariness?

\[ y^2 = x \]

Example of function:

I have a much broader understanding of functions... a broader view of what a function is and what it involves... And also, thinking about what a function was. But I don’t know that I could really define that before. And I try to think, “Could I have done that when I was maybe in 8th or 9th grade, when I was taking algebra classes?” And I don’t really know that I could have.

An ellipsis indicates deleted words.

Olivia:

Well, if you draw a vertical line through, you should only cross the graph once because if you cross it more than once, it means for that particular value of \( x \), there’s more than one value for \( y \). Like here, say my value for \( x \) was 2. I could have this value of \( y \) and this value of \( y \)... say (2, 1) or (2, –1). So because you have those two values, it means it’s not a function.

Olivia’s responses in the second interview suggest that she did not merely memorize the procedure of using the vertical line test to determine whether a relationship is a function. Olivia acknowledged her narrow view of function at the beginning of the course and described how her understandings changed through engagement in particular course activities:

Olivia:

I have a much broader understanding of functions... a broader view of what a function is and what it involves... And also, thinking about what a function was. But I don’t know that I could really define that before. And I try to think, “Could I have done that when I was maybe in 8th or 9th grade, when I was taking algebra classes?” And I don’t really know that I could have.
Interviewer: So were there places that helped you with defining a function and thinking about what a function really is?

Olivia: Well, creating a definition of a function. I remember “Function as a Mail Carrier” because the idea that every \( x \) can only have one—that idea that when you put in a value for \( x \), you should always get one \( y \). It shouldn’t be you put in 4 one time, and you get 6 for \( y \), and you put in 4 another time, the same number, and get 8. You can’t have that. So I think that was important, too.

Olivia noted that the course enhanced her understanding of function and identified four particular activities as being instrumental in her learning. She recognized that her broadened view of function was influenced by participating in discussions based on the guiding inquiry and by various types of mathematical and pedagogical activities. When asked about the pedagogy of the course as a factor in her learning, she described specific features related to Principle 3:

Olivia: One thing I also liked about the class is that we really worked on developing our understanding of math AND connecting it to teaching, like through the case studies. And there aren’t very many classes that do that... I think the two go hand in hand—really learning about the math and understanding it, then looking at how is that taught in the classroom? We looked at the tasks first, so we understood... what this task was about, the math that was involved, and then how a teacher was presenting the task, and how students in the task interpreted it, and maybe compare in your mind, “Well, you know, that’s how I thought of it.” It’s effective, I think, for teachers because both of them are really important and connecting them [is] important.

In sum, Olivia entered the course with substantial confusion about function from a content standpoint. Her work in interacting with the mathematical tasks and the development of the definition enabled her to successfully define and identify examples and nonexamples by the end of the course. Moreover, she linked the content learning to the work in considering the cases and student work, describing the ways in which moving between learner and teacher was important to her development as a teacher.

We now consider the ways that the same course supported a teacher with a different background by looking at the case of Carl, a preservice secondary mathematics teacher.

**The case of Carl.** Carl was a preservice secondary teacher completing a yearlong internship in a suburban middle school. He had earned a bachelor’s degree in mathematics from a major public university and took the course as the capstone of a fifth-year master of arts in teaching program. Despite Carl’s mathematical background, his work early in the course suggested a muddled understanding of function. Carl’s pretest definition allowed for arbitrariness but did not reference univalence, as shown in Figure 6. In distinguishing examples and nonexamples in his pre-interview, he used the univalence criterion but incorrectly described it as “one-to-one correspondence.”

At the end of the course, Carl held a deeper and better connected understanding of function. His posttest definition fulfilled both criteria for a correct function definition. Interestingly, Carl did not use the input/output language that was often used in class discussions of the definition. This suggests that Carl had not merely memorized the class definition but held a conception of function that made sense to him. In reflecting on his learning, he described the differences in his understandings since the beginning of the course:

Carl: I have a clearer definition of what a function is... I think most of us came into the class having worked with functions before, obviously, and doing vertical line tests to see if something in the function maybe is not a function. But I don’t think a lot of us had a really solid definition in our heads of what a function is. And I think that the class kinda helped us revise our own inkling of what a function is. The thing about functions is that correspondence between two different sets of quantities. Before the course, if someone had asked me “What is a function?” I couldn’t say... I would’ve said something about the vertical line test. I would’ve said something about an equation [or] function notation. But I don’t think I could have really given a really direct answer. After the course, I think I can.

Carl noted that while he entered the course with ideas about the definition of function, these ideas were incomplete, and the course provided an opportunity for him...
to revise his thinking and develop a clearer definition of function. When pressed to identify specific activities in the course that supported his learning, Carl described the discussion in which teachers created a definition of function (Class 3):

**Interviewer:** So you just (identified) creating a definition of a function. How did that help you come to clarify the definition of function?

**Carl:** Well, I mean we were just really brainstorming the definition of a function, and I think what it actually did was make me analyze the specific kinds of things that make up a function. It doesn’t necessarily have to be an equation. You know, it could just be the two sets. . . . But what it really did is it really made me scrutinize my own definition of a function that I had coming into the class, and we could change it and alter it a little bit, due to the discussion of the definition.

Having that up there throughout class made me go back and see, “Well, is this a function? Is this a function?” Go through the criteria that we came up with ourselves.

When asked to reflect on how the structure of the course supported his learning, Carl’s answers differed from Olivia’s. Carl focused on the enactment of the tasks in the course as a model for his own future classroom and the cases as reinforcing the real-time modeling:

**Carl:** Class time was a good example of how a pattern task could be implemented in the classroom, and the level of mathematics was high. [The instructor] had us solve in groups, was able to ask some open-ended questions that didn’t necessarily guide the group directly to an answer. . . . [When] group discussion was over, she was able to bring the class together and have a whole-class discussion [and] pick out certain solutions that were beneficial.
to the class as a whole to see. It was really an example and another reinforcement of how to use pattern tasks and to keep the level of mathematics high in the classroom. [And] the cases that we read are examples.

Carl’s background differed from Olivia’s in that he entered the course with a stronger conception of function but with a definition that was in need of clarification. Through the same set of activities, Carl was able to make repairs to his definition rather than adopting the co-constructed class definition. Carl also entered as a preservice teacher looking for models of how to enact student-centered tasks, models that had been lacking in his internship placement. For Carl, moving between doing mathematics and considering cases of teaching helped him develop pedagogical knowledge related to how he might support his students in developing conceptual understanding.

For these two teachers, traversing the continuum of content and pedagogy created a common set of learning opportunities that led to different learning outcomes. Olivia’s mathematics background was such that she took advantage of opportunities to learn related to the content of function, integrating the new content knowledge into an existing framework about her own well-developed pedagogical practices. She specifically noted the cases as a place in which the content and pedagogy come together, and one might anticipate that seeing this connection would make her better able to integrate new content understandings into her teaching. Carl, with a stronger mathematical background but at the very beginning of his teaching career, was able to take note of the ways that larger-grained pedagogical structures can support the learning of content. His ideas about the ways in which the group discussions and sharing of solutions modeled in learning of content. His ideas about the ways in which larger-grained pedagogical structures can support the mathematical background but at the very beginning of understandings into her teaching. Carl, with a stronger
tion would make her better able to integrate new content
pedagogical practices. She specifically noted the cases
as a place in which the content and pedagogy come
together, and one might anticipate that seeing this connection
would make her better able to integrate new content
understandings into her teaching. Carl, with a stronger
mathematical background but at the very beginning of
his teaching career, was able to take note of the ways that
larger-grained pedagogical structures can support the
learning of content. His ideas about the ways in which
the group discussions and sharing of solutions modeled in
the course led to new mathematical understandings pro-
vided useful models for Carl’s early practice as a begin-
ning teacher. As such, Principle 3 provided these teachers
with opportunities to learn about teaching mathematics
that fit their differing needs at the time.

Discussion

The content-focused methods course is a promising mod-
el for supporting teachers in developing mathematical
and pedagogical knowledge and integrating those knowl-
dge bases in ways that build knowledge needed for
teaching mathematics. The examples presented from the
functions course demonstrate how the design principles
can come together to provide diverse groups of teach-
ers with opportunities to learn. The functions course,
however, is only one instantiation of the content-focused
methods course model and was enacted in a specific
institutional context that may differ from your own.

So how does one begin designing a content-focused
methods course? Selecting a mathematical focus (Princi-
ple 1) is a good starting point. Identifying narrative and/or
video cases (e.g., Barnett, Goldenstein, & Jackson, 1994;
Boaler & Humphreys, 2005; Merseth, 2003; Smith, Silver,
& Stein, 2005a, b, c) and student work (e.g., Lamon,
2005; Parke, Lane, Silver, & Magone, 2003) that relate
to the mathematical focus can suggest a specific guiding
inquiry. In the sections that follow, we discuss ways in
which the principles could be implemented in different
contexts and the affordances and constraints of bringing a
content focus to an existing mathematics methods course.

Varying the Mathematical Focus and Guiding
Inquiry: Principles 1 and 2

By varying the mathematical focus, and in turn, the guid-
ing inquiry, additional content-focused methods courses
for teachers of grades 7-12 could be developed. For ex-
ample, content-focused methods courses on proportional
reasoning (using the guiding inquiries What is propor-
tional reasoning? and Are all fractions ratios? Are all ratios
fractions?; Hillen, 2005) and geometry and measurement
(using the guiding inquiry What is a proof?; Steele, 2006,
2008) have been developed and studied. A content-
focused methods course on reasoning-and-proving (Smith
& Stylianides, 2010; Hillen, Smith, & Arbaugh, 2011) is
currently under development. The guiding inquiries to
frame this course will include a mathematical question
(What is reasoning-and-proving?) as well as ones that
could be considered more pedagogical in nature (How
do secondary students benefit from engaging in reason-
ing-and-proving? How can teachers support the develop-
ment of students’ capacity to reason-and-prove?). While
the current principles reflect a secondary population, the
content-focused methods course model could be used
in courses for teachers of the elementary grades. Similar
principles have also been used to structure professional
development opportunities for teachers and their prin-
cipals (Steele, Johnson, Herbel-Eisenmann, & Carver, 2010).

Varying the Focus on the Learner-Teacher
Continuum: Principle 3

By shifting the focus on the learner-teacher continuum,
additional courses could be created that would serve
a variety of purposes. By placing an emphasis on the
learner end of the continuum, a mathematics content
course could be created, which could consist primarily
of activities in which teachers solved mathematical tasks
and occasionally examined student work or read a prac-
titioner-oriented article. Such a course could conceiv-
ably meet the forthcoming recommendations from the
The content-focused methods course provides a generalizable, adaptable model for integrating the study of content and pedagogy. The course described here resulted in teacher learning that varied in beneficial ways across teachers. Activities that traverse the content-pedagogy spectrum, grounded in a specific slice of mathematical content, can provide teachers opportunities to enhance areas of their knowledge across that spectrum.

**References**


**Authors’ Note**

This article is based on work supported by National Science Foundation grant ESI-0101799 for the ASTEROID (A Study of Teacher Education: Research on Instructional Design) Project, Margaret S. Smith, Principal Investigator. Any opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

**Authors**

Michael D. Steele, Department of Teacher Education, Michigan State University, East Lansing, MI 48824; mdsteele@msu.edu

Amy F. Hillen, Kennesaw State University, Kennesaw, GA 30144; hillen@kennesaw.edu