Developing Arguments About Congruence

Activity 1: Investigating the Varignon Area

In this activity, you will compare the area of a quadrilateral to the area of another quadrilateral constructed inside it. You will formulate conjectures and use congruence arguments to verify the conjecture.

COMMON CORE STATE STANDARD – HS GEOMETRY

Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence* in terms of rigid motions to decide if they are congruent.
* A *congruence* between two geometric objects is a rigid motion of the plane that maps one onto the other.

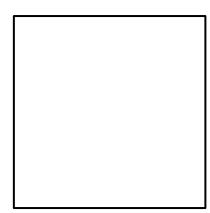
MATHEMATICAL PRACTICE

Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures.

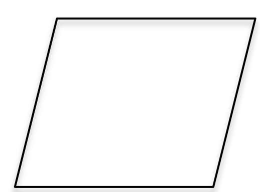
1.1 Exploring Area by Folding and Cutting Paper

For each part of the problem, start with the indicated shape and make folds and/or cuts to construct a new shape. Then explain how you know the new shape you constructed had the specified area.

1. Construct a square with exactly ½ the area of the original *square*. Explain how you know it is a square and has ½ of the area.



2. Construct a parallelogram with exactly ½ the area of the original *parallelogram*. Explain how you know it is a parallelogram and has ½ of the area. Compare your technique with the one used for the square.



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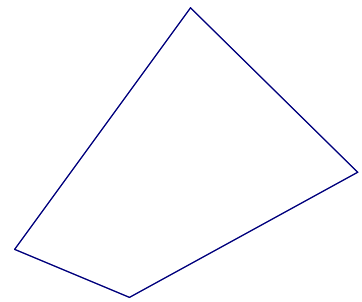
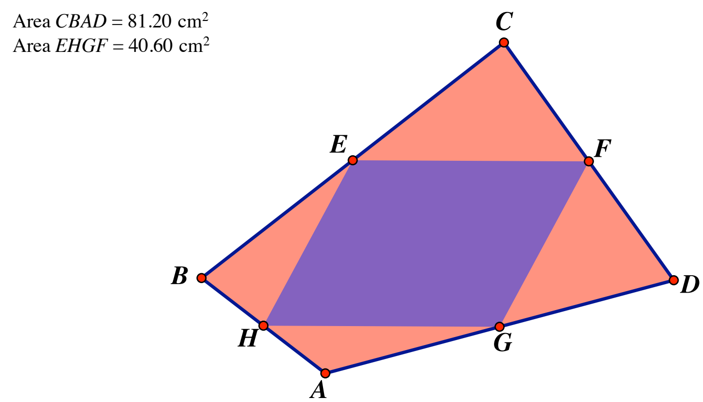
3. Construct a parallelogram with exactly ½ the area of the original *quadrilateral*. Explain how you know it is a parallelogram and has ½ of the area. Compare your technique with the ones used for the square and rectangle. Is there a specific way to construct the parallelogram that works for all cases?



1.2 Exploring Area using Graph Paper and Ruler OR Dynamic Geometry Software

1. Construct a parallelogram with exactly ½ the area of the original *quadrilateral*. Explain how you made the construction and how do you know it is a parallelogram.
   Label the parallelogram *EFGH*.

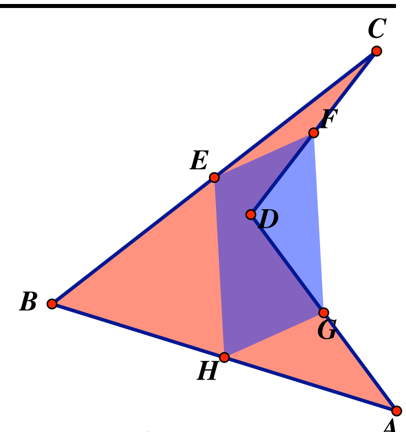


2. Estimate the areas of the quadrilateral *ABCD* and the parallelogram *EFGH* to strengthen your arguments about the relationship between the areas.

   Try a different quadrilateral or drag any of the points *A, B, C,* and *D* if working with dynamic software and observe if the relationship between the areas still holds.

   Make a conjecture about the relationship you observe:

3. Try a concave quadrilateral or drag a vertex of *ABCD* until it is concave. Does this change the ratio of the areas? How does this change your original conjecture?



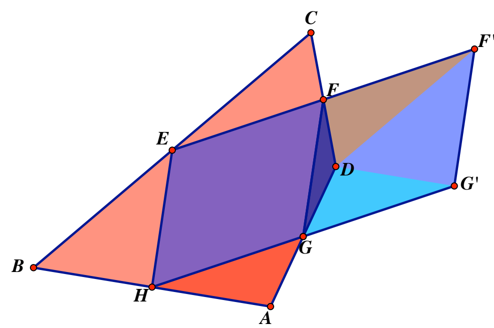
1.3 Explaining the conjectures

In the preceding section, you probably made a conjecture that goes something like this:

> *The area of the parallelogram formed by connecting the midpoints of the sides of a quadrilateral (concave or convex) is half the area of the quadrilateral.*

This conjecture matches a theorem of geometry that is sometimes called **Varignon's Theorem**. Pierre Varignon was a priest and mathematician born in 1654 in Caen, France.

Work through the steps that follow for one possible explanation as to why parallelogram *FGHE* has half the area of quadrilateral *ABCD*.



1. Translate the parallelogram in the direction of the vectors *EF* and *HG*. How is the area of the translated parallelogram *F'G'F'G'* related to the original parallelogram *FGHE*? Why?

2. Construct segments *F'D* and *G'D*.
   How is the triangle *ECF* related to the triangle *F'DF*? Explain why this relationship must be true using rigid motions.

   How is the triangle *HAG* related to the triangle *G'DG*? Explain why this relationship must be true? Is the reason the same as before?

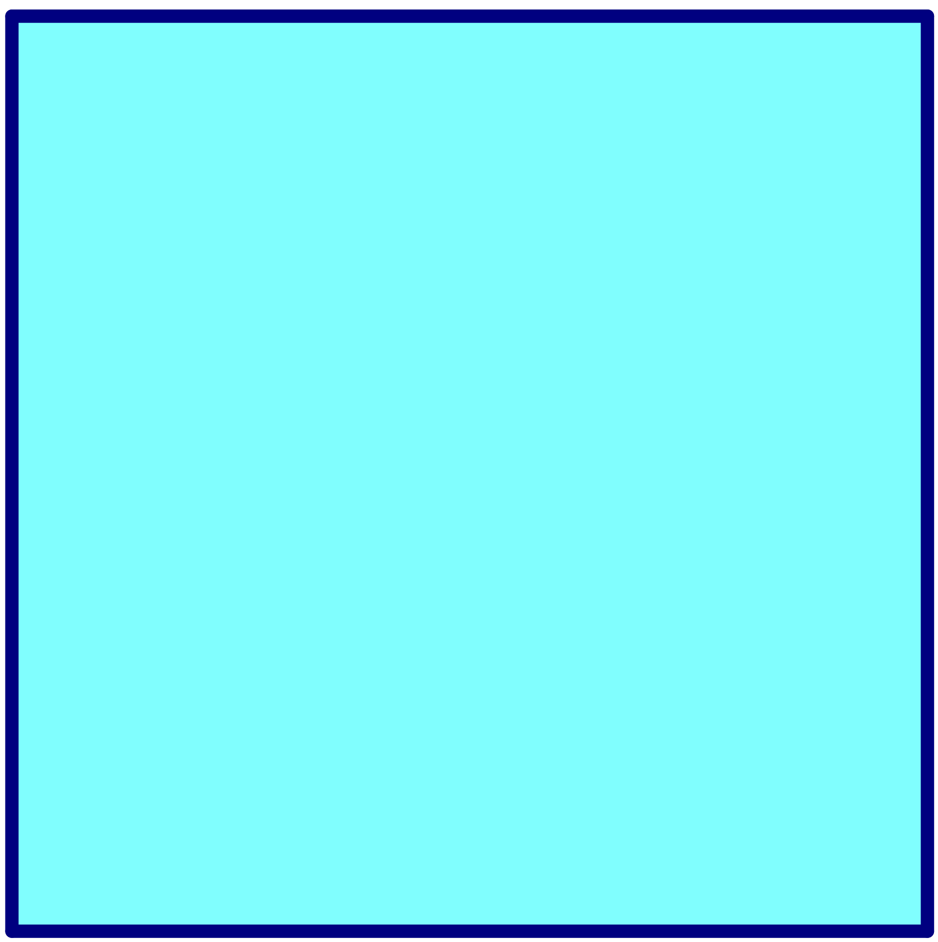
   Finally, how is the triangle *BEH* related to the triangle *DF'G'*? Is the explanation similar for these triangles as for the ones before?

3. Create a summary of your proof of the **Varignon's Theorem** from steps 1 – 4 using logical progression of statements.

Developing Arguments About Congruence

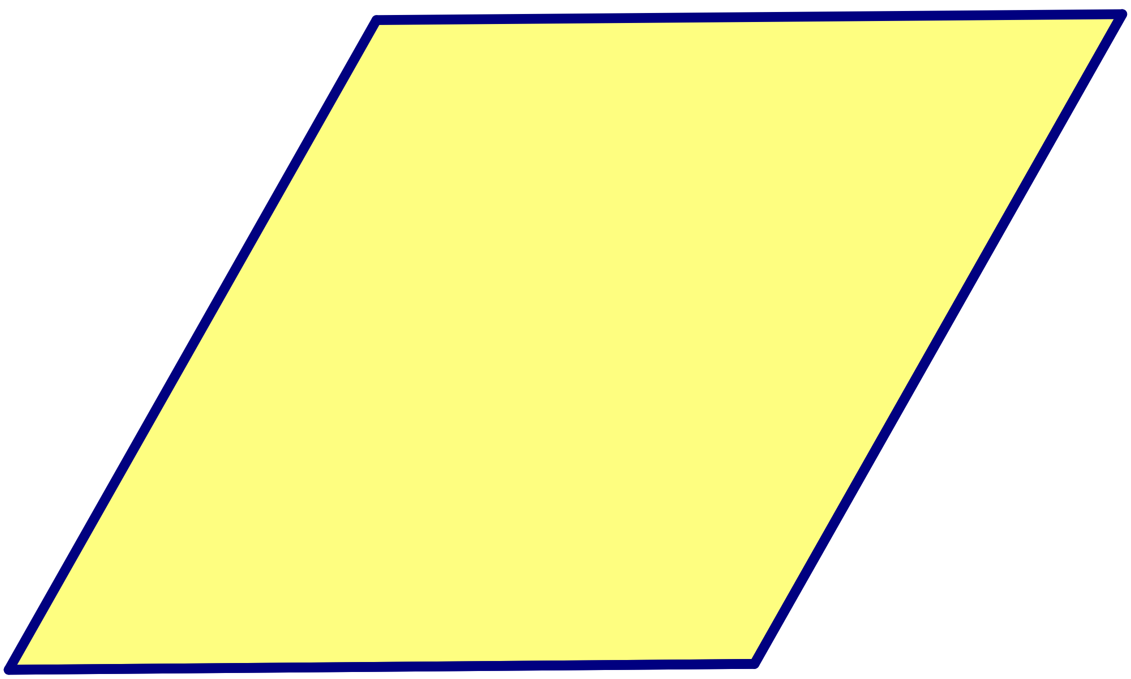
TASK 1

Cut out the square below.
Make folds to construct a new square with exactly ½ the area of the original *square*.
Explain how you know it is a square and has ½ of the area.



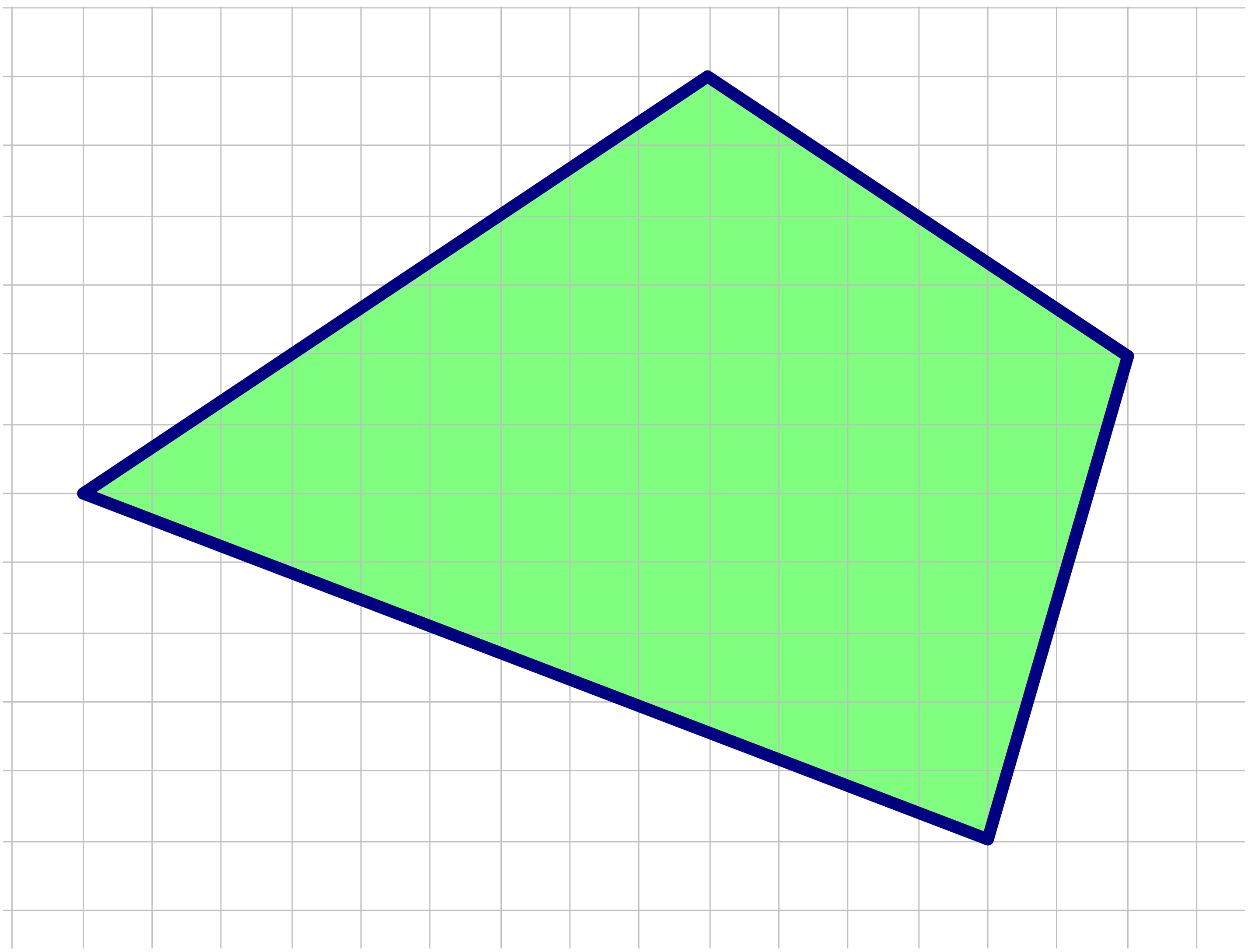
TASK 2

Cut out the parallelogram below.
Make folds to construct a new parallelogram with exactly ½ the area of the original *parallelogram*. Explain how you know it is a parallelogram and has ½ of the area.



TASK 3

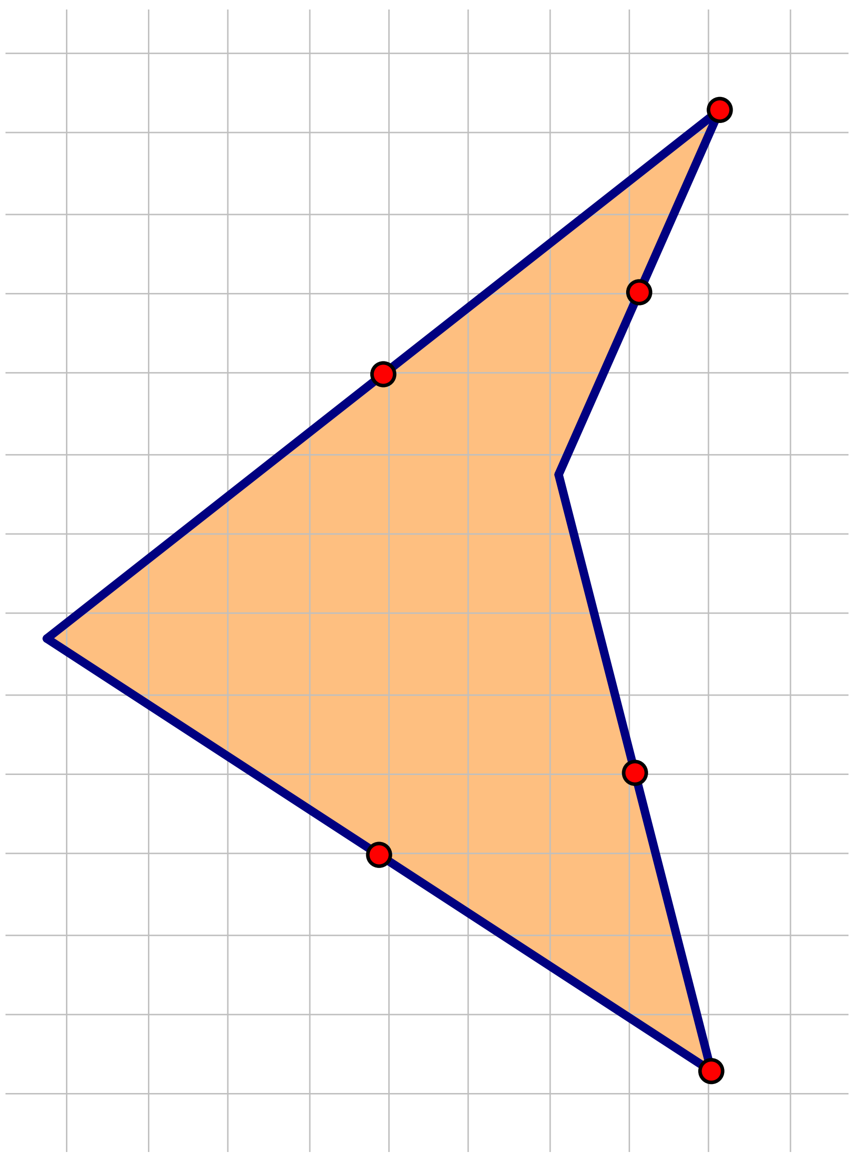
Draw an inner quadrilateral with exactly ½ the area of the original *quadrilateral*. Cut out the left over pieces and cover the inner quadrilateral to show that is ½ of the area.



TASK 4

Join the midpoints of the sides of the concave quadrilateral.

Is the parallelogram formed ½ of the original quadrilateral? Explain why or why not.



Developing Arguments About Congruence

Activity 2: Rotating Square

In this activity, you will use two congruent squares and explore their overlapping areas when the vertex of one of square is located in the center of the other. You will formulate conjectures and use congruence arguments to verify the conjecture.

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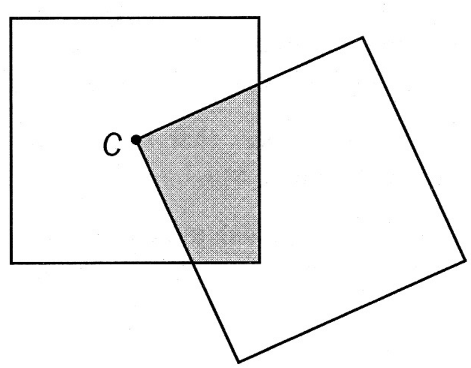
1.1 The Task (Adapted from *Focus in High School mathematics: Reasoning and Sense Making in Geometry,* NCTM, 2010, pp 5-14)

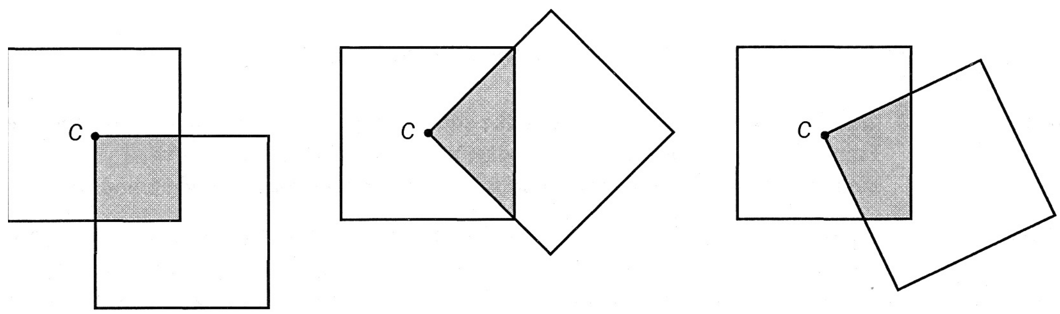
Two congruent squares (n units by n units) overlap as shown in the figure below. Vertex C of one square is at the center of the other square. If the square with the vertex C is allowed to rotate about the center, C, or the other square, what is the largest possible value of the overlapping shaded area?



1.2 Developing a conjecture

1. Construct two congruent squares from grid paper and use the tip of your pen to hold a vertex of the rotating square at the center of the other square. If available, you can also use dynamic geometry software to do the construction.

2. Explore all possible overlapping areas by visualizing how the shape of the shaded region changes as the top square rotates. What are the different shapes formed by the shaded region? What is the value of their corresponding areas? What are your arguments for finding the value of the area?



3. Write a conjecture that answers the Rotating Square task.

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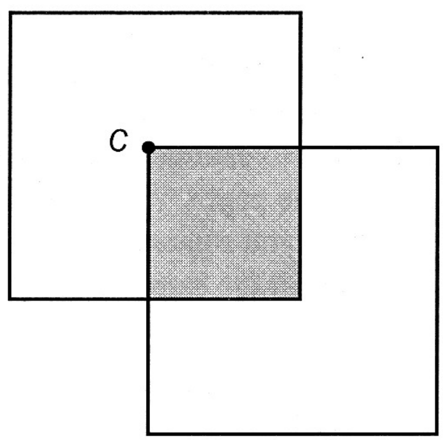
1.3 Exploring the truth (or proving) of the conjecture

In the preceding section, you probably made a conjecture that goes something like this:

The area of the shaded region is always ¼ of the area of the non-rotating square with center C and does not depend on the shape of the shaded region.

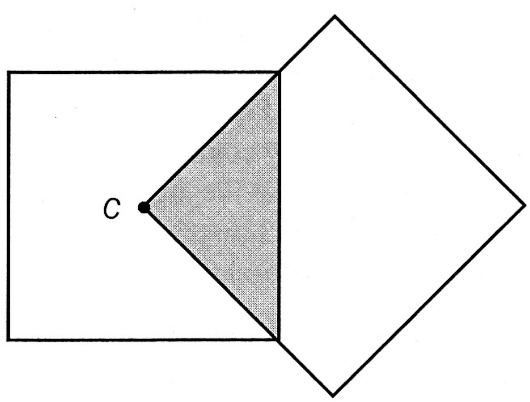
Now develop logic arguments to proof each case.

1. The area of the shaded square is ¼ of the area of the non-rotating square with center *C*.



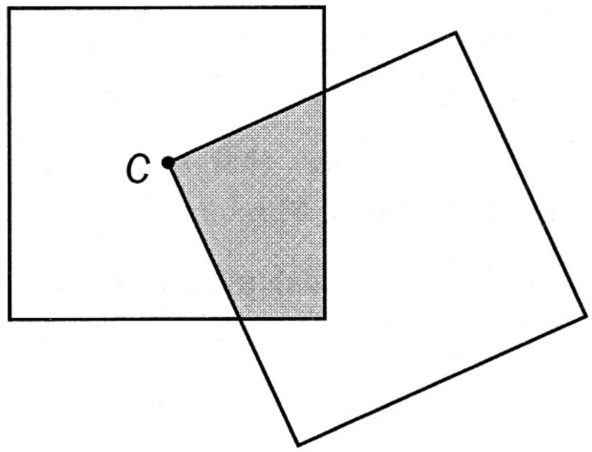
Proof

2. The area of the shaded triangle is ¼ of the area of the non-rotating square with center *C*.



Proof

3. The area of the shaded general quadrilateral is ¼ of the area of the non-rotating square with center *C*.



Proof

References

De Villiers, M. (2003). *Rethinking Proof with the Geometry's Sketchpad.* Emeryville: Key Curriculum Press, CA.

Driscoll, M. (2007). *Fostering Geometric Thinking. A Guide for Teachers, Grades 5 – 10*. Portsmouth: Heinemann, NH.

McCrone, S. King, J., Orihuela, Y., Robinson, E. (2010). *Focus in High School Mathematics: Reasoning and Sense Making in Geometry*. Reston, VA: NCTM.

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