Reshaping Teachers' Mathematical Perceptions: Analysis of a Professional Development Task

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As the focus of mathematics education moves from memorization toward reasoning and problem solving, professional development for in-service teachers must model these activities while simultaneously increasing participants' mathematical knowledge. We examine a representative task from a mathematics professional development course that uses rational number operation as an opportunity for problem solving and modeling. Transcripts exemplify the growth teachers make in deeply understanding the content—division of fractions—while engaging in guided reinvention and classroom discourse. We propose 4 interconnected qualities of this task that allow participants to engage in and reflect on the process of guided reinvention: (1) authentic context with multiple solution methods, including visual; (2) cognitive dissonance; (3) deep engagement; and (4) impact on mathematical knowledge for teaching.

Keywords: Content knowledge; Fraction division; Mathematical knowledge for teaching; Professional development

Since the publication of *A Nation at Risk* in 1983 (Gardner, 1983), multiple educational institutions and researchers within the United States have called for increased rigor in mathematics instruction and a shift from a strict focus on procedural fluency to incorporation of problem solving and critical thinking (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 1989; National Council of Teachers of Mathematics, 2000; Stigler & Hiebert, 1999). Several authors have suggested that tasks that progressively formalize number operations can be used as sense-making activities in and of themselves (Fosnot & Dolk, 2002; Gravemeijer & van Galen, 2003; Hiebert, 1997). The outcome for students of such lessons would be a combination of procedural fluency and conceptual understanding specific to the topic, as well as general experience with the practices of mathematics, including conjecture, reasoning, justification, and modeling.

But such classroom activity can be difficult for teachers who themselves experienced mathematics instruction that focused on memorization of formal algorithms (Ball & McDiarmid, 1989; Schoenfeld, 1988). Teaching mathematics rather than computation requires several different types of knowledge, collectively referred to as mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008; Hill, Rowan, & Ball, 2005). Teachers must understand the conceptual basis for number operations *and* be able to construct and facilitate problem-solving situations for students around those operations. Building upon student thinking also requires a fundamental shift in teachers' beliefs about the nature of mathematics itself, from that of mathematics as a set of rules and algorithms to mathematics as a product of quantitative reasoning (Ernest, 1989; Schoenfeld, 1988). The widespread adoption of the Common Core State Standards (CCSS) and the accompanying assessments has, to some degree, forced this shift because students must demonstrate content knowledge and engage in authentic practice of mathematics.

Professional development for in-service teachers has been widely seen as one method for increasing teachers' mathematical knowledge for teaching (Borko, 2004; Hill & Ball, 2004). Here we describe a single activity from a mandated professional development course—the Mathematical Thinking for Instruction (MTI) course. Within this single problem, we identify attributes of the task (and thus the course) that allow teachers to experience the process of guided reinvention as though they were students. The purpose of this paper is to (1) describe the MTI professional development course in order to situate the task; (2) describe the task and its facilitation, including class transcripts and participant-generated figures; and (3) examine participants' course evaluation reflections in order to identify attributes of the intervention that make it effective in modeling guided reinvention.

The Mathematical Thinking for Instruction (MTI) Course

In 2008, based on a call for improved mathematics instruction by the Idaho State Department of Education, the Idaho legislature mandated that all teachers who might potentially teach mathematics, as well as administrators (about 11,000 people), take the MTI course for recertification by 2014. The pool of MTI participants includes all teachers in the state with a K–8, special education, or secondary mathematics certification, and all those with administrator certifications. Although the course is offered at three levels with largely overlapping content (early elementary, intermediate, and secondary), a single class may include individuals with a very broad spectrum of mathematics backgrounds. It is not uncommon, for example, to have a middle school English teacher, a high school calculus teacher, and a district administrator all in the same secondary course. The 45-hour course focuses on early number, rational number, operations, and algebra. More detail on the development and implementation of the course can be found in Brendefur, Thiede, Strother, Bunning, and Peck, 2013; Brendefur, Carney, Hughes, and Strother, 2015; and Carney, Brendefur, Thiede, Hughes, and Sutton, 2014.

Changes in educators' mathematical content knowledge were measured precourse and postcourse, using items around number and algebra from the Learning Mathematics for Teaching project at the University of Michigan (Learning Mathematics for Teaching, 2008). In addition, retrospective changes in teachers' beliefs about mathematics and self-efficacy in teaching mathematics were measured postcourse with a survey. (Survey items were designed by RMC Research and Math in the Middle project staff at the University of Nebraska–Lincoln in 2005.) For example, participants were asked whether they agree, disagree, or neither agree nor disagree with the statements "All students can learn challenging content in mathematics" and "Mathematics should be learned as sets of algorithms or rules that cover all possibilities." Items were administered retrospectively postcourse in order to avoid pre- to post-intervention response-shift bias, in which participants' understanding of the survey items changes in response to the intervention (Bray, Maxwell, & Howard, 1984), in this case the MTI course. For example, prior to the course, many participants were unfamiliar with the word *algorithm* and may have based their definition of "challenging content" solely on procedural complexity rather than conceptual depth. Using a retrospective presurvey/postsurvey design has been documented as addressing this issue (Aiken & West, 1990; Lam & Bengo, 2003).

Data from the first 3 years of the MTI project show statistically significant gains in participants' content knowledge as well as significant shifts in their beliefs about the nature of mathematics and how it should be taught (Carney, Brendefur, Thiede, Hughes, & Sutton, 2014). These initial results led us to think more deeply about what aspects of the course and its facilitation might be responsible for shifts in participants' content knowledge and beliefs. In this article we analyze participant work, course transcripts, and evaluation feedback around the facilitation of a single task that exemplifies the overall course structure.

MTI Course Theoretical Background

The MTI course uses social-constructivist learning theory and was built around the concepts of teaching for understanding as described by Hiebert and others (Hiebert & Carpenter, 1992; Hiebert, 1997), in which learners construct their mathematical knowledge through their own mathematical activity and interactions with their peers and their teacher. More specifically, the course seeks to build teachers' mathematical knowledge for teaching through a series of tasks that require them to take part in the process of progressive formalization based on the Dutch concepts of "guided reinvention" and "realistic mathematics education" (Freudenthal, 1973; Freudenthal, 1991; Gravemeijer & Doorman, 1999; Treffers & Vonk, 1987). In guided reinvention, students are given an experientially "real" task that allows them to use the mathematics they do know as an entry into a new mathematical concept (horizontal mathematization). From these initial strategies, students then move through tasks, classroom discussion, and instruction to move toward more efficient and abstract mathematics (vertical mathematization).

Guided reinvention can still take place in professional development interventions because many teachers have memorized algorithms but have little understanding of why they work. When pressed to solve tasks without those algorithms, teachers with limited conceptual understanding will go through the process of reinventing mathematics, albeit in an accelerated form, connecting their own initial informal methods to progressively more formal methods introduced by their peers and instructor. Teachers with a more conceptual mathematical background are pressed to consider what initial informal methods might look like and how they connect to more formal methods.

The actual facilitation of the MTI course largely mirrors the orchestrating classroom discussions framework, which consists of (1) anticipating how students will approach a task; (2) monitoring students' progress, in part through

questioning; (3) selecting students' solutions for discussion in order to highlight specific mathematical ideas; (4) sequencing the student work in order to press connections; and (5) connecting across strategies to highlight key ideas and connect to formal mathematics (Smith, Hughes, Engle, & Stein, 2009; Stein, Engle, Smith, & Hughes, 2008). Participants in the MTI course are also asked to solve or examine a given task by creating and connecting between enactive representations (physically constructing or acting out the situation), iconic representations (visually modeling the situation with a drawing or visual tool such as a number line), and symbolic representations (numbers, variables, tables, and equations) that characterize various levels of formality (Bruner, 1964).

The Developing Mathematical Thinking Framework

The MTI course highlights five central practices for developing mathematical thinking that combine socio-constructivist theory and progressive formalization: (1) take students' ideas seriously, (2) press students conceptually, (3) encourage multiple strategies and models, (4) focus on the structure of mathematics, and (5) address misconceptions (Brendefur, 2008). This framework forms the basis of both our work with teachers in their classrooms as it pertains to students and for professional development with teachers. Each practice is discussed briefly below; detailed descriptions can be found in related publications (Brendefur et al., 2015; Carney et al., 2014).

Take students' ideas seriously. Of centrality is the idea that instruction should be based on student thinking. By monitoring students' approaches to a mathematical problem, the teacher determines what students know, what misconceptions they may have, and the breadth of understanding or experience across the class. One must also anticipate students' strategies so they can be quickly identified during the monitoring process. In addition, by selecting student (or participant) representations for class discussion, students can take ownership of the mathematical knowledge that emerges.

Press students conceptually. Once students (or participants) have applied their own thinking to a problem, the teacher must press them to generalize and abstract. This can largely be accomplished by pressing students to make connections between different student representations or solution strategies and by further connecting those representations to more formal concepts or representations. In this way, students gain familiarity with formal mathematics but see its direct connection to their own initial mathematical knowledge. This reflects a cyclic process of guided reinvention whereby students are pressed from their initial thinking to invent more formal efficient methods, which in turn become future initial strategies.

Encourage multiple strategies and representations. In the process of solving a rich contextual problem, a class may generate several solution strategies and ways of representing them. The course stresses that formal algorithms are simply one strategy for solving problems and that working with multiple approaches and enactive, iconic, and symbolic models strengthens students' understanding of the mathematics.

Focus on the structure of mathematics. There are fundamental mathematical ideas that run throughout mathematics. By focusing on these structural ideas and connecting them to student solutions and representations, teachers ensure students make connections across their mathematical experiences, which helps them become problem solvers. For example, decomposing numbers (e.g., 5 equals 3 + 2) is necessary for developing derived facts strategies and for seeing equivalent forms of algebraic expressions (e.g., $5x$ equals $3x + 2x$). Tricks, such as the "add a zero" rule for multiplying by 10, circumvent structural mathematical understanding and can thus be forgotten or misapplied. The MTI course intends to increase teachers' awareness of these structural components of mathematics.

Address misconceptions. We frequently observe deep misconceptions that stem from a lack of structural mathematical understanding. For example, a student who performs $1/2 + 1/3 = 2/5$ is missing a central understanding of what fraction notation indicates and possibly the deeper idea of "unit." This misconception would traditionally be addressed by simply restating the algorithm for adding fractions. The MTI course addresses how such mistakes indicate deeper mathematical misunderstanding and need to be addressed at a foundational level.

These five practices for developing mathematical thinking are central to the facilitation of the entire course, particularly in the division of fractions task described in this article, called the Maren's Garden problem. The Maren's Garden problem was also chosen because comments from participants and related feedback on the course evaluation (discussed under Implications: A Framework for Tasks that Exemplify Guided Reinvention) led us to consider that the attributes of this particular task may be useful for those creating mathematics professional development courses.

Task Facilitation

Prior Work

Prior to this task, participants spend 20–30 hours investigating number, operations, and algebra. Precursors relevant to this task include the following.

1. Understanding the power of context when making sense of operations. Participants investigate how computation algorithms develop from the need to model real situations and are based on foundational properties of numbers.
2. Developing multiple models for division from informal solution strategies.
3. Modeling fractions and fraction operations with number lines, bar models, and Cuisenaire® rods.
4. Investigating the following three concepts foundational for number understanding in general and fraction work in particular.
    - *Iterating and partitioning:* Iterating refers to the process of repeating a single unit, such as counting by thirds up to four, whereas partitioning refers to the opposite process of breaking into equal-sized pieces.
    - *Units and unitizing* (Lamon, 1999): Fluently and flexibly changing the unit being considered. For example, conceptualizing and operating with two 1/3 pieces as a new unit of 2/3.
    - *Equivalence and relationships:* Comparing and equating rational numbers, such as determining whether two fractions are equal.

These concepts overlap substantially in practice. For example, one might change the units of a given quantity by partitioning the unit (e.g., moving from fourths to eighths) in order to establish equivalence to another quantity. The Maren's Garden task is generally part of the culminating set of activities for this topic. This task would address fifth-grade standards in the CCSS around division of a whole number by a fraction and using visual representations and equations to solve real-world problems.

The Maren's Garden Task

The task is presented to participants as follows:

> Maren ordered 4 bags of soil for her raised flower gardens. Each garden needs three-fourths of a bag of soil. How many gardens can she fill completely with soil? How much soil does she have left?[1]

Participants are asked to solve the problem with two different methods of their choosing, but at least one method must incorporate a visual model. The answer to the task is 5 gardens with 1/4 of a bag of soil left over, but the traditional multiply by the reciprocal algorithm yields 5 1/3. This discrepancy is intentional and leads to cognitive dissonance and engagement as seen in the participant solutions below.

Video of the Maren's Garden problem facilitation was taken for two MTI courses. Segments of this video thought to exemplify participants' responses to the problem were transcribed, and pictures of participants' work are used where possible. Participant names have been changed.

Initial Participants' Solutions

Participants tend to approach the problem in one of two ways. Many begin by drawing a visual representation (informal iconic) of the four bags, divide them into fourths, and count how many groups of the three 1/4 pieces there are, as shown in participant Jane's work (Figure 1).

Participants using this method generally arrive at the answer of 5 with 1/4 left over. In Transcript 1 Jane describes her representation to the whole class.

Transcript 1

Jane: I just had four bars there; each bar represented a bag. Then I divided each bag into fourths. I thought, OK, a garden would take—first garden [gesturing to leftmost bar] would take up the first three [pointing to vertically striped boxes], then I did different pattern for the second garden, then I just kept on taking threes, um, until I got to where I couldn't take any more threes. And then went back and found I had one, two, three, four, five gardens with one quarter of a bag left.

1 This task was first used in a professional development workshop funded by a Mathematics and Science Partnership grant for Brendefur in 2004.

Figure 1. **Jane's picture—showing gardens in bags via different shading.**

Of particular note is that she begins to call the three 1/4 bags that make up a garden just "three," dropping the unit of "1/4 bag" altogether until the very end of her explanation. In essence, she has turned the problem into

(1) $$4 \div \frac{3}{4} = \frac{16}{4} \div \frac{3}{4} = 16 \textit{ quarters} \div 3 \textit{ quarters} = 5 \; R \; 1$$

where the remainder is in the unit of quarter bags. Thus, in participants' answer of 5 and 1/4, the 5 refers to gardens, whereas the 1/4 refers to bags. Most participants, however, simply see their solution in the picture itself rather than as this common denominator algorithm.

Participants using more traditional mathematics tend to jump to the algorithm for solving 4 divided by 3/4, multiplying 4 by 4/3 and ending with an answer of 5 1/3.

(2) $$4 \div \frac{3}{4} = 4 \times \frac{4}{3} = \frac{16}{3} = 5\frac{1}{3}$$

Because we require a visual model, many participants discover this discrepancy within their own work, while others identify it within their group.

Lines of Questioning

Depending on their thinking and models, groups are then presented with a series of questions and prompts.

1. Why is there a discrepancy between the answer from the picture and the answer from the algorithm?

Subquestion: Represent the problem with one of the other models for division or fractions used in the course (e.g., Cuisenaire rods, double number line, ratio table, partial quotient).

Subquestion: What part of a garden could you make with the leftover soil? How much of another bag would you use to fill another garden?

2. How does the traditional algorithm make sense in context? What does 4/3 mean in the context of the problem, and why does it make sense to multiply it by 4?

Subquestion: If you have created a ratio table that starts with 1 garden = 3/4 bags, create a ratio table that starts with 1 bag =__ gardens.

Subquestion: If you have created a picture or double number line, where can you see the 16/3?

These questions have three goals with respect to teachers' content knowledge. First, we want participants to make the connection that 1/4 of a bag of dirt is equivalent to 1/3 of a garden and thus realize that the unit one chooses to operate with is flexible. Second, they should be able to articulate that the "multiply by the reciprocal" algorithm is operating with the inverse of the original unit, changing from dividing by bags per garden to multiplying by gardens per bag (equation 2). And lastly, participants should be able to develop a common denominator algorithm for fraction division (equation 1) by recognizing that in many of their strategies they are converting the dividend into the same units as the divisor, namely quarters. Pedagogically, we hope that this process of questioning groups, whole-class discussion, and making connections between different representations takes teachers through the process of guided reinvention as though they were students, demonstrating the Developing Mathematical Thinking framework.

The transcript below exemplifies the line of questioning the instructor might take once a participant sees the discrepancy between the leftover 1/4 and 1/3 but is unsure about why it exists. We use symbolic versions of numbers when people are referring to values within an equation and express numbers as words when people refer to those values more within context. The instructor here begins by asking the participant to reiterate what each quantity means within the context of the problem.

Transcript 2

Instructor: [Referring to the numbers written on the participant's page] What is the 5; what is the 1/4; what is the 3/4; what is the 1/3?

Sue: 5 is the number of gardens we are filling . . . 1/4 is what was left out of the bag, out of four bags of soil . . . 3/4 is how much was required for each garden . . . So 1/3 is . . . hmmm . . .

Molly: A third of this [pointing to the 3/4]? Of the 3/4?

Sue: 1/3 is what is actually left in the bag, right?

Instructor: OK, so go back to your drawing here [similar to Figure 1]. Where's the 3/4?

Sue: Right here [pointing to one of the gardens in the picture].

Instructor: OK, so that's this [pointing to the 3/4]. What is this [pointing to the 1/4]?

Sue: 1/4 is this [again pointing to leftover quarter in the picture], what's left over after . . .

Molly: There's just a quarter left out of the bag.

Instructor: OK, let me do it this way then. Take that last quarter that's left. You have a quarter left. Throw it into the last garden. What's happened to that garden?

Molly: Too much soil.

Instructor: No, you just have . . . how much soil do you have left? A quarter . . . and you throw that into the last garden. How much of the garden is full?

Sue: Only a third of the—if we were to have another garden . . .

Molly: 1/3 of our sixth garden!

The instructor uses physical imagery (throwing soil into an additional garden) to press the participant to make a connection between the two different units of bags and gardens. A common tactic among instructors, as illustrated above, is to bring the participant's attention back to the context of the problem and the numbers related to that context. This tactic is meant to both help participants who have difficulty making sense of the numbers and operations they have written down and to press more advanced participants to look beyond known algorithms.

Additional Representations

Facilitation of this task relies on participants generating and connecting between multiple representations of the problem in order to meet the three goals described above. We therefore examine participant work for each of the two most common representations aside from the traditional algorithm and informal picture.

Double number line. Instructors often ask questions encouraging participants to translate between units. The double number line shows equivalence between units in proportional relationships and thus can be used to model multiplication and division. It is also called for in the CCSS in sixth grade as part of proportional reasoning. Jim created the number line in Figure 2.

Transcript 3

Jim: We had the four bags on a number line, uh, since it was three fourths of a bag per garden, we divided the bags into fourths, then just counted basically 3/4, 3/4, 3/4, 3/4, and 3/4 and got five gardens. And, uh, it was just the one fourth of a bag left over.

Instructor: And how do you see that that one fourth is also one third of a garden?

Jim: Well, uh, this unit [motions to the leftover quarter] is the same as this unit right here [motions to first quarter bag on the line], which was one out of the three that was included in the bag.

In this model, equal length indicates equal value. Jim's double number line indicates how many bags of soil are

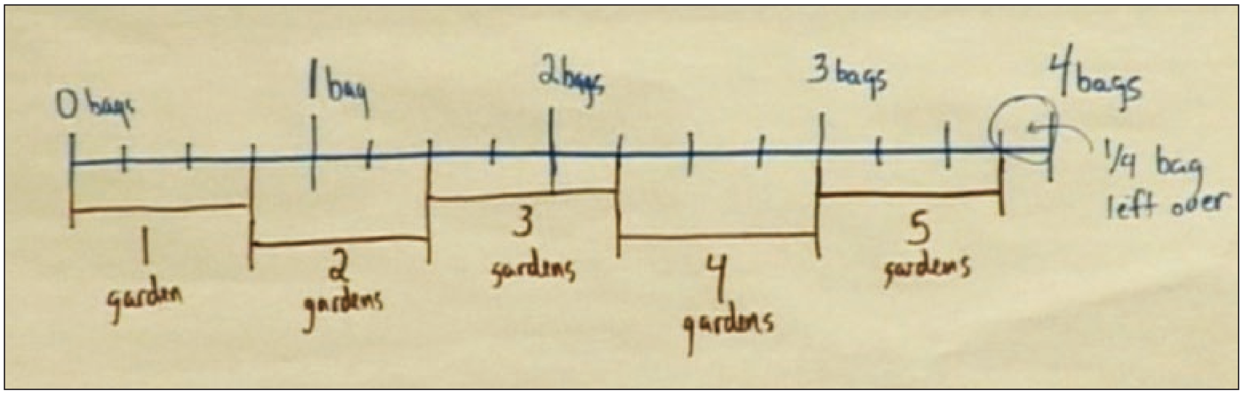

Figure 2. **Jim's double number line representation**

in a given number of gardens and vice versa. Of particular interest is the interpretation of the final quarter bag. One can compare the length of the 1/4 bag to the 3/4 bag required to create one garden. That 1/4 bag length is thus equivalent to 1/3 of a garden.

Ratio tables. Tables of equivalent ratios (or ratio tables) are used to model multiplication or division and to examine proportional relationships. They are explicitly called for in the sixth grade CCSS. In this problem, two different ratio tables can be used to model the problem (Figure 3). Ratio table (a) models the problem as it is given—the number of bags per garden. But ratio table (b) examines how many gardens one could make from each individual bag.

Sasha was asked to relate each of her two ratio tables back to the unit in which she is operating. In doing so she (at this point, unknowingly) is describing the two different strategies for dividing fractions.

Transcript 4

Sasha: So I did my ratio table both ways. So in the first one [pointing to table (a)] G is gardens, and B is bags. And then the arrows—this is still the bags column, I just wrote it so we can see it as 1 1/2 garden . . . so I mean bags. 1 garden is 3/4 of a bag, 2 gardens is 1 1/2 bags, 3 gardens is 2 1/4 bags, so you get down to 5 gardens is 3 3/4 bags. . . . So you see that we have 5 gardens and we had to use 3 and 3/4 bags, so we have 5 gardens and a quarter bag left over. And then going this way [pointing to table (b)], if you say bags, you have 1 bag, that's 4/3 of a garden, which is really 1 1/3 gardens, 2 bags is 2 2/3; 3, 4 gardens, so I have 4 bags, is 5 1/3 gardens, so I have five gardens and then a third of a garden left over.

Connecting representations. Facilitators model the practice of connecting between models to press participants conceptually toward more abstract or efficient methods while connecting back to less formal representations. For example, in this transcript, the participant initially explains the ratio table (Figure 3), but then the facilitator asks him to relate the ratio table to the number line (Figure 2).

Transcript 5

Sam: So that's 12/4, which is 3, right. So—and 4 = 3 [pointing to the 4 bags soil to 12/4 garden in the ratio table]. So then if we go to 5, if we want 5 bags, then we have 3 and . . . it's like doing that counting... so it's 3 3/4. So that means you have a quarter left. It's like counting up by 3/4 and counting down by 3/4. So this would be—2 would be 1 1/2, 3 would be adding another 3/4 again, and that would be 2 1/4. Then it's 3, and then it's 3 3/4 on the ratio table.

Instructor: How does that relate to your number line there?

Sam: I counted up . . . I took a bag, 3/4, and I made 3/4 be 1. Another 3/4 is a unit 2. Another 3/4 is a unit 3. Another 3/4 is a unit 4. These were left over, and so I just moved one over there and got to 5. And this was 1/4 left. On the number line, I made the units be—each unit be 3/4.

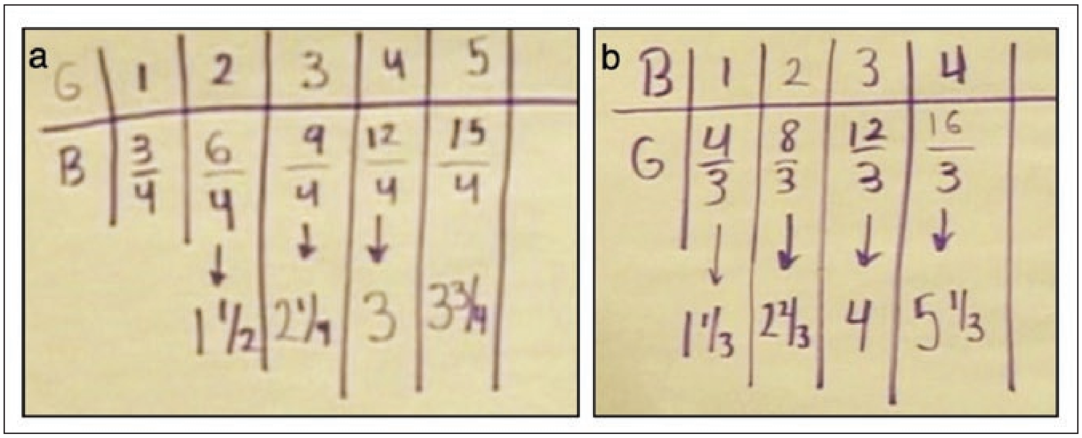

Figure 3. **Sasha's representation—ratio tables a) counting in units of gardens; b) counting in units of bags.**

Developing Algorithms

Figure 4 shows two *possible* progressions of formalizing through multiple representations, indicating how different strategies might build to two formal algorithms.

Multiplying by the reciprocal. The left column of Figure 4 connects the traditional algorithm for fraction division to the other representations. The 3/4 in 4 / 3/4 represents 3/4 bag per garden. The reciprocal can be thought of as either 4/3 gardens per bag or 4 gardens for every 3 bags. Both interpretations are evident in the ratio table and visual representations, where each bag contains 1 1/3 gardens' worth of soil. Once participants have shared their models, the instructor generally asks a participant to explain the connection between the visual models and the traditional algorithm, as exemplified by the following transcript where, during whole-class discussion, Chris connects the traditional algorithm to the ratio table in Figure 3b.

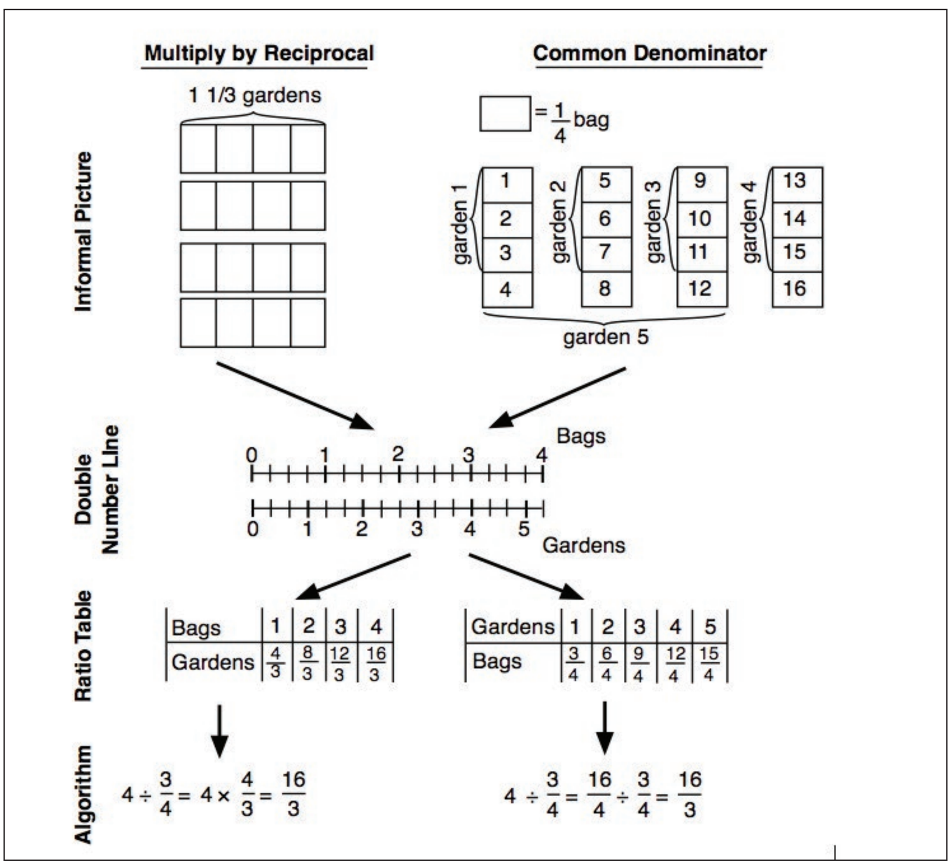

Figure 4. **Possible ways of progressively formalizing toward fraction division algorithms.**

Transcript 6

Instructor: Where is this 4 times 4/3 up here [gesturing to representations on the board]?

Chris: It's right here [pointing to first column of Figure 3b]. 4/3 times four to get . . . 4/3 of a garden out of one bag. 4/3 times 4 bags gives you 16/3 [pointing to fourth column], which is 5 1/3. So this 4/3 [pointing to 1:4/3 in ratio table] is this 4/3 [pointing to 4/3 in traditional algorithm].

Common denominator algorithm. Both the double number line and the informal picture bring out a second division algorithm, as seen in the right column of Figure 4. In this problem, we can view the four bags of soil as containing a total of 16 quarter-bags of soil. Viewing quarter-bags as our unit, we then measure out groups of three units to determine how many gardens can be made. There are five groups of three or five full gardens with one unit left over. We are changing all parts of the problem into the same fractional unit, allowing us to essentially ignore the unit altogether. In this case, 4 divided by 3/4 becomes 16/4 divided by 3/4, which we can treat as simply 16 divided by 3, as in Equation 1.

Technically, one need not find common denominators and could instead simply divide straight across in the same manner in which we perform fraction multiplication. But in even fairly simple problems, this can result in ugly fractions that have to be extensively manipulated

$$(\text{e.g.,} \frac{3}{4} \div \frac{2}{3} \text{ ends up being } \frac{3/2}{4/3}).$$

Teachers seldom develop the common denominator algorithm on their own during the task facilitation. Rather, after the problem has been largely debriefed, the instructor will put up the method on the board, asking teachers whether it is mathematically valid and what it represents within the problem context and the multiple representations. Some teachers do, however, develop the 16 units on their own within the drawing (Figure 1), as this transcript indicates.

Transcript 7

Tammy: This is my area model here, so I decided—so these are all bags. Bag, bag, bag [pointing to her drawing, similar to Figure 1]. I divided each bag into four. We have 16 total parts of soil. Each garden takes three parts, so I just decided to divide these up into three. Each shaded area is a garden . . . so there's a garden, there's a garden, there's a garden, there's a garden, there's a garden. So there's my three parts and then this little guy's left over [pointing to the unshaded quarter garden]. So that's 1/16 of soil left over. Or if I had two more pieces of my area I could fill a garden. So it's 1/3 of a garden completed.

Instructor: OK, and when you call it 1/16 . . .

Tammy: Parts of soil.

Instructor: Parts of? What is your whole then?

Tammy: 16 . . . my whole is four bags of soil, which is 16 parts of soil if I divided it into four. So I have 16 total parts.

Tammy is seeing the problem as four bags divided into sixteen equal pieces, each of which represents one fourth of a bag. This recognition is essential for developing the common denominator algorithm.

Course Evaluation Data

In order to capture participants' thoughts about this task, we analyzed their responses to three evaluation questions for courses taking place during the spring and summer semesters of 2013, pulling those responses that mention this task specifically (either by name or as a division of fractions task). This time frame was chosen for convenience because at this time course evaluations moved online, making it easier to filter participant responses. The specific qualitative evaluation items reported in this study are:

1. Describe the activities you found most useful and why (please be specific).
2. What new mathematics did you learn during the course? Comment on any mathematical "aha" moments you experienced.
3. Describe any new teaching practices you learned (specific to mathematics) and how you intend to use them in your classroom.

Although the course evaluation responses represent a temporal snapshot of the 5-year administration of the MTI course and were made anonymously, we consider them to be representative of the course participant population. This sample includes participants from all three levels of the course (797 early elementary, 417 middle grades, 120 secondary).

Of the 1,145 participants who responded to at least one of the three evaluation questions of interest (about 60% of all participants), 27% mentioned the fractions unit in one of their responses, and 59 people specifically mentioned the Maren's Garden problem. The fractions unit represents approximately one eighth of the course. Given the open-ended nature of the questions and the fact that over a quarter of responding participants mentioned the fractions unit generally and 5% specifically mentioned this problem, the tasks contained within seem to have been particularly impactful. Quotes from the evaluation questions in which participants specifically cite the Maren's Garden problem are used to support and exemplify the attributes of the task that have implications for mathematics professional development more generally.

Implications: A Framework for Tasks that Exemplify Guided Reinvention

Based on evaluation responses and the authors' collective experience in facilitating the task over 90 times, we present a framework of factors that make this activity useful in exemplifying the process of guided reinvention, even for participants who know a formal algorithm. Each aspect of the framework is accompanied by quotes from participant evaluations in order to illustrate the factor from the participant's perspective (the number following the quote refers to which of the three above questions the participant was answering).

Authentic Context, Visual Situation, and Multiple Solutions

This problem can be solved with a variety of models because the context lends itself to visual representation. In addition, all of the different representations connect to one another (Figure 4). This allows teachers with more limited math backgrounds to feel validated in their initial reasoning while pressing more mathematically confident teachers to find their algorithm within the less formal solutions. It is through connecting representations that participants take part in progressive formalization firsthand. The idea that problems can be approached informally seems to resonate strongly with many teachers.

Quote 1: Being able to multiply and divide fractions without using an algorithm (because of course I don't remember them) and getting the right answer by using an iconic model and actually understanding what I did and why it worked!! (Question 2)

Quote 2: Why the traditional algorithm for division of fractions requires that you use the reciprocal. It opened me up to the realization that not everything needs a computation, sometimes logic thinking is best. (Question 2)

In addition, this task lends itself to developing multiple algorithms because of its structure. This is a measurement division task for which two different broad solution methods are possible (Figure 4). We found partitive fraction division contexts to be less fruitful for initially developing multiple strategies because they tend to lead to the reciprocal algorithm. For example, questions of the form

> If 4 bags of soil fill 3/4 of a garden, how many bags are required to fill one whole garden?

tend to lead participants toward iterating 4/3 four times. Thus, selecting problems for which multiple informal strategies could be used allows for greater flexibility in how the mathematics is reinvented.

Cognitive Dissonance

In professional development teachers' time with a given topic is likely limited to a few hours, and engaging a wide range of teachers in the same problem can be difficult because some have formalized content knowledge and some do not. Given these constraints, we seek to ensure that everyone has access to the problem and can be cognitively engaged. Intentionally creating cognitive dissonance is one way to ensure that teachers are pressed beyond just "doing" the problem.

In this case, the multiple solution paths lead to two seemingly different answers: 5 1/3 (both in the unit of gardens) or 5 and 1/4 (where the 5 is the unit of gardens but the 1/4 is in the unit of bags). This creates temporary confusion that generates discussion and questions amongst participants. This dissonance is particularly valuable in generating discussion between participants who drew informal pictures and those who used an algorithm because their answers appear not to match—but both camps are confident in their approach. It is this apparent dissonance in a professional development setting that can help illustrate the value of discourse as participants slowly realize the commonalities and connections between their solutions.

Quote 3: When we were solving Maren's Garden I wanted to go directly to the standard algorithm but I drew my picture and then solved the problem with the standard algorithm. The answers didn't match so I was trying to figure out what I did wrong that made the answers different. It was when someone put a different visual on the board that I realized why drawing a picture gives you a different answer than the algorithm. Then for fun we got

the fraction rods out and solved the problem with those and saw an even better visual. (Question 2)

Quote 4: It was very eye opening to see that the algorithm's fraction did not match the picture model and a number line was used to understand the meaning of the fractions. (Question 2)

Deep Engagement

Because of the multitude of representations and the cognitive dissonance, this task typically takes about 1.5 hours to facilitate. This extended focus on a single problem may encourage teachers to rethink the nature of mathematics and its instruction, requiring them to see the development of number operations as a problem-solving opportunity in and of itself and, more broadly, what it looks like to do mathematics rather than computation.

Quote 5: I kept dreaming last night about 3/4 of a bag of soil per garden and that 1/4 is left over to cover 1/3 of another garden and so on . . . and this dream was neverending! My mind is exhausted! But I'm really learning a lot and have been trying to teach math from a different point of view now! It's been great! Thanks! (From uninitiated email correspondence with a participant, reported here with permission.)

Changing Mathematical Knowledge

This activity in particular focuses on both building content knowledge and changing knowledge about mathematics and how it is taught. The activity is couched within a series of smaller, content-specific tasks that build teachers' conceptual understanding of fractions, their representations, and the more general concepts of units and unitizing (see **Appendix A** for the unit overview). We hypothesize that our professional development would be far less effective if it consisted only of small tasks that focused on specific concepts or, conversely, only included extended problem-solving tasks such as the Maren's Garden problem. It is this combination and the sequencing of tasks that allows participants to model the problem themselves (nontraditionally) and connect between models, thus building both their mathematical content knowledge and pedagogical content knowledge. We see both of these ideas in the quotes below.

Quote 6: I had some "aha" moments with multiplying and dividing fractions. I was able to see how they are connected to real world situations. I learned the algorithms and was able to compute accurately but could not understand what was happening. Thanks to MTI I have a better understanding. (Question 2)

Quote 7: In my work with classroom teachers, I intend to really push them on de-emphasizing a specific method and taking the time to develop the concept through student-developed (teacher influenced) models. That was probably my greatest take away from this training. . . . It really does take a lot of time to develop a deep understanding and it is important that teachers take the time. (Question 3)

Implications for Teacher Educators

In planning for work with preservice or in-service teachers, we suggest that mathematics teacher educators might use these four attributes in evaluating potential tasks. Problems from curricula that teachers are familiar with can be used to reinforce content knowledge but may lack multiple points of entry or the cognitive dissonance that deeply engages adults and has the potential to change their perceptions about mathematics. But one might alter such problems to ensure that multiple representations are possible by changing the number set (e.g., from large values to whole numbers or simple fractions) or changing the context to elicit specific visual representations (e.g., from money to distance or area). Likewise, one can foster cognitive dissonance by altering the task such that there is a single solution that has multiple, seemingly different, forms. Removing directions such as "Create an equation that . . ." or "Use a table to . . ." can give the flexibility to approach the problem in different ways, leading to different methods and solution forms. Conversely, mathematics professional development sometimes introduces advanced topics such as combinatorics or group theory in tasks that are useful in helping teachers develop problem-solving strategies. But we have found that using familiar mathematical content can help teachers make strong connections to their own practice, resulting in changes in their mathematical knowledge for teaching.

Conclusion

The goal of this article is to provide a detailed analysis of a single professional development task that engages teachers in mathematical discourse that can challenge and change their specific content knowledge. But based on participants' comments and reported changes in beliefs, such tasks may also change teachers' perceptions about the nature and teaching of mathematics itself by allowing them to take part in the activity of guided reinvention by using context and models to develop and make sense of computation algorithms. At a time when teachers are seeking opportunities to increase content knowledge, we argue that professional development focused on mathematics content must also change

foundational beliefs about mathematics, pressing teachers to see mathematics as a sense-making activity. It is notable that this task is relatively simple; it does not require a great deal of setup, uncommon materials or manipulatives, or any technology. Rather, it is through the facilitation and questioning that deep mathematical connections are made. This problem and problems that have similar traits (e.g., multiple representations and cognitive dissonance) might thus be used in a wide variety of professional development settings to increase content knowledge while modeling how the teaching of computation can be utilized as a problem-solving activity.

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Appendix A: Fraction Unit Sequence for MTI Course

The sequence of activities below represents 8–12 hours of coursework in the 45-hour class.

1. Defining fraction
    a. Enumerating a denomination
    b. Part–whole interpretation
2. Connecting to whole number
    a. Composing and decomposing fractions to add and subtract
    b. Skip counting by fractional units
3. Iterating and partitioning on the number line
4. Unitizing within a bar model and connection to fraction multiplication
5. Unitizing within an area model
6. Defining unit and "whole" with Cuisenaire rods (e.g., if a given color = 1, what do other color rods represent)
7. Maren's Garden task (division of fractions)
8. Multiplication of fractions in an area model
9. Analyzing student strategies for subtraction of mixed numbers

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