Multiplication by 10_{five}: Making Sense of Place Value Structure Through an Alternate Base

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Prior research has shown that preservice teachers (PSTs) are able to demonstrate procedural fluency with whole number rules and operations, but struggle to explain why these procedures work. Alternate bases provide a context for building conceptual understanding for overly routine rules. In this study, we analyze how PSTs are able to make sense of multiplication by 10_{five} in base five. PSTs' mathematical activity shifted from a procedurally based concatenated digits approach to an explanation based on the structure of the place value number system.

Keywords: Alternate bases; Place value; Preservice teachers; Whole numbers/Natural numbers/Counting numbers

Mathematical knowledge is based on both convention and logic. However, convention in this case serves as a shelter for those who don't have a conceptual understanding of a mathematical procedure (Ma, 1999, p. 31).

Setting the Stage: An Example for the Reader

When school children encounter base ten numbers, they often have limited knowledge of the structure of our base ten system. To simulate a similar experience for preservice elementary teachers (PSTs), we often ask them to construct numbers in a different base for which they do not have an explicit structure yet in place. To allow the reader to experience some of what the PSTs experience, we begin with an example situated in base five. In base five we collect units until we have 5 units, then we create a group of 5 to form a new unit. We continue this way, grouping 5 of each unit type to form the next larger unit. Consider the number 123_{five}, which refers to 1 group of twenty-five, 2 groups of five and 3 ones (see Figure 1a). To provide some language to talk about the numbers in base five, a group of 5 ones will be referred to as a *long*, a collection of 25 ones (or 5 longs) will be referred to as a *flat,* and a collection of 125 ones (or 5 flats) will be referred to as a *long flat* (see Figure 1b). The numeral 23_{five} can be read aloud as 2 longs 3 ones. While it is possible to convert a base five number like 123_{five} into base ten (it is 38), we ask the reader to take a minute and try to solve the following problem without converting out of base five. This will allow you to experience working in an unfamiliar base before reading the remainder of the paper.

Problem: *Find the product* $3_{\text{five}} \times 23_{\text{five}}$.

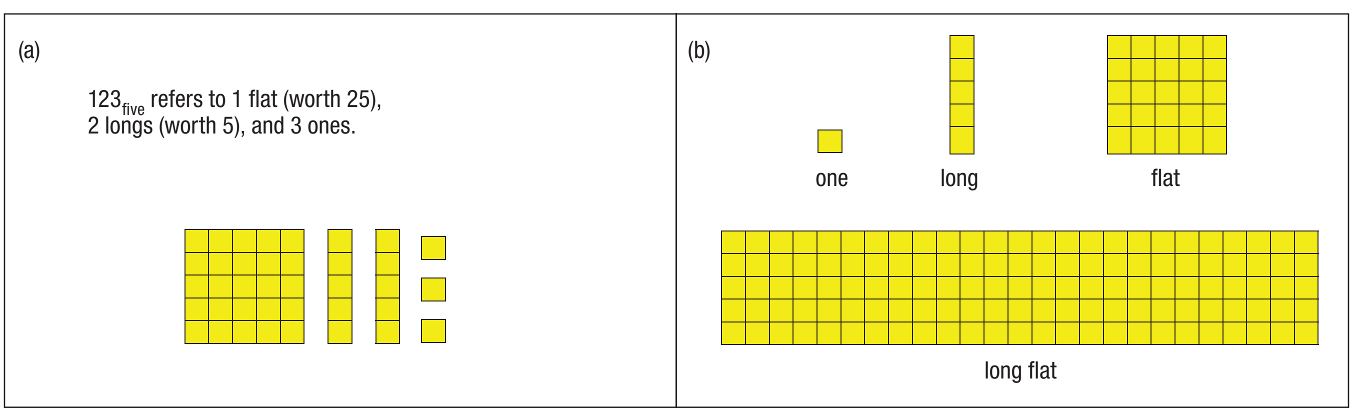


Figure 1. **An introduction to base five language and imagery.**

Introduction

Current educational policy documents in the United States advocate that children develop a conceptual understanding of mathematics, focusing on sense making in addition to procedural fluency (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). We will adopt Hiebert and Lefevre's (1986) characterization of conceptual and procedural knowledge. Conceptual knowledge is knowledge where "the linking relationships are as prominent as the pieces of information" (1986, pp. 3–4). For example, conceptual knowledge of base ten might include an understanding of the relationship between adjacent unit types (ones, tens, hundreds), and thus a number like 300 might be understood as being 300 *ones*, 30 copies of *ten*, 3 copies of *hundred,* etc. In contrast, Hiebert and Lefevre identify procedural knowledge as knowledge of symbols, rules, and algorithms. Knowledge that the 3 in 345 is in the hundred's place because that is the name of the place (label) rather than the value would be considered procedural knowledge. In order to support their students' development of conceptual knowledge, teachers also need to have conceptual knowledge of the mathematics they teach.

Preservice teachers (PSTs) often demonstrate the ability to use procedures efficiently but lack the knowledge to explain why these procedures work (Ball, 1988; Lo, Grant, & Flowers, 2008; Ma, 1999; Thanheiser, 2009a). Further, the PSTs may be unaware that there is a rationale (a "why") for each step in the procedure (Thanheiser, Philipp, Fasteen, Strand, & Mills, 2013). However, the PSTs' lack of conceptual knowledge may be masked by their use of mathematical terminology, including procedural use of place value language (Ma, 1999; Thanheiser, 2009a). For example, a PST who is multiplying the multidigit numbers 24 x 38 may state that the product of the 2 and 3 is a 6 that is placed in the hundreds column, using the term *hundreds* as a label, but without understanding that the 2 refers to 2 tens or 20, the 3 refers to 3 tens or 30, and the 6 refers to 6 hundreds.

In this article, we examine how an alternate base context might contribute to the development of conceptual knowledge of the multiplicative structure of place value in a preservice mathematics content course on number and operation. By understanding place value we mean understanding that a digit has a face value (value of a digit) and a place value (value of the unit type the digit represents) and that there is a consistent 10 to 1 multiplicative relationship (in base ten) between adjacent unit types. The place value and face value are combined to determine the value of the digit. We use the context of base five to allow PSTs to work in a system in which they do not have access to preestablished labels for place values or preestablished procedures that might interfere with or mask their sense making.

We leverage base five to provide an opportunity to compare across bases. This may enable PSTs to see and understand more clearly the structure of place value number systems in general and base ten in particular. Prior researchers indicate that, rather than being trapped within base ten, students may see that base ten is an example of a type of place value number system (Treffers, 1987; Vygotsky, 1962). In particular, we will focus on the mathematical task of multiplying by the base (10_{five}) in base five in order to focus on the multiplicative structure of place value number systems.

In base five, the numeral 10_{five} refers to 1 group of five. To avoid confusing base ten and base five, we do not refer to 10_{five} as ten. Rather, we may call 10_{five} a *long*, following the notation given in Figure 1 (Bennett, Burton, & Nelson, 2011). The difference between ten (written 10) and a long (written 10_{five}) was a source of disequilibrium for the PSTs. This disequilibrium was explored within our task, where the PSTs compared the role of multiplying by 10_{five} in base five with the role of multiplying by 10 in base ten, examining the multiplicative structure of both number systems. We explain this in more detail after the solution to the earlier example problem.

A Solution to the Example 3_{five} x 23_{five}

We now solve the problem posed to the reader above. The problem 3_{five} x 23_{five} can be modeled as repeated addition, creating three copies of 23_{five} (Figure 3a). To group like units we rearrange the base five manipula-

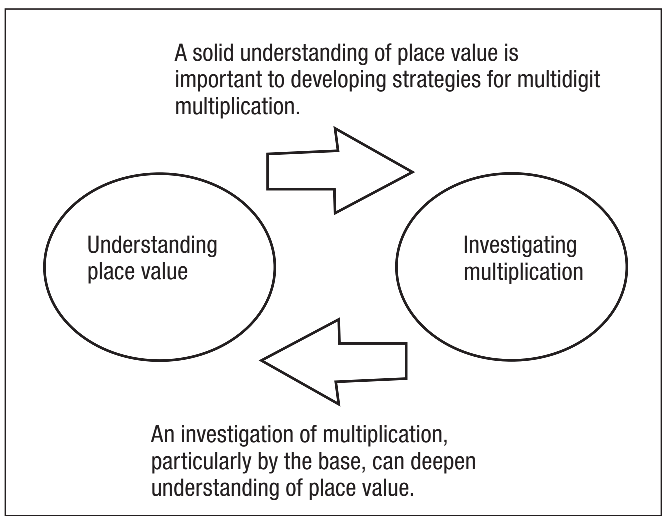


Figure 2. **Linking growth in place value understanding to investigating multiplication.**

tives to place the longs together and to place the ones together (Figure 3b). One characteristic of place value systems is that we need to create minimal collections (i.e., no more than 4 of any one type of unit) for notation, as there are only symbols for values less than the base. To regroup, we look for 5 of one type of unit. Five longs are regrouped to create 1 flat and 5 ones are regrouped to create 1 long (Figure 3c). Once a minimal collection has been established, the solution can then be recorded symbolically as 124_{five} (Figure 3d).

The reader may want to try to find the product of 10_{five} x 23_{five}[1] and reflect on the solution before reading the remainder of this paper. This problem is the central focus of the rest of paper.

Why Times 10?

In base ten, multiplying a number by 10 causes all of the digits in that number to be shifted to the left one place value and a zero to be appended at the end of the number (in the ones place). This is also true in base five when multiplying a number by 10_{five} (or in any base when multiplying by 10_{base}). This phenomenon is important for PSTs to make sense of for two reasons. First, unpacking why digits shift (unchanged) to the left places a focus on the multiplicative structure of base five and base ten place value number systems. Second, multiplication by the base is a key phenomenon to exploit when developing or making sense of multidigit multiplication strategies.[2] Both of these reasons are discussed further in this section, but first we discuss what to name this phenomenon.

What's in a name? There are many options when selecting a name for the multiplication by 10_{five} rule. Possibilities include the *Times Ten Rule, Times One Zero Rule, Times Five Rule*, and *Times Base Rule*. Each of these titles has its affordances and its limitations. In base ten, the Times Ten Rule is an appropriate title, but in base five it is a misleading use of base ten language (Danielson, 2010). The Times One Zero Rule is independent of base, but implies a concatenated digits conception by ignoring place value when naming the number 10_{five}. The Times Five Rule is a name specific to base five rather than being a name that could be applied across bases. Further, five has an associated symbol representation in base ten, namely 5, that is not transferrable to base five. The Times Base Rule is a more mathematically accurate and general title but may not be authentic to the PSTs' own informal language.

For the purpose of this article, we use the Times Base Rule with the caveat that this is not the language that was utilized in the classroom.[3] The Times Base Rule is mathematically accurate, captures the generalizability, and does not have the conceptual limitations of the alternative titles. The Times Base Rule refers to the generalization that multiplying a number by the base (written 10_{base}) of that number system causes the digits of that number to move one place to the left and a zero to be written in the ones place.

Multiplication by 10 and Multiplicative Structure. Multiplication by 10_{base} in any base means that we make *base* copies of that number. In base ten this means 10 copies; in base five this means 5 copies. Having base copies of a unit results in regrouping the *base* copies to 1 copy of the next larger unit. Thus all digits are moved one place to the left in a number and a zero is appended at the end (in the ones place). (See Figure 4.)

In base ten, each digit's place value is 10 times larger than that to its right. For this reason, multiplication by 10 can serve an essential role in probing understanding of

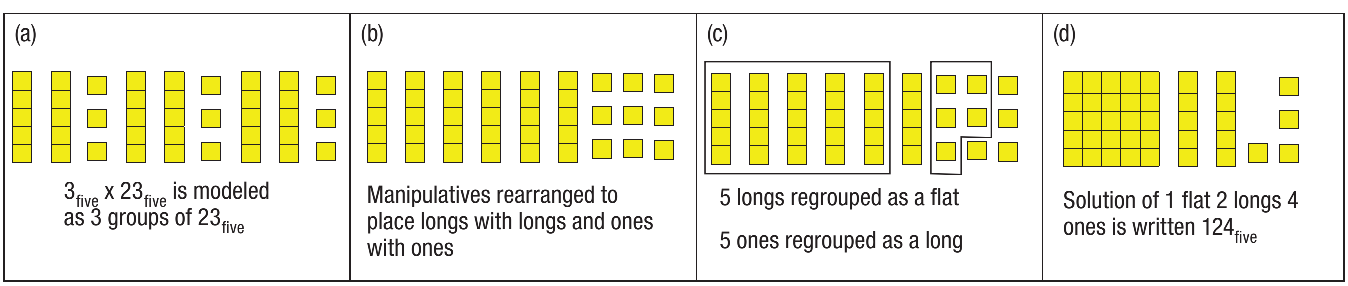


Figure 3. **A solution to 3_{five} x 23_{five}, modeled with manipulatives.**

1 Note: In the task given to PSTs, the numbers in the problem were given in the opposite order, $23_{\text{five}} \times 10_{\text{five}}$. Nevertheless, the PSTs interpreted the problem as 10_{five} groups of 23_{five} rather than 23_{five} groups of 10_{five}. In the updated version of the task, the prompt is changed to $10_{\text{five}} \times 23_{\text{five}}$.

2 In fact, this is an essential step in the standard U.S. multiplication algorithm, where numbers are implicitly broken into multiples of powers of ten.

3 The students referred to this rule as the Times Ten Rule or "adding a zero."

The Times Base Rule: Multiplying a number by the base causes all of the digits to shift to the left and a zero to be appended to the end of the number.

(e.g., 10 x 45 = 450 and 10_{five} x 32_{five} = 320_{five})

Note: This rule is true in any base. In this study, PSTs investigate multiplying by 10_{five} in base five.

Figure 4. **Statement of the Times Base Rule.**

the base ten system. Baturo and Cooper (1998) illustrated this point by utilizing *times ten* tasks to show conceptual gaps in students understanding of the multiplicative base ten system. Focusing on multiplication by 10_{base} brings PSTs attention to this consistent multiplicative relationship between unit types (i.e., between columns in multidigit numbers).

Utility of multiplication by 10. Multiplication by 10 and multiples of 10 in base ten also plays an important mathematical role in student generated strategies for multidigit multiplication (Ambrose, Baek, & Carpenter, 2003). Because of the multiplicative structure of the place value system, students often split multiplication problems along place value to make use of the efficiency of multiplying by 10. For example, consider the problem 12 x 23. A student may split the 12 into 10 and 2, changing the problem from 12 groups of 23 to 10 groups of 23 plus 2 more groups of 23. This creates two easier problems (10 x 23 and 2 x 23), the products of which can then be added together. The student generated strategic use of the distributive property may lead to a more general multiplication strategy of splitting numbers along place values.

The role of multiplying by 10 in base ten is essential but is often hidden within the procedure for the standard algorithm. (See Figure 5 for an example of multiplying 23 by 42 using the standard algorithm.) During the first set of steps of the standard algorithm, 3 is multiplied by 42 to result in 126. In the second set of steps, it appears that the digit 2 is multiplied by 42 to get 84. The 2 from 23 is actually 20 (i.e., 10 x 2). Because the value of 2 x 42 is 84, the value of 20 x 42 (which is 10 times more than 2 x 42) can be found by appending a zero to 84 to arrive at 840. Because of the efficiency and compactness of the algorithm, it is possible for students to follow the correct steps and place 84 in the correct location (moved to the left one place) without understanding the relationship between multiplying by 2 and multiplying by 20. Unpacking the meaning of multiplication by 10 may help PSTs to make sense of how multiplication by 20 is related to multiplication by 2. Further, this relationship can serve to highlight the underlying multiplicative structure of place value that allows for the efficiency of the standard algorithm.

Brief Literature Background

Preservice Teachers' Understanding of Place Value, Number, and Operation

A small body of research exists on PSTs' conceptions of number and operation within the traditional base ten context (Thanheiser et al., 2014). A conceptual understanding of multidigit whole numbers requires an understanding of place value and the relevant 10 to 1 relationship found in base ten (Fuson, 1990; Fuson et al., 1997; Kamii, 1986; Thanheiser, 2009a). For example, the 2 in 234 represents 2 groups of a *hundred*. Since a *hundred* is equivalent to 10 *tens* (adjacent unit types), the 2 also represents 20 *tens;* and since a *hundred* is equivalent to 100 *ones* (each *ten* is equivalent to 10 *ones*), the 2 also represents 200 *ones*. PSTs should be able to move flexibly between these three interpretations using the 10 to 1 multiplicative rela-

$$\begin{array}{r} 42 \\ \underline{\times\ 23} \\ 126 \\ \underline{+\ 840} \\ 966 \end{array}$$

Consider the problem 23 x 42. In the standard U.S. algorithm, the multiplicand (42) is multiplied by each digit of the multiplier (23) and then added up with all of the digits properly shifted. The written algorithm is shown on the left. First the 3 is multiplied by the 2, resulting in 6, which is noted below the 3; then the 3 is multiplied by the 4 in 42, resulting in 12, which is noted under the 2, thus resulting in 126 in the first line of the result. Next, the 2 in 23 is multiplied by the 2, resulting in 4, which is shifted one column to the left and noted under the 2 of the 126; then the 2 in 23 is multiplied by the 4, resulting in 8, which is noted to the left of the 4. Sometimes a zero is placed in the ones column after the 84, either to indicate that the 84 refers to 84 tens or merely as a placeholder to remind the algorithm user to shift to the left. Finally, the partial products are summed to arrive at the total of 966.

Figure 5. **A brief description of the standard U.S. algorithm for multidigit multiplication.**

tionships between columns. This flexible understanding of multidigit numbers is referred to by Thanheiser (2009a) as a reference units conception. This conception of number was the most sophisticated held by PSTs in that study, followed by another correct conception in which PSTs were able to see the digits representing the correct values but did not have the flexibility to move between unit types. Thus for the number 234, PSTs would be able to state that the 2 represents 200, the 3 represents 30, and the 4 represents 4, but they would struggle finding alternate interpretations. Thanheiser also identified two incorrect conceptions based on seeing the values of the digits as concatenated. PSTs either saw the digits in 234 as 2 ones, 3 ones, and 4 ones, or combined this incorrect concatenated digits conception with aspects of the correct conception, thus seeing some digits in terms of correct unit types but others incorrectly in terms of ones (i.e., 2 as 2 *hundreds,* 3 as 3 *ones,* and 4 as 4 *ones*). Thanheiser found that two thirds of the PSTs she interviewed held one of the two incorrect conception of multidigit numbers at the beginning of their content (Thanheiser, 2009a) and methods (Thanheiser, 2010, 2014) courses.

PSTs and teachers with an incorrect conception of multidigit numbers struggle to explain the mathematics embedded in standard algorithms (Ball, Hill, & Bass, 2005; Ma, 1999; Thanheiser, 2009a, 2012), such as regrouping, which is often termed *carrying* or *borrowing* in the PSTs' language. Simply informing PSTs about the 10 to 1 relationship between adjacent place values is not sufficient to build conceptual understanding of the structure of the base ten number system (Thanheiser, 2009b, 2014) or to make sense of commonly used multidigit algorithms. Ma's (1999) study of in-service teachers revealed that U.S. teachers were proficient in performing multidigit multiplication, but 61% were unable to provide a conceptually based explanation. The teachers focused on how to perform the algorithm, rather than why the algorithm worked. This is consistent with other research on operations that indicates that PSTs or teachers may believe that knowing how to find an answer is equivalent to understanding (Graeber, 1999; Lo et al., 2008). Despite this documented lack of conceptual understanding, little research has been done to support PSTs' development of place value understanding (Thanheiser, 2014). This study focuses attention on developing an understanding of the multiplicative structure of place value by investigating multiplication by the base in an alternate base context (base five).

Prior Research Using Alternate Bases

Alternate bases have been used as a research and educational tool to both uncover and deepen PSTs' understanding of number and operation. Zazkis and Khoury (1993, 1994) used the context of base five decimal fractions (such as 12.34_{five}) to reveal PSTs' conceptions of the underlying place value structure of numbers that would have been masked by procedural use of base ten language and symbols. The study showed that PSTs struggled to maintain a consistent 5 to 1 multiplicative relationship between place values, sometimes shifting to a 10 to 1 or 2 to 1 relationship instead. The base 20 Mayan numeral system has been used as a context to explore shifts in the value of digits when comparing the values of a "one" with 1, 2, and 6 zeros attached at the end (Thanheiser & Rhoads, 2009; Thanheiser, 2014). This study showed that PSTs struggled to make sense of three-digit and larger numerals within an alternate base.

McClain (2003) and Yackel, Underwood, and Elias (2007) used a base eight context in preservice teacher content courses, focusing on number, coordinating units, and addition and subtraction. Both studies found that learning trajectories from prior research with children were informative in creating learning trajectories for PSTs. The base eight context in those two studies provided an opportunity for PSTs to experience a sense-making approach to early arithmetic. This context provided an alternate view from the procedural approach previously experienced by the PSTs. The work of McClain and Yackel, Underwood, and Elias focused on counting and single and multidigit addition and subtraction of whole numbers. In this study, we build on that work and leverage alternate bases in a multiplication setting. Particularly, we address multiplying by the base as a way to highlight the multiplicative structure of place value number systems.

For this study, our primary research question is: How might PSTs build understanding for multiplication by 10_{base} and how does this impact their understanding of place value structure? In answering this, we (1) show that making sense of multiplication by 10_{base} is not a trivial task, (2) provide insight into how one group of PSTs was able to make sense of multiplication by 10_{five}, and (3) discuss shifts in the PSTs' understanding of place value.

Methods

Description of the Course

The multiplication task discussed below was used in the first quarter of a three-term mathematics content sequence designed for preservice K–8 teachers at a large urban university in the Pacific Northwest. The class met twice a week for 10 weeks, with each class 1 hour and 50 minutes in length. The focus of the first course in the sequence was whole number and operation. The second author was the teacher for the class, while the first author

designed the multiplication tasks and played the role of observer during the class sessions.

In the first five weeks of the term, PSTs investigated number and operation in base ten and alternate bases focusing on counting, addition, and subtraction. The alternate base sequence began with an investigation of a fictional base five context called Alphabitia[4] (Bassarear, 2011), in which PSTs invented and compared number systems using base five manipulatives. The goal of this activity was for the PSTs to develop an understanding of a grouping system and a place value system and the difference between those. In the first two weeks of the term the PSTs also explored historic number systems, including the base 20 Mayan system and the base 60 Babylonian system, as well as the alternate base context of time (hours, minutes, seconds) to examine the underlying structure across the systems (grouping by base, identifying place values, exploring the relationship between place values, etc). Then the PSTs investigated addition and subtraction in base ten and other bases in weeks 3 and 4 and began multiplication in base five in week 5. The central base five multiplication task is shown in Figure 6.

Description of the Task Design

The design of classroom mathematical tasks is a cyclic process of creating, trying, and modifying tasks (Liljedahl, Chernoff, & Zazkis, 2007). This study is part of a larger project that focuses on the design of tasks that leverage alternate bases for supporting PSTs in developing understanding of multidigit multiplication strategies. This article focuses on a portion of one cycle, related to the task of multiplying by the base, and provides insight into how the task was modified for a subsequent cycle.

The design of the initial multiplication task was based on the design heuristic of guided reinvention, taken from Realistic Mathematics Education (RME) (Freudenthal, 1991; Gravemeijer & Terwel, 2000). Students begin in a familiar context where they can use informal strategies to make sense of the situation. Then tasks are provided that move students' mathematical activity to more general and formal strategies. "The idea is to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (Gravemeijer & Doorman, 1999, p. 116). By working in base five, the students had the opportunity to reinvent an algorithm for multiplication and develop conceptual knowledge with a degree of ownership. Within this article, we focus on a key stage in the reinvention: making sense of multiplication by the base. In particular, we focus on shifts in PSTs understanding of the structure of place value as they investigated multiplication by the base.

The task was designed to move from simpler to more complex multidigit multiplication problems, with the intention that students would develop more sophisticated strategies based upon the structure of the place value system in order to cope with larger numbers. The multiplication task was designed with the intention of allowing the PSTs to invent their own sense-making solution strategies by placing the PSTs in the nonroutine context of base five rather than base ten. A classroom culture of sharing and explaining ideas was key to the implementation of the task and had been established by the classroom teacher in the prior weeks of class. In the task directions, the PSTs were invited to choose problems to share that would purposefully highlight some aspect of what they learned. The prompt for this task is given in Figure 6. This study

Multiplication Task

Create a poster that demonstrates and explains how to multiply in base 5 (without converting to base 10). Choose two of the following problems to present in your poster, along with an additional problem that highlights something that you find interesting/ challenging/awesome about multiplication in base 5.

Make sure you are showing what you are doing and why. We are interested in the CONCEPT of multiplication. Use the base 5 manipulatives and try to include pictures.

23_{five} x 3_{five}

23_{five} x 10_{five}

42_{five} x 31_{five}

243_{five} x 12_{five}

Figure 6. **The multiplication task given to PSTs.**

4 Alphabitia is similar to the base five investigations in Xmania (Schifter & Fosnot, 1993) and Orpda (Hopkins & Cady, 2007).

focuses on one group of PSTs who chose to investigate the multiplication by 10_{five} subtask.

Description of the Data Collection

The first author observed the two and a half class periods devoted to the multiplication tasks. She took detailed field notes with a focus on the PSTs' strategies and areas of struggle. A videographer captured the work of a focus small group during the first two days of the multiplication study. During the third day, the camera was focused on the whole class discussion rather than the small group. Photographs were taken to capture student work and posters. A brief, written presurvey was collected to provide context for the PSTs' knowledge of the Times Base Rule prior to the start of task (Figure 7).

The small group observed for the study consisted of two women and three men: Alice, Karen, Danny, Lee, and Joe (all names are pseudonyms). One of the five PSTs, Karen, was absent for the second day of the multiplication task.

Description of the Data and Data Analysis

The primary data source was video recordings of small group work within the classroom. In addition, we analyzed responses to a written presurvey (see Figure 7) to determine whether each student demonstrated an awareness of the Times Base Rule in base ten. Rather than focusing on pre- and post-snapshots of knowledge, this study focuses more heavily on the PSTs' learning process, following a recent commendation for more *motion studies of learning* (Thanheiser et al., 2014).

Presurvey

1. Solve the problem 23 x 42 IN TWO DIFFERENT WAYS and explain your reasoning for each step. Include both the how and why for each step, if possible.
2. (a) Show how to solve 24 x 10 and explain your reasoning.

   (b) Show how to solve 24 x 13 and explain your reasoning.

   (c) Are 2(a) and 2(b) the same difficulty level, or is one easier or harder for you? Explain why.

Figure 7. **Prior knowledge about multiplication and Times Base Rule in base ten.**

The focus small group video was analyzed retrospectively with the following steps adapted from Lesh and Lehrer (2000). (1) Each video was watched by the first author, who created a detailed log of events, marking critical events (Maher & Martino, 1996), which provide insight into the generation of student strategies. Critical events included episodes of struggle, discussions of procedures, teacher presses, explanations or justifications within the small group, and comparisons between base five and base ten. For example, one critical event occurred when students misrepresented the number 10_{five} as ten groups rather than 5 groups. (2) Episodes containing critical events were transcribed. (3) The detailed log and transcribed episodes were examined for themes within the base five multiplication data. Themes were identified if they appeared as a common idea or thread linking a number of critical events. Examples of themes include making sense of multiplication by 10_{five}, reliance on procedural knowledge, and comparing base five and base ten. The first of these themes, making sense of multiplication by $10_{five,}$ is the focal theme of this paper. (4) The detailed video log was reanalyzed to create a narrative of all events related to multiplication by 10_{five}, including transcripts of student conversations and analyses of those transcripts. (5) This narrative was analyzed to find key points in the learning sequence, allowing for the identification of both areas of struggle and conjectures for overcoming struggle.

Results and Discussion

The results section is structured in four parts. These parts focus both on how the PSTs made sense of multiplication by the base and on the PSTs' shifting conceptions of place value structure.

1. We provide brief background information on PSTs' prior engagement with multiplication by 10 in base ten.
2. We show evidence that PSTs struggle to provide a conceptual justification for the Times Base Rule in base five. Their struggles demonstrate a lack of understanding of the multiplicative structure of the place value system.
3. We tell the story of how the PSTs built a conceptual justification for the Times Base Rule by shifting toward viewing 10_{five} as a quantity rather than as two concatenated digits.
4. We highlight how both repeated addition and area imagery were used by the PSTs to develop a better understanding of the Times Base Rule and the multiplicative structure of the place value system.

Background About Small Group's Prior Knowledge

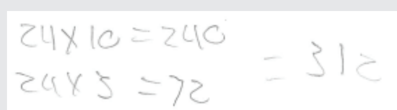
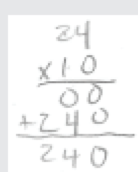
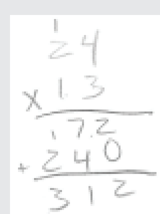
By considering the baseline surveys, we provide a brief snapshot of the PSTs' general awareness of the Times Base Rule in base ten. When comparing 24 x 10 and 24 x 13, three PSTs in the group (Karen, Lee, and Joe) appeared to be aware of the Times Base Rule. The remaining two PSTs (Danny and Alice) did not use the Times Base Rule when comparing the difficulty of the problems. (See Table 1 for the PSTs' responses.) Because several of the PSTs already used the Times Base Rule in base ten, they were primed to notice the same rule occurring during the base five task.

The small group's initial conceptions were similar to the whole class, where 12 of the 20 PSTs used the Times Base Rule and the other eight did not. While the task prompt for the baseline data included a request for explaining their reasoning, the PSTs did not show evidence that they could explain why the Times Base Rule worked. If any explanation was attempted, it tended to involve statements of the following form: 24 x 10 = 240 "because we can move the 0 from the 10 to the 24," using the Times Base Rule as the justification rather than justifying the Times Base Rule.

Struggling to Justify *Times Base Rule* (Procedural Justifications)

During the two days of small group work, the PSTs were asked to attempt four specific base five problems (Figure 5), and they were invited to try out multiplying other numbers to investigate their ideas. Despite teacher encouragement for sense making and an opportunity for inventing algorithms, the small group's strategies initially relied heavily on the standard U.S. algorithm for multiplication, which the PSTs borrowed from base ten and

Table 1
Analysis of each student's level of awareness of a Times Base Rule

PST	Use of Times Base Rule	Evidence, when comparing 24 x 10 to 24 x 13.
Karen	Uses Times Base Rule	"When multiplying a number by 1, 10, or 100 . . . simply add as many or as little zeros as the 1, 10, 100 . . . has." (ellipses were included by the student)
Lee	Uses Times Base Rule	"[24 x 10] is easier because it is a multiple of ten number. I just take the zero from ten and put it to the right of 24 for shortcut math in my head."
Joe	Uses Times Base Rule	 "You can break multiplication down to easier problems. I broke it into base 10 then the remainder 3, then added." "[24 x 10] is easier [than 24 x 13]. When big #'s are multiplied by more than 10 it is hard for me and I have to break it down to groups of 10."
Alice	No evidence of use of Times Base Rule	"They are similar for me. If you think of it as groups, it is the same."
Danny	No evidence of use of Times Base Rule	a)  b)  "[24 x 10] is less difficult than [24 x 13] as it is multiplied by 10, [24 x 10] makes for an easier process and the problem does not include any carrying."

modified to work in base five. (An example of the standard U.S. algorithm for multiplication was given in Figure 5 and its modification for base five is given in Figure 8a.)

The group's first solution for 23_{five} x 10_{five} relied entirely on the standard algorithm, which was written out to include a row of zeros, followed by 230, and summed to 230 (Figure 8b). The teacher pressed the PSTs to make sense of their solution. The following episode demonstrates the PSTs' reliance on the standard algorithm to justify why the Times Base Rule works for this example.

Teacher: So why can we just add a zero to the end?

Joe: Because if you do it the standard algorithm way it works and that's why I do that.

Karen: When you multiply by 1 you always know it's going to be the same number. So when it's a ten or hundred or a thousand[5] you add as many zeros as are behind that number.

Teacher: Why does that work? Why do you always add a zero there?

Joe: Because you put a one, a ones column there, and the zeros are just place holders.

The PSTs' explanations treat the algorithm as the source of justification. The standard multiplication algorithm treats each digit in a multidigit problem as a single digit independent of place value. The efficiency of the algorithm allows its users to solve large problems knowing only single digit facts. Perhaps it is this aspect of the algorithm that invites the PSTs to see the 0 and 1 in 10_{five} as two separate issues to explain, rather than looking at multiplication by 10_{five} as a number of copies of a specified number. This can be seen in Karen's statement above about multiplying by 1 causing a number to remain the same. Continuing the conversation, Karen deals with the zero separately.

Karen: When you multiply things by zero it's zero. So I think that's why we're getting that whole "place holder thing" [student gestures air quotes]. 'Cause it's just adding zeros. When you go down the line you have to add more zeros anyway. [She makes hand gestures as if she's writing out the algorithm, pointing at the zeros.]

While the group has developed a type of procedural explanation that relies on the standard algorithm, they continue to express discomfort with the explanation. We see evidence in the conversation that follows Karen's previous explanation.

(a) $\begin{array}{r} 1 \\ 42_{\text{five}} \\ \underline{\text{x } 23}_{\text{five}} \\ 231 \\ \underline{+134} \\ 2121_{\text{five}} \end{array}$	In order to modify the standard algorithm to base five, PSTs regroup whenever they have 5 or more of something. For example, 3 ones x 2 ones = 6 units, which is regrouped as 1 long 1 ones, with the 1 long "carried" above the 4 and added to the product of 3 units x 4 longs. This results in thirteen longs, which is, in turn, regrouped to 2 flats 3 longs. It is unknown whether the PSTs consider the values as units, longs, and flats, or merely concatenated digits.
(b) $\begin{array}{r} 23_{\text{five}} \\ \underline{\text{x } 10}_{\text{five}} \\ 00 \\ \underline{+23} \\ 230_{\text{five}} \end{array}$	The steps of the standard algorithm are dutifully followed, resulting in a row of 0s when 0 is multiplied by 23_{five}. In the second step, the 1 in 10_{five} is multiplied by 23_{five}, resulting in a 23 shifted one place to the left. The 00 is added to the 23 to arrive at 230_{five}. The separate steps of the algorithm were used as a justification for why multiplication by 10_{five} results in adding a zero to the end of a number.

Figure 8. **The Standard U.S. Algorithm for multiplication in base ten and modified to base five.**

5 Karen has incorrect use of base ten language in a base five context.

Danny: Is there another way to describe it, though? The one copies the number, the zeros just add. It's really easy to think of. And it works in base five.

Karen: You know if you just say it, a kid will be like, Oh that's what I have to do . . . but why? It's hard to explain.

Danny: See I can't even remember, I can't explain why in regular base ten.

We see students looking back to their base ten knowledge and actively making connections between the base ten and base five systems. What they know in one system, they may import to the other system (e.g., standard algorithm), while what they don't know (e.g., explanations) may become more apparent to both the teacher/researcher and the PSTs. Danny's acknowledgment that he cannot explain why the Times Base Rule works in base ten may demonstrate that he is reflecting on his own conceptual knowledge gaps as a result of investigating within base five.

Shifting from 10_{five} as Two Distinct Digits to 10_{five} as a Quantity

After a teacher suggestion that the group tries to unpack the Times Base Rule to make it more accessible to children, Karen returns to the group's bag of manipulatives and takes out 2 longs, 3 units, and 1 long, creating 23_{five} and 10_{five}. While Karen's selection of manipulatives (Figure 9a) represents both numbers in the base five problem, she struggles to use accurate language in describing the numbers.

Karen: If it's twenty times ten,[6] there's ten sets of twenty-three.

Joe: Yeah

Karen: This is going to take a while.

Karen's use of base ten language to talk about base five is problematic. Multiple group members initially make the mistake of gathering ten [base ten] copies of 23_{five} instead of gathering five (10_{five}) copies.

The small group works through this confusion by inventing a strategy based on repeated addition, which Danny voices below (visual provided in Figure 9b.)

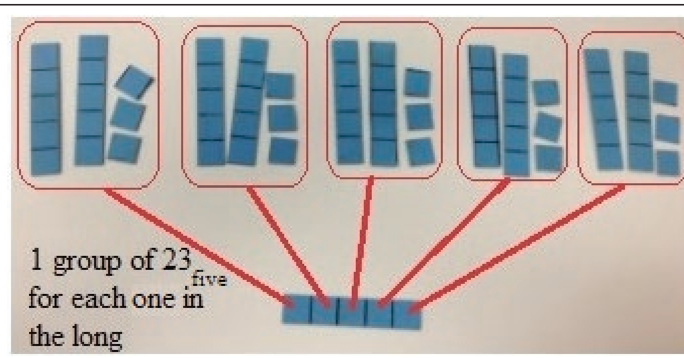
Danny: If we could line this [pointing to 1 long] up with the groups we are using, there would only be one, two, three, four, five groups. [As he talks he points to one unit at a time on the long, until he gets to the fifth unit.]

Joe: Of two longs and three units.

Joe: Yeah, I like how you are touching it. [Referring to Danny counting out on the long manipulative to find the number of groups of 23_{five}.]

The process of touch counting with the long (10_{five}) manipulative allows the group to unpack the meaning

(a) Karen represents 23_{five} with 2 longs 3 ones, and she represents 10_{five} with 1 long. Verbally, she misnames the 10_{five} as ten.



(b) Danny associates each single square in the long with 1 group of 2 longs 3 ones, allowing him to make explicit use of both numbers in the product.

Figure 9. **Using manipulatives to visualize 23_{five} x 10_{five}.**

6 Karen refers to 20_{five} x 10_{five} but uses base ten language inappropriately, calling 20_{five} twenty and 10_{five} ten.

of 10_{five} both as 1 long and as 5 units. This represents a shift from the previous strategy of treating 10_{five} as two concatenated digits, "1" and "0." The strategy also allows the group to overcome the confusion between 10_{ten} and 10_{five}, differentiating between the meaning in base ten and the meaning in base five.

Multiplication by 10_{five}: Assembling a Conceptual Justification

The small group discussion has shifted from treating 10_{five} as a 1 and a 0 and is now treating 10_{five} as a quantity that can be considered either as 1 long or as 5 units. With this conception in mind, the teacher presses once again for making sense of the Times Base Rule. This time the teacher prompt shifts from asking why a zero is placed at the end of a number and focuses on why the rest of the digits have shifted (without change) to the left.

Teacher: So what happens when you move it over? What does that mean?

Alice: It's holding a place or something. The fact that there's nothing in that column.

Teacher: So what caused it to move over?

Joe: Because we have enough.

Alice: We had enough to go to the next column.

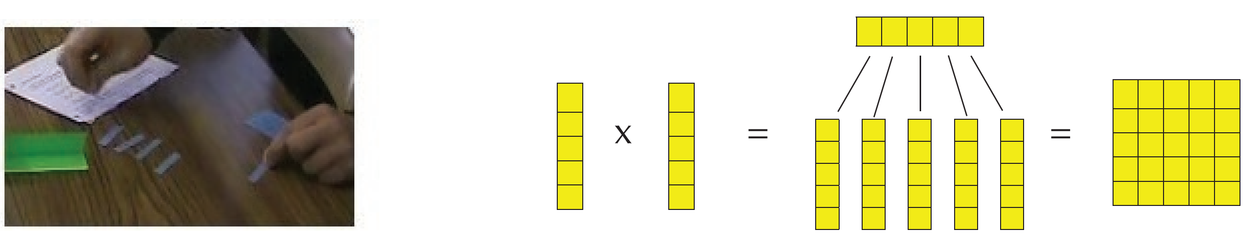
Joe: So in this case you have five groups of this which bumps it over to the next group and you have five groups of this which bumps it over to the next group. Because that's what it takes, the qualification to move over to another group is to completely be full with five units of that, so if you multiply by five you have five units of this which puts you over here [gesturing at an empty table] the same amount of five, just, you know, five units of it, so it goes in the next one and then you have five units of this thing so that goes to the next one.

This new explanation is not based on the standard algorithm, but instead relies on the consistent multiplicative structure of the place value system. The PSTs' language highlights the consistent 5 to 1 multiplicative relationship between adjacent unit types as a result of investigating the Times Base Rule. This can be seen in Joe's language about having five groups and in his explicit attention to the number of items needed to allow for regrouping. Next Joe explains why a long times a long is a flat.

Joe: For every one block here [pointing to each unit on the long] you get one of these [points to a long]. So you get one is one [associates a long for each unit he counts on the original long] two equals two, three equals three, four four, a fifth one gets you up to these [pointing to a flat] so if you were to do that for three [pointing to 3 longs], you'd just get three of these [flats].

The PSTs assemble two key pieces of information to create this conceptual explanation.

1. Multiplication by 10_{five} means repeatedly add a number 5 times.



For each unit he touches in the first long (the multiplier), Joe counts out another long. This results in a collection of 5 longs, which he regroups to 1 flat.

Figure 10. **Joe devises a touch counting strategy for multiplying 10_{five} x 10_{five}.**

2. Adding 5 copies of one unit type allows you to upgrade to the next larger unit type.

While students may have held the two pieces of this idea separately, combining them into a conceptual explanation for the Times Base Rule occurred only after continued teacher press for justification. The conceptual explanation created by the students' touches on the very structure of the base five number system: that each column or place value represents a grouping of 5 times the previous column. This explanation stands in contrast to the earlier procedural explanations that treated the 1 and 0 in 10_{five} as two separate digits. This indicates a shift from discussing concatenated digits toward explaining the multiplicative structure of a place value system.

Multiplication by 10_{five}: Making Generalizations & Comparing Strategies

After using the touch counting method to justify why a long times a long is a flat, the small group begins to generalize this strategy.

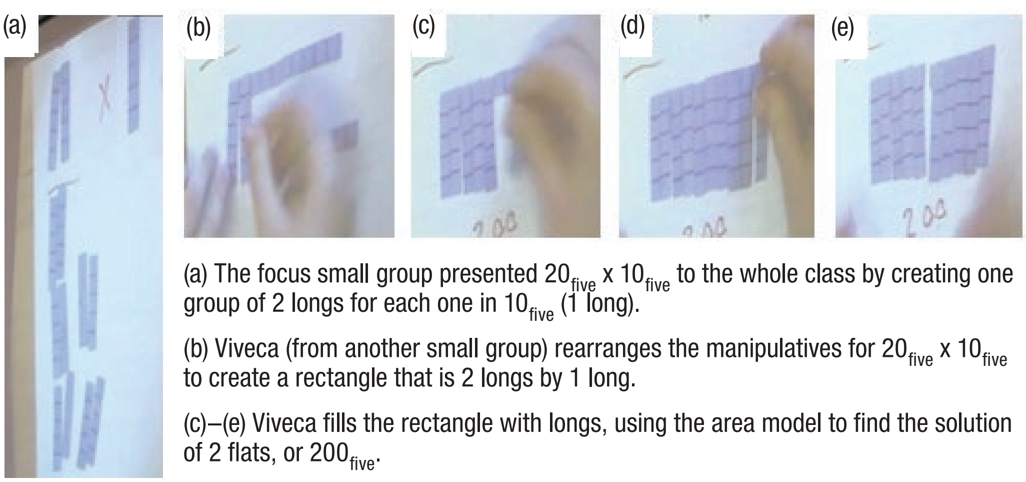
> *Joe:* So you used to have three of these [longs] living is this column and now you have three of these [flats]. So you just move over a column. And if you had these [flats] you'd move over to a different column.

Mathematically, Joe uses the fact that 10_{five} x 10_{five} $=100_{five}$ to conclude that 10_{five} x 30_{five} = 300_{five} and that 10_{five} x 100_{five} = 1000_{five}. Thus, it seems that Joe is explicitly aware that if a base five manipulative (or quantity of matching manipulatives) is multiplied by a long, it upgrades to the next larger type of base five manipulative.

Near the end of the second class period, each small group is asked to share their investigations with the whole class. Members of the focus small group share their exploration of multiplication by 10_{five} at the board, focusing on the touch counting approach for repeated addition using the example 20_{five} x 10_{five} (Figure 11a). Viveca, a member of a different small group, requests permission to rearrange the group's manipulatives. Viveca sets the two factors perpendicular to each other (Figure 11b–e) and creates a rectangular array of longs. The class spontaneously applauds this new area model, and Danny announces to his group: "That was a breakthrough." Having created a justification for the Times Base Rule through their repeated addition model of multiplication, they were primed to make sense of an area based justification for the Times Base Rule. This justification creates a visual model that focuses on the multiplicative structure of the base five system. The number being multiplied by the base (in Figure 11, this is 20_{five}) is treated as one dimension of the rectangle, while the base is treated as the other dimension. The visual model highlights how each long is multiplied by 10_{five} and becomes a flat. The structure of the number system, as well as the area model imagery, continued to be a focus of later class discussions of multidigit multiplication in both base five and base ten.

Overview of the Shifts in Place Value Understanding

When investigating why the Times Base Rule would make sense in base five, the small group initially focused on the separate digits of 1 and 0 in 10_{five}, pointing to the



(a) The focus small group presented 20_{five} x 10_{five} to the whole class by creating one group of 2 longs for each one in 10_{five} (1 long).

(b) Viveca (from another small group) rearranges the manipulatives for 20_{five} x 10_{five} to create a rectangle that is 2 longs by 1 long.

(c)–(e) Viveca fills the rectangle with longs, using the area model to find the solution of 2 flats, or 200_{five}.

Figure 11. **Rearranging the manipulatives to visualize the area model.**

separate ideas that multiplication by 1 causes no change to the other factor, while multiplication by 0 results in a zero. These two facts were assembled haphazardly as a justification for why a zero was placed at the end of a number when multiplying by 10_{five}. An important shift in understanding of the Times Base Rule appears to be related to reconceiving of 10_{five} as a quantity (equivalent to 5 units) rather than as two separate single digits.

Using the language of Thanheiser's framework (2009a) for conceptions of multidigit numbers, the PSTs seem to have shifted from one of the concatenated digits approaches toward a groups of ones and/or reference units approach. Moreover, the PSTs recognized that 10_{five} happens to be the special quantity that results in regrouping. That is, if you add 10_{five} copies of a single item you get one copy of the next larger item. For example, 10_{five} copies of a long results in a flat. The importance of conceiving of 10_{five} as a quantity is consistent with research by Kamii (1986), who notes a similar importance for children when thinking of ten.

Reflections

Modifications to the Task

The data presented in this study represents part of one cycle in the development of a classroom mathematical task (Liljedahl et al., 2007). Each cycle consists of *design, enact, reflect,* and *modify/redesign* phases. The last part informs changes to the task for future cycles. Two key changes will be made in the task for the next cycle. First, rather than giving the task as four subtasks, the subtasks will be given one at a time, to allow for classroom discussions and cross-pollination of ideas between subtasks. While the focal group for this study was able to make sense of multiplication by the base, the other small groups did not devote as much attention to the Times Base Rule and may have missed an opportunity to develop a better understanding of the multiplicative structure of place value systems. By breaking the task into four subtasks, all small groups would be encouraged to unpack the Times Base Rule. The Times Base Rule is an important phenomenon to investigate because it serves as a building block for making sense of multidigit multiplication as well as a tool for making sense of place value structure.

The second overall task change that will be made for the next cycle is a stronger focus on the guided reinvention of a general strategy for multidigit multiplication. (See Figure 6 for the prior task sequence and Figure 12 for the modified task sequence.) The PSTs will be asked to create a general strategy for finding the area of any rectangle in a box of base five rectangles. We anticipate that PSTs will make use of the Times Base Rule and the multiplicative structure of base five when reinventing their own multiplication strategies.

This refined task sequence will be subject to further study and revision in the next iteration of the task design study. Attention will be paid to prompts and investigations that promote PSTs' guided reinvention of multidigit multiplication strategies.

The Role of an Alternate Base

The exploration of base five as a context for number and operation has two intended pedagogical goals. The first goal is to provide an alternate base to compare against base ten. The differences between the bases can serve to draw attention to the meaning of place value and the relationships between columns. The similarities between bases, particularly the Times Base Rule, can draw attention to the multiplicative structure of place value number systems. In alignment with the Common Core Mathemati-

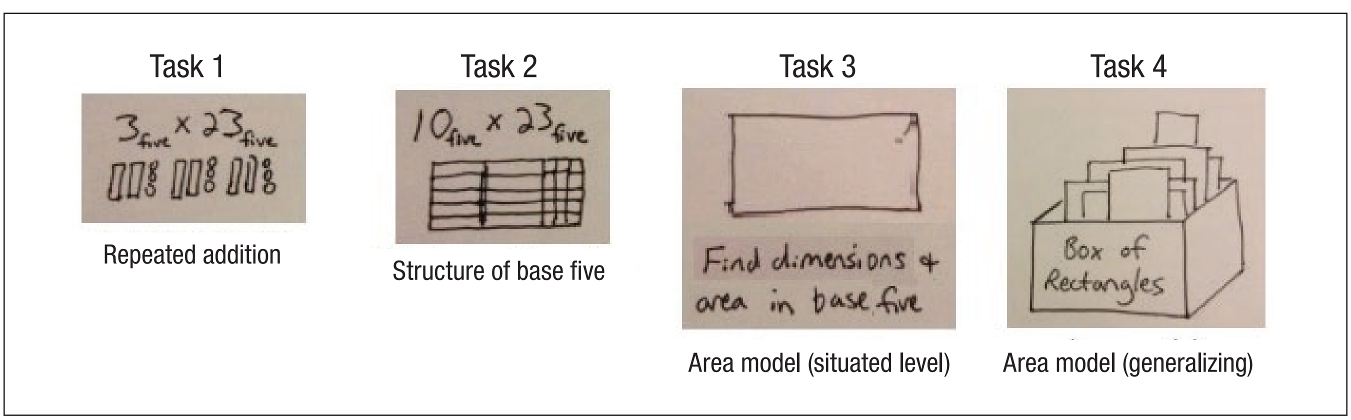


Figure 12. **Modified instructional sequence for multidigit multiplication in base five.**

cal Practice *Look for and make use of structure* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), the structures of base five and base ten become a focal point of class discussion and student reasoning.

The second goal in using base five was to provide PSTs with the type of sense-making learning experience advocated by the current reform movement and by the tenets of RME. This was done by placing the PSTs in a context where they might explore ideas in number and operation without relying on their prior wealth (or burden) of memorized number facts and rules. This opportunity was successful in leading PSTs to focus on sense making and the structure in mathematics, rather than on memorizing and practicing rules. While base five was a convenient base to use for this teaching experiment (it aligned with the course textbook), we imagine that investigating other alternate bases could also serve the purposes of (a) providing a comparison to base ten and (b) providing a context for sense-making experiences. The discussion of which base is best is a discussion we will leave for the readers.

Take-away

While PSTs may be aware that when you multiply by 10 you can just append a zero, they may not be able to justify why this is true. Investigating the Times Base Rule in alternate bases serves to bring forward the question of why this phenomenon occurs generally and what that indicates about the structure of place value number systems. Creating a valid justification for the Times Base Rule is not a trivial task for PSTs, but it is a worthy task for two mathematical reasons. First, making sense of the Times Base Rule helps PSTs develop a conceptual understanding of the multiplicative structure of place value number systems. Second, the Times Base Rule is a fundamental part of making sense of multidigit multiplication strategies, including both student-invented strategies and standardized algorithms.

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