Elementary Teacher Candidates' Use of Number Strings: Creating a Math-Talk Learning Community*

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This article presents the results of an exploratory study detailing 4 teacher candidates' initial implementations of a number string protocol in which they presented sequences of related problems to 3rd graders. We detail how the teacher candidates were taught the components of the protocol in their methods course and describe the math-talk (student-participation) levels that occurred during their 1st number string experience with their students. We coded the lesson transcripts for math-talk levels, which range from teacher-led to student-driven, and provide examples of the number strings and excerpts from the teacher candidates' reflections to illustrate our results. Results indicate that number strings are a supportive structure for beginning teachers as they facilitate math talk.

Key words: Elementary teacher candidates; Math-talk levels; Number strings; Sequences of related problems

With the introduction of the Common Core State Standards for Mathematical Practice, there is renewed momentum for mathematical discussion in classrooms. Teachers are expected to elicit students' reasoning and explanations as well as help students listen and respond to each other's mathematical thinking (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). These processes are difficult to manage, especially for new teachers who often experienced teaching as telling (Smith, 1996; Nicol, 1999): "There is a strong tendency for novice teachers, once they have entered the profession, to revert to their default model of teaching as they were taught" (Borg, 2004, p. 275). Further, teachers may rely on "telling" to avoid the daunting task of making sense of students' thinking (Smith, 1996). Without purposeful planning about how to respond to students' comments in a mathematical discussion, new teachers are often not able to give pedagogically meaningful responses to their students and often ignore unanticipated responses, hindering the meaning-making process (Inoue & Buczynski, 2011).

As teacher educators, one of our goals is to help teacher candidates move beyond telling to eliciting and building on students' thinking. One way we can do this is through modeling a specific routine, during which teacher candidates participate as students in the lesson and we as teacher educators periodically step outside our elementary teacher roles to discuss the elements of the practice. Along with this, we can provide classroom and clinical experiences where teacher candidates are directly involved in rehearsing this particular routine and reflecting on their practice (Grossman et al., 2009). In this article, we present results from an exploratory study on teacher candidates' use of a specific routine or practice—number strings—and explore its potential to support novice teachers' efforts to facilitate mathematics discussions.

Number Strings

Researchers have identified number strings[1] as a discussion routine that could help teachers focus less on *teaching as telling* and more on *exploring* the mathematics and *facilitating* conversations about students' solutions and reasoning (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010). Number strings consist of a series of related problems that highlight a particular mental math strategy or big idea. Each problem in a particular string is written horizontally and is completed one at a time; students mentally work out a solution to the problem and share their methods with the class (DiBrienza & Shevell, 1998; Fosnot & Dolk, 2001; Parrish, 2011). For example, students might first see 3 x 4, then 3 x 8, then 6 x 8, followed by 6 x 16. The underlying strategy here is doubling, although students might also use other mental

1 It may be more appropriate to call this routine *problem strings*; however, we use the term *number strings* to reflect popular terminology. For a more detailed account of number strings and how they differ from other forms of math talk, see numberstrings.com/2014/05/27/whats-in-a-name/.

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math strategies. When conducting the number string routine, teachers are encouraged to elicit and probe student strategies, record these strategies to facilitate clarity, foster an understanding of representations, encourage comparisons, address incorrect answers, and encourage students and their classmates to clarify or justify their strategies, especially in terms of efficiency and utility (Lampert et al., 2010).

There are a few accounts of teachers implementing number strings with their students (e.g., Fosnot & Dolk, 2001; Lampert et al., 2010; Parrish, 2011), and some include a focus on the types of strategies that students develop when given the opportunity (e.g., Murata et al., under review; O'Loughlin, 2007). However, there is little research on how teacher candidates implement the routine and whether those who use them can do so faithfully and maintain higher levels of math talk with elementary students. In the following section, we describe a framework for characterizing levels of math talk and discuss how the elements embedded in the number string routine have the potential to support higher levels of math talk.

Math-Talk Theoretical Framework

Hufferd-Ackles, Fuson, and Sherin (2004) illustrated four main components that contribute to a math-talk learning community: questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning. They describe each of these components within a series of levels, highlighting a trajectory toward effectively incorporating many of the Common Core mathematical practices into teaching. Scoring a zero across all four math-talk components constitutes the lowest level, at which teaching involves telling, asking right/wrong questions, dictating how to solve problems, and making little attempt to elicit student thinking. Math talk at levels 1 and 2 involves a shift in focus from answers and teacher-provided explanations to students providing explanations of their solution strategies. The teacher begins to ask probing questions and involve other students in helping make sense of each other's strategies. As teachers and students shift toward using discourse to "extend one's own thinking as well as the thinking of others" (p. 82), they advance to the highest level within the framework and a more learner-centered mathematics community, a level 3.

Questioning

Teachers may ask students questions in order to clarify terminology, lead students through a method, extend students' thinking, and involve other students in the discussion (Boaler & Brodie, 2004). Further, the style, substance, and quantity of teachers' questions greatly affect the classroom learning environment (Kemmerle, 2013; Hiebert & Wearne, 1993). In low math-talk classrooms, teachers do not leverage questioning; instead, they ask yes/no or quick computation questions (Hufferd-Ackles et al., 2004). Teacher candidates struggle with asking questions in order to learn more about students' thinking; instead, they often ask questions to move students toward the correct answer (Nicol, 1999). When students fail to answer a question or provide an incorrect answer, teacher candidates might ask leading questions or have difficulty responding (Inoue & Buczynski, 2011). Because one of the first steps in the number string routine involves additional questioning, our conjecture was that teacher candidates using the routine would move beyond telling and answer seeking; using the routine, teachers would elicit and encourage students to explain their strategies by using a low-press question (Kazemi & Stipek, 2001), such as "How did you get that?" Teachers' strategy-eliciting exposes students to a diversity of methods and helps them progress toward thinking about when particular methods are more efficient (National Research Council, 2001). Further, part of the number string protocol (see Kazemi, Franke, & Lampert, 2009, for details) involves asking additional low-press, clarifying questions (Chapin, O'Connor, & Anderson, 2009), such as "Where does the 2 come from?" or "How did you decompose the 12?" as well as some high-press questions that require more justification (Kazemi & Stipek, 2001), such as "Why did you add instead of subtract?" A final aspect of number strings that encourages high-press questions and elevates the level of math talk is having students consider connections among mathematical ideas, strategies, and representations (DiBrienza & Shevell, 1998; Kazemi & Stipek, 2001; National Research Council, 2001; Parrish, 2011). Teachers can do this by asking questions such as "How are these two strategies similar or different?" or "How can we change our representation to show the new strategy?"

Explaining Mathematical Thinking

Through questioning, teachers can elevate the class's mathematical discourse by encouraging students to explain their solution strategies. However, teacher candidates often define successful teaching as providing clear explanations (Minor, Onwuegbuzie, Witcher, & James, 2002) and may feel compelled to explain mathematical concepts and procedures in order to meet their perceptions of what teachers do (Parrish, 2011). This practice results in low math-talk discourse (Hufferd-Ackles et al., 2004). As teacher candidates shift toward letting students lead the explanations, they encounter the challenge of attending to and noticing the relevant details of students' mathematical explanations (Jacobs, Lamb, & Philipp, 2010). Moreover, if teacher candidates do not understand

a student's reasoning, they may lose confidence in their own understanding of mathematics (Nicol, 1999). The number string routine encourages teacher candidates to use revoicing and restating, which help students learn to explain more thoroughly (Chapin et al., 2009). When revoicing, teachers repeat or summarize what students say using more precise language and ask students to evaluate whether the statement accurately represents their thoughts. This practice helps students justify and take ownership of their ideas, and "the revoicing move entails a collaborative effort at building and explicating a complex idea" (O'Connor & Michaels, 2007, p. 281). As more students restate (Chapin et al., 2009) or clarify each other's ideas, the class moves to higher levels of math talk (Hufferd-Ackles et al., 2004).

Source of Mathematical Ideas

Many teacher candidates espouse a view that their role is to dispense information (Brookhart & Freeman, 1992), which can lead to low math-talk classrooms where teachers do the explaining and are the source of mathematical ideas (Hufferd-Ackles et al., 2004). Because teacher candidates also value being student centered (Minor et al., 2002), they may compensate by using "show and tell" (Stein, Engle, Smith, & Hughes, 2008)—having multiple students explain their strategies without providing input, without supporting other learners to learn from the explanation, or without attempting to help them make connections among their strategies. In classrooms with higher levels of math talk, teachers use students' ideas to a greater extent; therefore, students' ideas guide mathematics lessons. Additionally, students' more detailed explanations engender comparisons (Hufferd-Ackles et al., 2004), especially when they are given time to grapple with a variety of strategies (Murata et al., under review).

The number string routine can help teacher candidates transition to building on students' thinking and to giving students more control over the conversation. One important element of the number string routine is accepting all student answers, even incorrect ones, when first posing a problem. This encourages students to be the source of the ideas rather than the teacher. At the same time, teachers can use the incorrect answers as an opportunity for students "to confront their thinking" (Parrish, 2011). Another integral part of the number string routine, representing students' strategies, ensures that students have access to each other's methods and also provides a means for comparing students' strategies visually (Fosnot & Dolk, 2001). Although teacher candidates may have difficulty using representations to highlight similarities and differences *among* problems (Bofferding, 2012), with the visuals as a guide, students can talk about each other's strategies, resulting in higher math talk. The comparisons are important because making connections among strategies and the underlying mathematics is a key purpose of number strings.

One prevalent model for representing student thinking is the empty number line, which encourages multiple strategies (Klein, Beishuizen, & Treffers, 1998), helps highlight potential shortcuts (Bobis, 2007), and can improve students' procedural competence (Klein et al., 1998)—all of which are goals of number strings. Empty number lines best illustrate jumping strategies (adding on or finding a difference) or showing addition and subtraction together, such as when students round one number and compensate for the change at the end (Fosnot & Dolk, 2001). Some strategies, however, such as splitting, do not lend themselves to illustration via the empty number line (Bobis, 2007). Splitting involves decomposing numbers according to their place values and adding the parts in stages. In these situations, the use of branching or recomposing notation (drawing lines from the number to show how it is broken up) is more appropriate (Fosnot & Dolk, 2001). (An example of an empty number line and the use of branching notation can be found in Table 1 for String B.)

Responsibility for Learning

When teachers are the source of mathematical ideas—that is, their classrooms are at a lower level of math talk—they also hold responsibility for the learning of others (Hufferd-Ackles et al., 2004). In an effort to encourage students to take a more active role in conversations, work together to make sense of the mathematics, and feel more comfortable with sharing, teachers can use a think-pair-share strategy or partner talk (Chapin, O'Connor, & Andersen, 2009; Reinhart, 2000). Aside from making all students responsible for solving the problems, talking with a partner could improve students' explanations, making them more confident in sharing. Further, as students participate in a classroom where their thinking is valued, they will begin to take a more active role in their learning (Hufferd-Ackles et al., 2004).

Based on a survey of 134 teacher candidates, Minor, Onwuegbuzie, Witcher, and James (2002) found that 28.4% believed the role of the teacher is to transmit information to students while only 12.7% held the progressive belief that students should be actively engaged in solving learner-generated problems. Experiences with number strings could support progressive beliefs and higher levels of math talk because an important element of the number string routine—applying reasoning (Chapin et al., 2009)—supports a shift in responsibility from teacher to student by encouraging students to help their classmates under-

stand the mathematics, even agreeing or disagreeing with other students' reasoning when necessary.

Research Questions

Based on the elements that are part of the number string protocol, we hypothesized that utilizing them could help teacher candidates facilitate higher levels of math talk. However, even though the number string protocol includes steps for addressing different levels of questions, focusing on explanations, and letting students take a more central role, novice teacher candidates might struggle to enact all parts of the protocol. Therefore, it is important to know what parts of the protocol teacher candidates do enact and to what extent they can support higher levels of math talk, leading us to pose the following research questions:

1. To what extent do elementary teacher candidates enact the elements of the number string protocol in an initial lesson?

2. What levels of math talk are they able to facilitate across the four categories?

3. What are their successes and struggles in enacting number strings meaningfully in terms of the levels of a math-talk learning community?

Methods

Participants and Setting

The participants were 24 master's degree students taking a mathematics methods course as part of a 1-year, full-time elementary education teacher training program in California. The teacher candidates assisted in placement classrooms during the morning and took classes at the university in the afternoon. In their placement classrooms, the teacher candidates worked with individual students or small groups around mathematics and taught several mathematics lessons to the whole class. The mathematics methods course met once per week for 3 hours. We selected four teacher candidates (Zoe, Lana, Becca, and Stephanie—all names are pseudonyms) for our focus because they all worked in third grade, English-speaking classrooms. According to Zoe and Stephanie, students in their classrooms often did mental math problems where they could write their work on white boards and compare answers, but only students in Zoe's classroom regularly *shared* their strategies. Students in Lana and Becca's classrooms did not regularly participate in mental math activities as a class. Three of the four teacher candidates designed and used the same number string. Eliminating variation helped us focus on the implementation of the number string in the teacher candidates' placement classrooms as well as the math-talk levels that occurred during the number strings.

Modeling Number Strings in Methods Class

A major goal of this methods class was to prepare the teacher candidates for leading a student-centered series of mathematics lessons with a focus on eliciting student thinking, and we felt the number string routine had the potential to help teacher candidates elicit and build on student thinking in an accessible way. At the beginning of each class session throughout January and February, the instructors conducted a number string (see **Appendix A** for examples) with the teacher candidates to familiarize them with the protocol, as described by Kazemi, Franke, and Lampert (2009). The number string problems were selected to be hard enough to give the teacher candidates an authentic number string experience. In Table 1, we present excerpts from two example number strings conducted with the teacher candidates that highlight how the instructors incorporated elements of the number string protocol.

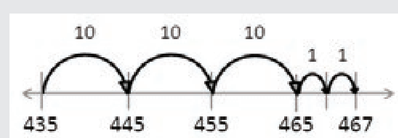
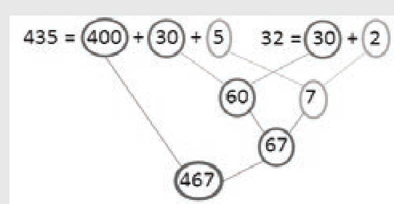
As seen in the excerpt from String B (see Table 1), the instructors introduced and talked about open number lines and branching notation as key representations to use to illustrate students' strategies. In String A and other multiplicative examples, they highlighted the use of arrays.

Data Sources

After participating in six number strings, during which the instructors explicitly discussed the components of the routine, the teacher candidates designed and eventually taught a number string in their practicum placement. For the assignment, they had to (1) identify the big idea for their string, (2) develop or find a number string, (3) anticipate students' strategies (both correct and emerging), (4) identify possible representations they could use to illustrate students' strategies, and (5) list possible questions they could use to facilitate the discussion around the problems. The teacher candidates had read sections of Sherry Parrish's book, *Number Talks: Helping Children Build Mental Math and Computation Strategies (Grades K–5)*; however, most candidates designed their strings based on the examples from class and the level of their students in their practicum placements. They worked together in grade-level teams to design strings appropriate for their students.

Table 1

Class Number String Descriptions (I = instructor, TC = teacher candidate)

Excerpts from Demos		Instructor's role
String A: *Instructor writes 15 x 18 on the board.*		Writes problem horizontally.
I:	Think about how you might mentally find the product.	Invites participation.
TC:	First, I knew 10 x 18 is 180. Then I knew 5 x 18 was half that, so 90. Then I added 180 + 90 to get 270.	*Instructor writes equations on the board.*
I:	Is there another way to think about the problem?	Elicits multiple strategies.
TC:	I knew that 15 x 20 is 300, and that is two too many fifteens, so I subtracted 30 from 300 to get 270.	*Instructor writes equations on the board.*
I:	It's wonderful how many different ways there are of thinking about this problem.	Places an emphasis on the celebration of diverse thinking.
TC:	I cut 18 in half to get 9 and doubled 15 to get 30. Then I multiplied 9 x 30 to get 270.	*Instructor writes equations on the board.*
I:	That is an interesting approach. Will you please tell me more about why you did that?	Demonstrates how to handle unclear strategies by probing.
TC:	[*Repeats statement without adding anything.*]	
I:	Who can use their own words to explain the strategy?	Invites others to restate.
Two more candidates explain the doubling and halving strategy using different words.		
I:	Thank you to everyone for helping us all understand this strategy. At first I wasn't sure about the strategy, but our discussion has helped clarify things more for me.	Emphasizes class participation.
TC:	[*Pretending to be a student*] I got 140 because 10 x 10 is 100 and 8 x 5 is 40.	
I:	Thank you for sharing. [*The instructor drew an array to illustrate the components of the product and show the missing parts before writing the next problem: 18 x 27.*]	Accepts incorrect answers and uses representations to illustrate strategies.
String B: *Instructor writes 435 + 32 on the board.*		Writes problem horizontally.
TC:	I made three jumps of ten and then two jumps of one.	Shows the jumping strategy using an empty number line. 
TC:	First, I added 30 + 30, then 5 + 2. I took that 67 and added it to 400 and got 467.	Shows the recomposing strategy using branching lines. 

Once candidates developed their number strings, the instructors provided feedback on them. The most common feedback was to begin the string with an easier problem—selection of problems that were too difficult came perhaps from the fact that the number strings done in the methods class were designed for the teacher candidates. Other feedback included ideas on connecting possible student strategies as well as using visual representations to capture possible student thinking. In their written reflections, teacher candidates stated that this feedback, especially suggestions for the first problem of the string, was very helpful. Each candidate then taught the number string in her placement classroom and submitted a video of her enacted number string and a reflection that described (1) the actual strategies students used during the number string, (2) any changes they made to the problems as the string progressed, (3) descriptions of their experiences using representations to highlight student thinking, and (4) a discussion of changes they would suggest or insights they had that would be helpful to others using their number string.

Data Analysis

The second author transcribed each number string video. After dividing each lesson video transcript into chunks consisting of a problem introduction or a strategy request plus a response, we used the descriptions of the math-talk levels (see Hufferd-Ackles et al., 2004) to code the level of questioning, explaining, source, and responsibility evident in each exchange. It should be noted that the teacher candidates were not familiar with the math-talk levels; these were used for research purposes only. As we coded the first participant, we found that some responses were not solidly at one level but fell in between two sequential levels. After discussion, we marked these as transitioning between the two levels to capture these nuances (see Table B1 in **Appendix B** for descriptions of the levels). We individually coded the remaining participants and had 77%–81% agreement across the four categories. Next, we discussed the discrepancies until we reached consensus (see Table 2 for an example of our coding).

The teacher candidates' planning notes provided insight into the big idea for their strings, and their reflections were coded for references to the four math-talk categories as well as the elements of the number string protocol, which enabled us to triangulate the results of our coding with the teacher candidates' thoughts about their number strings.

Results

Enacting Number Strings

Lana, Becca, and Stephanie used the problems 4 + 7 and 44 + 7 to encourage students to break apart numbers, regroup them, and notice similarities in the ones places.[2] Zoe, at her cooperating teacher's request, taught a different string and emphasized a common difference strategy using 25 – 3, 25 – 13, 26 – 14, and 28 – 16. All four candidates successfully focused on students' strategies rather than just their answers and were able to use a variety of representations to make the strategies visible to the students. Although the teacher candidates facilitated some conversations around making connections among strategies, highlighting the big idea of the number string was more challenging. Table 3 provides a summary of number string elements that the candidates enacted.

Math-Talk Levels Across Categories

Overall, the number string routine provided a helpful structure for all four teacher candidates to progress beyond a level 0 teacher-focused community toward a level 1 student-focused math-talk community. Their average percentage of math talk for each level of the four categories is shown in Table 4. The prevalence of level 1 codes for *questioning* reflects the candidates' emphasis on asking students about how they solved the problems; if students had asked each other more questions, the level for questioning would be higher. Similarly, the high percentage of *explaining* at level 1 occurred because the candidates frequently had to ask several follow-up questions in order to obtain a complete explanation; students did not offer full explanations on their own. The most common level of coding for *source* (1–2) reflects the candidates' efforts to build on students' thinking by representing their strategies in various ways. The most common level of coding for *responsibility* (0–1) occurred because although students were listening to each other's thinking, they were not yet initiating or leading the conversations.

Next we give examples of the candidates' use of the math-talk components and discuss practices that supported or hindered their efforts to move toward higher levels of math talk.

2 None of the three teacher candidates got to their other three planned problems: 49 + 7, 49 + 17, and 56 + 18.

Table 2
Example Codes for a Transcript Excerpt

Transcript for lower levels of math talk	Codes based on Hufferd-Ackles et al. (2004)
[*One student shares a strategy for solving 7 + 4 prior to this excerpt.*]	**Questioning:** Level 1 (Teacher allows student to explain and is interested in the response.)
Stephanie: Did someone do it differently?	
Student: I looked at 7 plus 4 and you could break 4 into 3 plus 1.	**Explaining:** Level 1–2 (Student gives a more complete explanation with some details missing initially.)
Stephanie: Okay, so you knew that 3 plus 1 is 4.	
Student: . . . and then I did 7 plus 3 and then added the 1.	**Source:** Level 1–2 (Teacher expands on student's explanation.)
Stephanie: So you knew that 7 plus 3 is?	
Student: 10.	**Responsibility:** Level 0–1 (Teacher elicits multiple strategies, but other than that, students are not involved in talking about strategies.)
Stephanie: So you knew that was 10. And then you added the 1 to get 11?	
Student: Yes.	
Stephanie: Oh, this is exciting—we're solving it in interesting ways.	

Transcript for higher levels of math talk	Codes based on Hufferd-Ackles et al. (2004)
[*After discussing how 28 – 16 and 26 – 14 are related*]	**Questioning:** Level 1 (Teacher allows student to explain and is interested in the response.)
Zoe: Okay, last comment, Dorothy?	
Student: I just wanted to say to Alicia that her answer of 22 could not be possible because you have 28 and you're minusing 16. There is no possible way to get 22.	**Explaining:** Level 2–3 (Student gives an unprompted statement that stakes a position, but teacher does not advance it.)
	Source: Level 3 (Teacher allows students to interrupt and provide their own explanations or interpretations.)
Zoe: We're just off by a one [ten]. We're all really close today.	**Responsibility:** Level 3 (Students agree or disagree with each other without prompting and work together to establish the answers to problems.)

Questioning

All four candidates asked level 1 questions, such as "How did you get your answer?" Notably, all candidates let students' explanations dictate correctness. Also, because they asked, "Who solved the problem a different way?" students began to take responsibility for listening to their peers so as not to repeat strategies.

Low press. All candidates used low-press questions to encourage clarification: "So you started with seven and put four in a group. What did that look like in your head?" (Becca) "You counted by ones. What did you start with?" (Stephanie)

High press. All four candidates used at least one high-press question during their number string. When one student noticed that 12 was the answer to all of the problems, Zoe asked the students to determine the reason for the pattern. However, most high-press questions focused less on patterns and more on clarifying the mathematics in explanations, especially around splitting or decomposing and recomposing numbers. For 7 + 4, one student said that she added 7 + 3 to get 10 and then added 1 more to get 11. Becca asked, "So how did you know to add one more after you added three?"

These high-press questions drew attention to the decompositions and set an expectation for the students to justify their strategies, and the low-press questions encouraged students to explain their strategies in more detail. By pursuing students' strategies in more depth, these teacher candidates communicated to the class that their ideas formed the basis of their learning. Further, Zoe was

Table 3
Number String Elements Enacted by the Teacher Candidates

	Zoe	Lana	Becca	Stephanie
Video length (in minutes)	25	22	18	28
Number of coded chunks	24	12	10	9
Number of strategies elicited	13	10	8	10
Probed student thinking	12	35	24	29
Addressed mistakes	4	4	0	0
Highlighted friendly numbers	1	3	1	2
Made strategy connections	1	1	1	3
Use of representations				
Number lines	3	0*	2	8
Recomposing lines	5	0	0	0
Pictures (e.g., dots)	0	4	1	0
Written equations only	4	5	5	2

* In her reflection, Lana indicated that she used number lines, although they were not visible in the video.

Table 4
Average Percentage of Math Talk Across the Four Teachers

Math Talk Levels	Categories			
	Questioning	Explaining	Source	Responsibility
0	11%	13%	11%	0%
0–1	19%	8%	6%	68%
1	64%	46%	9%	9%
1–2	6%	13%	57%	13%
2	0%	17%	15%	8%
2–3	0%	2%	0%	0%
3	0%	0%	2%	2%

successful in transitioning toward a level 2 as she asked, "Should we ask our friends if they know anything? Do you want to call on a friend?" to encourage peers to ask each other questions.

Challenges. Although low-press questions came naturally to the candidates, they sometimes asked leading questions, limiting students' responses. For example, one student said that she saw 4 + 7 as seven dots and four more. Lana asked her if she counted (assuming this was her method); a more generative question would have been, "What did you do next?" Lana then referred to an illustration of dots on the board, demonstrating, "I'm going to start [counting] from here: 1, 2, 3 . . . ? Or did you say 7, 8, 9, 10, 11?" Asking the student, "How did you count?" would have given the student an opportunity to explain more deeply.

Explaining Mathematical Thinking

Across the candidates' number strings, 46% of their explanations of mathematical thinking were characteristic of level 1 math talk, with 30% of their explanations transitioning to or reaching level 2. As the candidates continued to probe, students' explanations became more detailed with less prompting. Lana explained, "When students explained their thinking as 'just knowing' the answer, I pushed them further by asking them to picture what they saw in their minds and really think about the process they went through to get the answer. . . . When this [more detailed explanation] happened, other students

started to recognize their own processes and the discussion became richer."

Revoicing. The teacher candidates played a strong role in eliciting student thinking and then revoicing students' explanations to encourage more complete answers. The candidates' use of revoicing fell into several categories. In some cases, they repeated students' statements verbatim; this served to make sure all students heard their peers' ideas. Other times, candidates revoiced and named a student's strategy or used more precise mathematical language, which gave legitimacy to the students' explanations:

(Solving 25 – 3)

Student: I pretended that the 2 wasn't there, and I knew that 5 minus 3 equals 2 and then I just added the 2 and it equaled 22.

Zoe: So you kinda used our *canceling* out strategy up there. So you pretended that it was 5 minus 3 and pretended that [2] wasn't there and then you put it back.

Zoe acknowledged that the student was using the canceling strategy that the class had talked about previously for non-regrouping problems, making connections to the class's prior knowledge.

Challenges. Even though all four candidates probed their students' explanations during the number strings, they often over-revoiced, which took away the opportunity for students to do the bulk of the explaining. This shifted the focus away from the students and back to the teacher and moved the discourse community back toward level 0 math talk, where the teacher is considered the source of mathematical ideas and holds the responsibility for student learning. Although the candidates' methods instructors had talked to them about listening to the students, revoicing, asking questions, and not telling too much, the candidates had not discussed what is considered too much prompting or revoicing.

The candidates' over-revoicing may have resulted from the difficulty they thought students experienced in response to their probing. For instance, in her reflection, Zoe wrote, "I asked a student to explain how she knew what to do with the remaining numbers, but she seemed a bit confused by the question, so I moved on." Becca said, "I'm not sure if the students' difficulties explaining their answers result from a lack of experience doing mental math with various strategies, or if they were confused about their own thought processes." Despite students' struggles explaining their mathematical thinking, however, the teacher candidates seemed optimistic about the effect of consistently doing number strings. Becca hypothesized, "I expect that [students] would develop the needed mathematical vocabulary with more frequent math string activities."

Source of Mathematical Ideas

A large percentage (57%) of the segments in the number strings were coded Level 1–2 in the "source" category, making it the most successful component. During their number strings, the teacher candidates elicited multiple strategies. More importantly, the candidates allowed students to use their own strategies and make errors and gave students opportunities to identify their miscalculations rather than telling them the correct answers. This shifted the source of mathematical ideas away from the teacher candidate and onto the students.

Representing student thinking. The candidates legitimized students' strategies by representing them visually. Overall, the teacher candidates used empty number lines, branching/recomposing lines, pictures, and/or equations to illustrate students' strategies, especially when the students' strategies were easy to understand. Zoe used empty number lines when students talked about counting back or making jumps and used recomposing or branching lines whenever students talked about breaking apart a number, even if the student only broke apart one number and then made jumps. When solving 26 – 14, one student suggested subtracting 4 from the 26 and 4 from the 14 and then subtracting the two resulting answers. Zoe wrote the horizontal equations 26 – 4 = 22, 14 – 4 = 10, and 22 – 10 = 12 vertically stacked on the board.

Connections among strategies. After students shared their strategies, the candidates found ways to help students make connections among strategies. For example, when one student explained an important strategy for solving 28 – 16, Zoe capitalized on it to make connections to 26 – 14 and the big idea of the number string:

Student 1: I put 28, and right next to 28 I put a 16, and then minused 6 [from 28] and it equaled 22, and then 16 minus 6 equals 10, and then 22 minus 10 is 12.

Zoe: So [Student 1] noticed that we can do the same thing to both numbers. So 28 minus 6 and then 16 minus 6 to get 10, a friendly number. And then he subtracts the two numbers. What did we do to get from 26 to 28?

Student 2: We just added 2.

Zoe: What did we do to get from 14 to 16?

Student 2: We also added 2.

Zoe: That's kind of what [Student 1] did, only in reverse.

Lana also helped students make connections among the ways to make 10 when adding 4 + 7, depending on if the student started with 7 or 4. According to her reflection, she felt that "this was a great opportunity to compare the two equations and show how making tens in regrouping seemed to be a trend." Later, for 44 + 7, when a student made a multiple of 10 by adding 44 + 4 = 48 and then 48 + 2 = 50, she told the class, "Are you noticing a pattern here? People are wanting to make tens."

In other cases, students made connections to each other's strategies. While working on the problem 4 + 7 and after one student had found 10 as a friendly number (7 + 3) and then added 1 to get 11, Stephanie asked her students:

Stephanie: Did anyone do it differently?

Student: I did it like [the previous student], but I did 6 plus 4 is 10. And then added on.

The student was able to connect his thinking to another student's regarding ways of finding 10 as a friendly number, which further solidified that the students were the ones presenting the mathematical ideas. In her reflection, Stephanie said, "Thirty minutes and two problems into this number string—*after* I had informed the class that it was time for recess—many students were still sharing their ideas." Her statement shows that not only were students able to be the source of mathematical thinking during the number string, but they were also enthusiastic about doing so.

Challenges. Although the teacher candidates helped students *make* some connections among strategies, they had a difficult time *showing* these connections with their representations. For example, writing the equations 26 – 4 = 22, 14 – 4 = 10, and 22 – 10 = 12 vertically stacked on the board did not help Zoe show why 26 – 14 and 22 – 10 have a common difference. If she had written each problem vertically side by side (Bofferding, 2012) or on a number line, she could have highlighted that this student was keeping a common difference (see Figure 1).

Showing connections requires that candidates know multiple ways to represent each strategy and can decide which representation will be most effective for a particular situation, which may require additional experience.

Responsibility for Learning

Overall, 68% of the exchanges were coded at a Level 0–1 for the responsibility component. Zoe had a nice moment, however, near the end of her number string that met the criteria for a Level 3 exchange. She put the problem 28 – 16 up on the board, and students initially presented three different answers (12, 22, and 11). Four students shared their strategies, and the consensus seemed to be that the answer was 12. Then one student raised her hand, and Zoe called on her:

Zoe: Okay, last comment. Dorothy?

Student: I just wanted to say to Alicia that her answer of 22 could not be possible because you have 28 and you're minusing 16—there is no possible way to get 22.

While this could, on the surface, be seen as an affront to Alicia, Dorothy's tone of voice indicated that she was trying to be helpful to her classmate. This depicts a subtle shift in responsibility away from the teacher and onto the

$$\begin{array}{l} 26 - 14 = 12 \\ \underline{-4} \quad \underline{-4} \\ 22 - 10 = 12 \end{array}$$

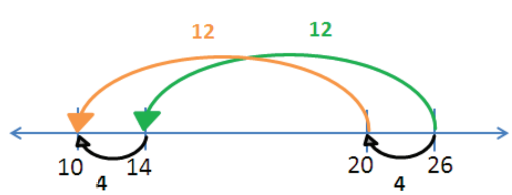


Figure 1. **Two ways to highlight how to maintain a common difference.**

students. They were beginning to care about each other's learning, and they wanted to help each other understand. In her reflection, Stephanie said, "I really enjoyed hearing some of the students—particularly those who normally like to work on their own—respond to their classmates' strategies and compare their solutions to the work on the board." While there is still much room to grow in this area, three of the candidates specifically encouraged collaboration during their number string. Zoe asked, "Do you want to call on a friend?" Becca asked, "Does anyone know where [Student] was going with this?" Stephanie asked, "Can anyone explain [Student's] way?" These types of probes encouraged students to take responsibility for understanding each other's mathematical thinking.

Challenges. This component of a math-talk community was the most challenging for the teacher candidates during their number strings. The prevalence of level 0–1 codes indicates that teacher candidates held onto the notion that they ultimately were responsible for managing and facilitating the mathematical discussion. While there were moments when students directed their comments to each other or referenced a peer's strategy, the bulk of the exchanges were between teacher and student, with the teacher directing the flow of the conversation.

Discussion

Number String Elements and Math-Talk Levels

Overall, the results provide an encouraging account of the power of number strings in helping teacher candidates use more student-centered levels of math talk. Because the number string routine requires that teachers ask students how they solved the problems and record their strategies, teachers naturally asked prompting questions to make sure they understood how to represent students' strategies accurately. This practice helped them be better prepared to deal with unanticipated responses. Rather than ignoring responses (Inoue & Buczynski, 2011), the candidates tried to record students' thinking on the board and asked clarifying questions. Although the candidates did not always feel they truly understood the students' methods, they actively tried to and, by recording the methods, provided an opportunity for other students to try to clarify the methods as well.

Later in their number string conversations, students then began to reference each other's work or make explicit comparisons between their strategies. Rather than the teacher determining correctness (Muis, 2004), the students had a more active role. Another math-talk strategy that likely contributed to students' participation was candidates' specifically asking students to make sense of what their classmates said or to add on to each other's statements. Both experiences provided students with reference points on which to build. All this is evidence that even within one number string, all four candidates were able to move the class toward higher levels of math talk.

The teacher candidates used revoicing naturally (and liberally) throughout their lessons; however, they could increase their use of restating, having other students summarize each other's strategies. This would provide students with additional opportunities to actively engage with their peers' thinking. Although encouraging students to build on each other's thinking did help students start to take responsibility for their learning, the *responsibility* category was consistently coded the lowest in terms of the math-talk levels. This is not surprising because this category is the most dependent on establishing a classroom culture based on student-centered practice. Such extended collaboration is not likely to surface in one initial lesson. However, teacher candidates can encourage collaboration through partner talk and a more targeted focus on helping students make connections among strategies. Three types of comparisons emerged in these cases: the teacher relating students' strategies to the big idea of the string, the teacher comparing two or more students' strategies, and students making comparisons to peers' strategies. Beyond these, though, teacher candidates need to capitalize on the comparisons, especially in terms of having the class analyze the efficiency and utility of their strategies.

Teacher candidates can also encourage students to take responsibility by providing more opportunities for students to agree and disagree by addressing students' mistakes. Candidates often had students explain their strategies in the order that they volunteered their answers. If the second student they called on had an incorrect answer, it was often not probed because the correct answer was already discussed; therefore, two candidates did not address any mistakes. Candidates could have students who presented incorrect answers describe their solution processes, even though they were incorrect, to give the rest of the class a chance to disagree; alternatively, they could point out that the class got several possible answers and ask the students to agree or disagree with one of the answers and explain why. Having students think critically about the answers and strategies would help them transition to taking more responsibility for the conversation.

Common Core Mathematical Practices

In terms of the Common Core State Standards for Mathematical Practice (National Governors Association Center

for Best Practices & Council of Chief State School Officers, 2010), the results of this study provide some insights into ways to help students construct viable arguments and critique the reasoning of others (practice #3). One key practice that all teacher candidates used was recording students' strategies on the board using equations, number lines, and branching notation, which teaches students how to use appropriate tools strategically (practice #5) and how to model with mathematics (practice #4).

Using the Number String Protocol in Methods Classes

The ease with which the teacher candidates enacted the number string protocol suggests that this type of lesson is a powerful first step in learning to facilitate productive mathematical discussions with students. Well-thought-out protocols, such as the number string protocol, can provide a structure that helps beginning teachers incorporate more student-centered practices into their teaching.

However, the number string structure does not guarantee that all of the discourse will be rich. Lana in particular struggled to move away from the role of teacher as the source of mathematical ideas. Although she successfully moved away from teaching as telling, about half of her questions led students to answer in particular ways (Nicol, 1999). A discussion in methods courses about leading versus open questions could bring these challenges to the surface, especially when combined with rehearsals (Kazemi, Franke, & Lampert, 2009) focused on the use of questioning.

In contrast to prior reports about teacher candidates having difficulty understanding the mathematics content (e.g., Nicol, 1999), the teacher candidates in our study expressed frustration with understanding students' strategies, particularly as it related to representing their strategies. Providing opportunities for teacher candidates to rehearse and represent strategies "on the spot" in methods courses could give them experience representing while others are explaining yet also give them a space where they can stop and ask for suggestions and the class can problem-solve effective representations for various strategies.

Although all candidates were able to elicit multiple strategies from students, the end goal of the number strings was not always clear. Only Zoe specifically moved students toward thinking about the constant difference pattern among problems in their string. One suggestion to improve candidates' and students' attention to the relations among the problems and strategies would be to identify specific questions instructors could incorporate as necessary elements into the number string protocol to promote the higher levels of talk. For example, (1) after students share their strategies for a problem, ask them to describe how their strategy is similar to or different from another student's, (2) after solving a few problems, have students identify how two of the previous problems are similar and different, and (3) at the end of the string (if no one has noticed the pattern in the problems), ask, "How could you use the first problem to help you solve the second?" (Wickett, 2003).

Limitations and Future Work

While this study examines the math-talk levels that occur during four teacher candidates' first number string enactment with their students, there is no comparison data that allows us to look at the math-talk level of these teachers during other kinds of math lessons. We also do not know how their talk compares to average new teachers. A next step would be to not only videotape and code teacher candidates' number string lessons, but also other math lessons they teach to see if there are differences and/or spillovers from the number string protocol. Additionally, it would be interesting to compare the math talk in number strings and math lessons to discussion/talk in other subjects. Are there cross-curricular discussion skills that can help teachers engage their students in meaningful talk across subjects?

Given the success the teacher candidates had implementing number strings, we should continue to explore issues surrounding their use. Future work should also explore when (if at all) leading questions are beneficial, what factors influence the types of questions teachers ask, and explore instructional activities, such as rehearsals, that can help support teachers to move toward more open questions that will enable them to elevate the math talk in their classrooms. Additionally, it would be meaningful to follow teacher candidates into the first 2–3 years of their career to see to what extent they are able to maintain or build on what they have learned from the number string protocol.

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Appendix A: Example Number Strings Used in the Methods Course With the Teacher Candidates

1. Find Friendly Numbers (Decomposing Branching Notation)

   35 + 36

   75 + 36

   37 + 59

   175 + 46

2. Making Jumps to Helpful Numbers (Number Lines)

   1999 – 4

   2002 – 4

   2002 – 14

   1898 – 14

3. Compensating (x 20 and Subtract Groups); Using Place Value (Arrays)

   15 x 20 (*not given, but would help prompt the targeted strategy)

   15 x 18

   24 x 18

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Appendix B: Example Explanations of Codes

Table B1

Explanations of Codes Used in Data Analysis of Teacher Candidates' Lessons

Level	Questioning	% Math talk
0	Teacher asks question for sole purpose of getting answer (no inquiry into students' methods).	11%
0–1	Teacher asks a question that indicates an interest in students' explanations, but follows up by asking (mostly) leading questions instead of allowing students to do most of the explaining.	19%
1	Teacher asks questions about how students got their answers (and why they did what they did). Teacher allows students to explain.	**64%**
1–2	Teacher encourages students to think about questions they have about their peers' explanations.	6%
2	Teacher prompts students to ask each other questions.	0%
2–3	Student initiates questions but does not follow up on the responses.	0%
3	Student initiates questions about classmates' work and probes the responses.	0%
	Explaining Mathematical Thinking	
0	Student provides an answer to a math problem but offers no explanation of his or her thinking.	13%
0–1	Student names the strategy he or she used but provides little detail on what this entailed.	8%
1	Student gives more explanation about the details of the strategy, but teacher does a lot of helping (asks leading questions or adds too much detail when revoicing what the student said).	**46%**
1–2	Student gives a more complete explanation with some imprecise language.	13%
2	Student gives a full explanation; teacher doesn't have to clarify much.	17%
2–3	Student gives an unprompted statement that stakes a position, but teacher does not advance it.	2%
3	Student provides justification as part of an explanation and teacher encourages deeper thinking about strategies.	0%
	Source of Mathematical Ideas	
0	Student only gives answer and doesn't expand upon it.	11%
0–1	Student gives barely more than an answer, but it is evident that he or she is beginning to think about the strategy (instead of the answer only).	6%
1	Student gives an explanation with little response or building on by the teacher.	9%
1–2	Teacher expands on student's explanation by illustrating it with a representation, naming a strategy, or helping the students start to compare; may attempt to address errors but in a vague way. Student is more confident.	**57%**
2	Students are beginning to take charge and explain their thinking to each other. Teacher takes advantage of students' errors as opportunities for learning.	15%
2–3	Teacher begins to use students' ideas for building new understanding.	0%
3	Teacher allows students to interrupt and provide their own explanations or interpretations.	2%

Responsibility for Learning		
0	Teacher affirms students' answers as correct or incorrect (not observed).	0%
0–1	Teacher elicits multiple answers, and there is some sense that students need to listen (because they will have to say whether they have a different or similar answer). Students listen to each other.	**68%**
1	Student references or repeats what another student said but provides little further explanation.	9%
1–2	There is some sense of comparison or analysis between strategies. A student might reference a classmate's strategy and say something such as "I did another jump."	13%
2	Students work together to understand and get the answer at the teacher's prompting.	8%
2–3	Students agree or disagree with each other but need prompting, especially to decide on answers.	0%
3	Students agree or disagree with each other without prompting and work together to establish the answers to problems.	2%

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