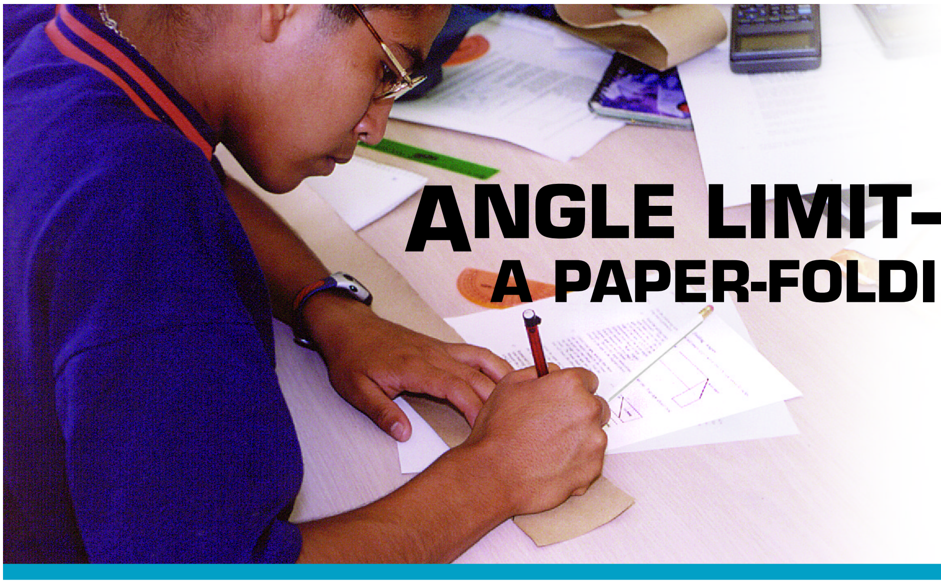

ANGLE LIMIT—A PAPER-FOLDING INVESTIGATION

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Students investigate the limit of angle measures formed by repeatedly folding strips of paper

In an interesting investigation in the 1997 College Preparatory Mathematics (CPM) Educational Program, students investigate the limit of angle measures formed by repeatedly folding a strip of paper. The idea originally came from *Build Your Own Polyhedra* (Hilton and Pederson 1994). What makes the investigation interesting is its blending of geometry, algebra, and limits. The activity works best as a discovery lesson in which students work in pairs to follow the steps, measure the angles, and make a table of their observations. They then analyze their results and try to make a conjecture as to what is happening.

Students begin with a long, rectangular strip of paper—adding-machine tape at least 18 inches long works best—and follow these steps for folding.

1. Fold the paper along a line similar to dotted $\overline{AB}$ shown in **figure 1,** creating an acute angle *TAB*. Make a clear crease along the fold, then unfold and use a pencil and ruler to draw a line segment along the crease. For discussion, a is used for the measure, in degrees, of angle *TAB;* but for the discovery part of the lesson, do not assign a variable—students actually measure the angles that they create. Angle *CAB,* which is the supplement of angle *TAB,* has measure $180 - a$.
2. Fold the paper so that $\overline{AC}$, part of the upper edge of the paper, lies along the first fold, $\overline{AB}$, as shown in **figure 2.** The new fold $\overline{AF}$ bisects the supplementary angle *CAB* to form angles of equal measure, $\angle CAF$ and $\angle BAF$, whose measures are

$$\frac{180-a}{2}=\frac{180}{2}-\frac{a}{2}.$$

Make a clear crease, then unfold the paper and use a pencil and ruler to trace over the fold line.

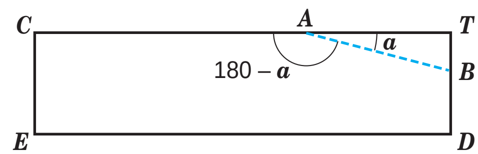

Fig. 1
First fold

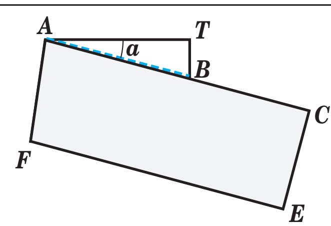

Fig. 2
Second fold

Note that $\angle AFD$ has the same measure as $\angle CAF$, since they are alternate interior angles formed by the transversal between the parallel

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sides of the paper strip. See **figure 3.** The supplement of ∠*AFD,* which is ∠*AFE,* has measure

$$180-\left(\frac{180}{2}-\frac{a}{2}\right)=180-\frac{180}{2}+\frac{a}{2}.$$

Again refrain from giving students information about the angles during the discovery part of the lesson. They should draw these conclusions themselves.

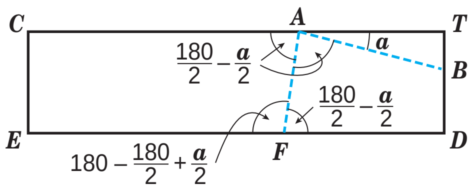

Fig. 3
Analysis of first and second folds

3. Fold so that $\overline{FE}$, part of the lower edge of the paper, lies along the second fold line, $\overline{AF}$. See **figure 4.** The new fold line $\overline{FG}$ bisects the supplementary angle *AFE* to form angles of equal measure *AFG* and *EFG,* whose measures are

$$\frac{180}{2}-\frac{180}{4}+\frac{a}{4}.$$

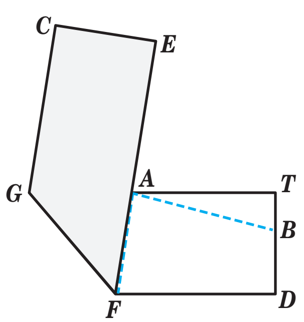

Fig. 4
Third fold

See **figure 5.** Make a clear crease, then unfold and use a pencil and ruler to trace over the fold. Note that ∠*AGF* has the same measure as ∠*EFG,* since they are alternate interior angles. Therefore, the supplement of ∠*AGF,* which is ∠*CGF,* has measure

$$180-\left(\frac{180}{2}-\frac{180}{4}+\frac{a}{4}\right)=180-\frac{180}{2}+\frac{180}{4}-\frac{a}{4}.$$

4. In the discovery phase of the lesson, have students continue to fold the paper strip at least six more times, by bisecting succeeding supplementary

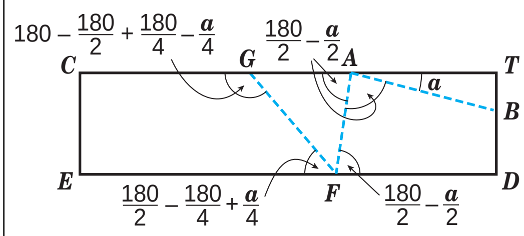

Fig. 5
Analysis of first, second, and third folds

angles, and then measure the angles that they produce. See **figure 6.** They should discover that no matter what the measure of the initial angle is, the measure of the bisected supplementary angles formed by the first fold, second fold, third fold, and so on, gets closer and closer to 60 degrees, although it is sometimes more than 60 degrees and sometimes less than 60 degrees.

The measures approach 60 degrees as the number of folds approaches infinity

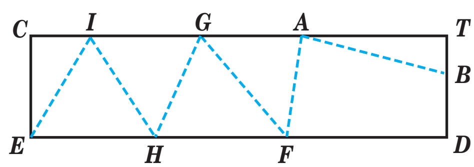

Fig. 6
View of several folds

In a geometry classroom, this activity should stimulate an analysis of the process. The teacher may need to help students begin the analysis by assigning a variable to the measure of the initial angle, *TAB,* but students can then begin to figure out the measures of the other angles produced by the folding and bisecting. As a hint, the teacher might have students begin with a specific value for the initial angle, *TAB,* say, 20 degrees, and analyze the measures of the angles formed by the folds on the basis of that number. If necessary, the teacher should point out that alternate interior angles are formed after a fold (see step 2) and that the next fold bisects the angle that was the supplement of the previous alternate interior angle formed.

Eventually encourage the students, working in pairs or groups of four, to record their results as variables in a table. See **table 1** for one possible way, but students can devise their own way to organize their results.

By using the information that they have collected, students can then prove that the limit of the measure of the angles formed by folding is 60 degrees. First, the angle measure follows the pattern

TABLE 1

Angle Measure Data

Angle Measure	Measure of Supplement to Angle	Measure of Bisected Supplement
a	$180 - a$	$\frac{180}{2} - \frac{a}{2}$
$\frac{180}{2} - \frac{a}{2}$	$180 - \frac{180}{2} + \frac{a}{2}$	$\frac{180}{2} - \frac{180}{4} + \frac{a}{4}$
$\frac{180}{2} - \frac{180}{4} + \frac{a}{4}$	$180 - \frac{180}{2} + \frac{180}{4} - \frac{a}{4}$	$\frac{180}{2} - \frac{180}{4} + \frac{180}{8} - \frac{a}{8}$
$\frac{180}{2} - \frac{180}{4} + \frac{180}{8} - \frac{a}{8}$	$180 - \frac{180}{2} + \frac{180}{4} - \frac{180}{8} + \frac{a}{8}$	$\frac{180}{2} - \frac{180}{4} + \frac{180}{8} - \frac{180}{16} + \frac{a}{16}$

$$\frac{180}{2} - \frac{180}{4} + \frac{180}{8} - \frac{180}{16} + \frac{180}{32} - \frac{180}{64} + \cdots + \frac{(-1)^{n-1}\,180}{2^n} + \frac{(-1)^n a}{2^n},$$

where $n = 1, 2, 3, \ldots$, which, except for the residual term,

$$\frac{(-1)^n a}{2^n},$$

is a geometric series with first term $a_1 = 180/2$ and $r = -1/2$. Since the infinite sum of a geometric series is

$$S = \frac{a_1}{1 - r},$$

the result is

$$S = \frac{\frac{180}{2}}{1 - \left(-\frac{1}{2}\right)} = \frac{\frac{180}{2}}{\frac{3}{2}} = 60.$$

The residual term,

$$\frac{(-1)^n a}{2^n},$$

has a limit of zero as n approaches infinity, that is,

$$\lim_{n \to \infty} \frac{(-1)^n a}{2^n} = 0$$

for all a. Therefore, the measures approach 60 degrees as the number of folds approaches infinity. Actually, after only ten folds, the error is less than a tenth of a degree.

REFERENCES

Hilton, Peter, and Jean Pederson. *Build Your Own Polyhedra.* Menlo Park, Calif.: Addison-Wesley Publishing Co., 1994.

Sallee, Tom, Judith Kysh, Elaine Kasimatis, and Brian Hoey. *Mathematics 4* (*Mathematics Analysis*). Sacramento, Calif.: CPM Educational Program, 2001.

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