GRAPHICAL TRANSFORMATIONS AND CALCULATOR GREETING CARDS

Greeting cards exist in many forms—homemade, store-bought, musical, and Internet cards—just to name a few. With the availability of graphing calculators, the creativity and fun of making greeting cards can be brought into the mathematics classroom to enhance students' understanding of functions. Graphing-calculator greeting cards can take on different characteristics and can be created by students at different mathematics levels. A central objective of the task is to use algebraic equations to create desired graphical designs that, along with strategically placed text, extend a calculator greeting. The algebraic equations used can range in difficulty from linear functions in rectangular coordinates to polar or parametric equations. Algebra students who are learning about linear equations, as well as advanced mathematics students who are working with a broad range of families of functions and relations, can create calculator greeting cards.

Figure 1 shows a greeting card that I created to extend a Valentine's Day sentiment to my students. The parent functions that I used for this heart were $f(x) = |\sin x|$ and $g(x) = |x|$; however, I could have created a heart with linear and quadratic functions. I used algebraic transformations of $f(x)$ and $g(x)$ to form the heart shape in the first quadrant of the graph. Also, I restricted the domains of $f(x)$ and $g(x)$ to clearly define the heart design, and I set the graphing mode to Dot. The details for creating the greetings on either Texas Instruments or Casio calculators are shown in **figure 2.**

Graphing-calculator greeting cards can be created by students at different levels

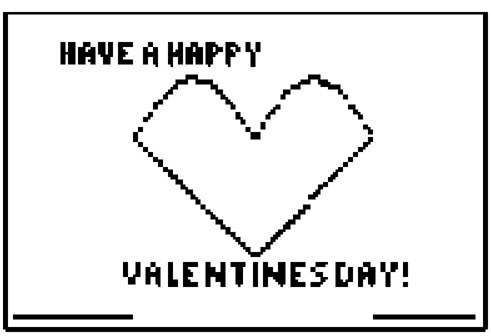

Fig. 1
Calculator greeting card

Creating these greeting cards involves mathematical reasoning, problem solving, and connections across representations. These processes are central to students' development of mathematical thinking (NCTM 1989, 2000). In particular, designing the graphic shapes can help students build a sense of functions, a major goal of the secondary mathematics curriculum (NCTM 1989, 2000). According to Eisenberg (1992, p. 174), "having the skill to visualize the graphs of functions is one important component in having a well-developed sense of functions." However, on the basis of research findings, Eisenberg reported that many students cannot connect the graphs of functions with their algebraic representations. Creating the graphic designs of greeting cards involves students in connecting graphs with corresponding analytic descriptions as they compose algebraic representations to form the desired graphs. This task engages students in developing their sense of functions by applying transformations of functions.

Typically, transformations of functions are taught by varying the coefficients of the algebraic representations and observing the resulting changes in the graph. After observing the resulting changes, students tend to formulate such syntactic descriptions as the following: Adding a number outside the parentheses of $f(x)$ moves the graph vertically. Adding a number inside the parentheses moves the graph horizontally. If the number is greater than zero, it moves to the left; and if less than zero, it moves to the right.

Research indicates that this approach encourages students to memorize rules without understanding their origin, fails to support students' ability to distinguish between similar symbolic forms, and causes student difficulties in generalizing among different families of functions (Borba and

"Sharing Teaching Ideas" offers practical tips on teaching topics related to the secondary school mathematics curriculum. We hope to include classroom-tested approaches that offer new slants on familiar subjects for the beginning and the experienced teacher. Of particular interest are alternative forms of classroom assessment. See the masthead page for details on submitting manuscripts for review.

Making a Calculator Greeting: HAVE A HAPPY VALENTINE'S DAY! Card
Read all directions before beginning.

TI Graphing Calculators

[WINDOW]
Xmin=0
Xmax=12
Xscl=1
Ymin=0
Ymax=10
Yscl=1

[MODE]
Dot

[Y=] setting
Y1=(abs (2 sin(x+π)) + 6)(x≥π)(x≤3π)
Y2=(1.3abs(x-2π) + 2) (x≥π) (x≤3π)
The ≥ and ≤ are under **[2nd] [TEST].**

Home Screen **[2nd] [QUIT]**
Text(4, 10, "HAVE_A_HAPPY")
Text(50, 25, "VALENTINES_DAY")

****Text** is under **[2nd] [DRAW]**.**
To type letters, remember to LOCK the ALPHA key with **[2nd] [A-LOCK].**

To determine the **placement of the words,** use **Text** (row#, column#, . . .)
The calculator screen is divided into rows and columns with corners:

(0,0)	(0,94)
(57,0)	(57,94)

Storing the Greeting Card:
[2nd] DRAW select **STO**
StorePic Pic 1 (or Pic 2, Pic 3, etc.)
****Pic#** is under **[Vars] Picture****

Casio Graphing Calculators (9850)

[MENU] select **GRAPH** icon
[Shift] [F3] (V-Window)
Xmin : 0 max : 12 scale : 1
Ymin : 0 max : 10 scale : 1

Store Window settings.
[F4] (STO) **[F1]** (V-W1) **[EXE]**

[F3] (Type) **[F1]** Y= equations
Y1=abs (2 sin (x+π)) + 6, [π, 3π]
Y2 = 1.3 abs (x-2π) + 2, [π, 3π]
*FOR Abs, **[OPTN] [F5]** (NUM) **[F1]** (Abs).*

[F6] (G <-> T) Toggles to Graph

To type Text on graph screen
[F4] (Sketch) **[F6]** (>) **[F6]** (>) **[F2]** (Text)
For letters, **[SHIFT] [ALPHA]** then type.
Move around the screen with arrows.

Store Picture to avoid losing Text.
[OPTN] [F1] (PICT) **[F1]** (STO) **[F1]** (Pic 1)
To recall the greeting card.
[OPTN] [F1] (PICT) **[F2]** (RCL) **[F1]** (Pic 1)

Create your own calculator greeting card! Here are some possible designs:

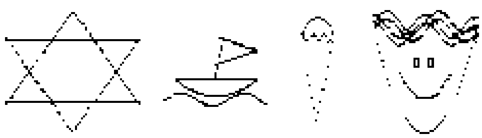

Fig. 2
Directions for greeting cards

Building graphical designs engages students in visualizing the graphs of algebraic equations

Confrey 1996; Confrey 1994). Creating the greeting-card graphs gives students a context that enhances their understanding of transformations by testing their conjectures while they compose algebraic equations to match the desired graphics and while they consider reasons behind the algebraic representations and the resulting transformations.

Ruthven (1990) found that upper secondary mathematics students with access to graphing calculators performed better than students who did not have access to graphing calculators in finding algebraic descriptions of given graphs, both in recognizing and in refining the algebraic forms. These findings concerning the use of graphing calculators and the negative outcomes from students' tendency toward syntactic descriptions of transformations of functions suggests that students need to be engaged in reasoning deductively about the transformations of functions observed on their graphing calculators. The greeting-card task encourages students to develop arguments beyond a syntactic level to explain the connection between parameters of the algebraic forms of functions and the resulting graphs. Building graphical designs engages the students in visualizing the graphs of algebraic equations as they compose the algebraic representations to match the desired graphs. The task also focuses students' attention on the domain and range of functions and the effect that restricting the domain has on the graph. ➔

GREETING CARDS IN THE CLASSROOM

Before introducing the greeting-card task, I had a group of advanced mathematics students use calculators to explore transformations of functions of the forms $y = f(x) + a$, $y = f(x + a)$, $y = af(x)$, and $y = f(ax)$, where a is a real number and $f(x)$ can be expressed with an algebraic equation that is somewhat familiar to the students. These familiar functions include $y = x$, $y = x^2$, $y = x^5$, $y = 2^x$, $y = |x|$, $y = x$, $y = \sin x$, and $y = \log x$. Similarly, with algebra students, I would first engage the students in investigating parameter changes of function families under study, such as changes in the slope and y-intercept for linear equations.

During these explorations, I expected the students to seek explanations for the transformations of the graphs connected with changes in the algebraic representations. As suggested by the previously discussed research, the students' explanations tended toward descriptions of what they observed without focusing on why it was occurring. **Figure 3** and **figure 4** show students' explanations for $y = f(x + a)$ and $y = f(ax)$, respectively. **Figure 3** shows the confusion that could occur with respect to $y = f(x + a)$ when the student's focus is on rules for shifting left or right. The student initially recorded an incorrect rule for this translation of $f(x)$. Students can easily confuse the direction of the translation when they focus only on descriptions of the movement of the graphs. **Figure 4** shows a rule for $y = f(ax)$ developed by a student from observations of transformations on quadratic functions. This rule does not generalize across function families.

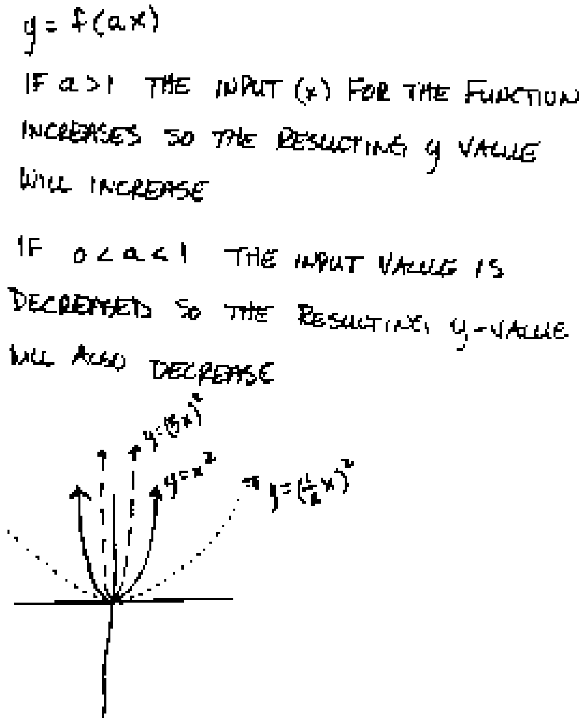

Fig. 4
One student's description of the transformation $y = f(ax)$ on the basis of limited observations

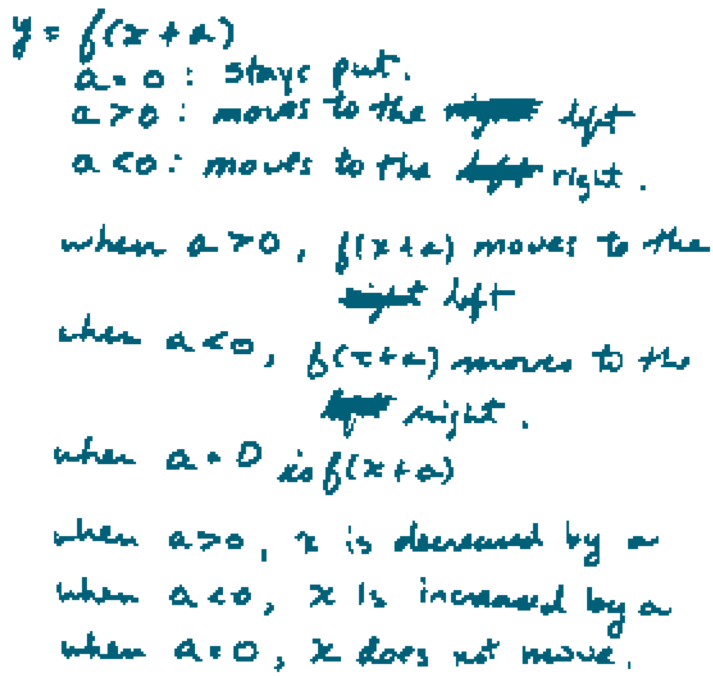

Fig. 3
One student's syntactic description of the horizontal translation $y = f(x + a)$

I then introduced the greeting-card task. After I extended the Valentine's Day greeting to my students on the overhead graphing calculator, we discussed its creation, focusing on functions that could compose the heart. I led the students through the process of creating the greeting card on their own calculators. This process helped them clarify the syntax used by their calculators and better understand the domain restrictions used to define the desired graphs. Some students with TI-85 calculators discovered that they could not type text on their graphs; however, they were able to use PEN under the DRAW menu to "free write" words on the graph. A few students using TI-82 and TI-83 calculators discovered that they could type the text directly on the graph instead of using the Text command. The Casio 9850 users found that they had to type text directly onto their design screens.

As homework, the students then designed and created their own calculator greeting cards and explained how they arrived at their creations. Their card designs involved a variety of function families and showcased their creativity. Typically, students began with a desired graphical design and then composed algebraic representations of functions that would produce the desired graphs. They had to visualize their design as a composition of functions that could be described algebraically, create the algebraic representations, and apply their understanding of transforming the functions as needed to complete their designs.

Figure 2 shows possible designs for the greeting cards. Examples of possible designs should be appropriate for the level of mathematics that is accessible by the students. My students used the given designs to varying degrees. Some students drew partly on the given designs and added their own touch of humor or personality, such as in **figure 5.** The WINDOW settings for **figure 5** were Xmin = 2, Xmax = 10, Xscl = 1, Ymin = 1.5, Ymax = 9, and Yscl = 1. The functions for composing the graph on a Casio 9850 graphing calculator are given below.

[Y=] settings:

$$Y1 = (X - 6)^2 + 2,[3.9, 8.1]$$
$$Y2 = \sqrt{(.4 - (X - 6)^2)} + 3.7$$
$$Y3 = -\sqrt{(.4 - (X - 6)^2)} + 3.7$$
$$Y4 = \sqrt{(.06 - (X - 6.8)^2)} + 5.5$$
$$Y5 = -\sqrt{(.06 - (X - 6.8)^2)} + 5.5$$
$$Y6 = \sqrt{(.06 - (X - 5.2)^2)} + 5.5$$
$$Y7 = -\sqrt{(.06 - (X - 5.2)^2)} + 5.5$$
$$Y8 = .3 \sin(8X) + 6.8,[3.9, 8.1]$$

The suggestions for possible designs gave the students a starting point. On an earlier occasion with a different group of students, I had not suggested possible designs, and the complexity and completeness of the designs were limited.

The students printed their cards by downloading the calculator images to a class computer with Program-Link for Casio calculators and TI-Graph Link for TI calculators. They shared their greeting cards with class members and explained how they created their graphs. The students also revealed that they shared their calculator greetings with family members and friends. Learning that they had shared their mathematics homework with others outside of class was wonderful.

After engaging the students in the greeting-card task, I returned to explaining why the parameter changes in the algebraic representations produced corresponding graphical changes across function families. During the greeting-card task, the students were visualizing algebraic representations to match desired graphs of functions. This activity was complementary to making algebraic transformations and observing the graphical changes. Thinking about transformations of functions within the greeting-card task helped the advanced mathematics students expand their explanations of the transformations beyond their original syntactic descriptions. **Figure 6** shows a student's explanation of the transformation of $y = f(x + a)$, where a is any real number. This explanation reveals spatial and deductive reasoning on the part of the student

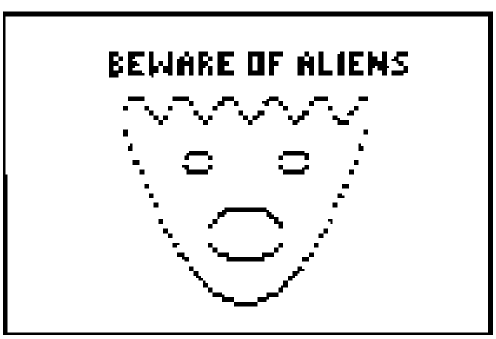

Fig. 5
A student's humorous card involving three different function families

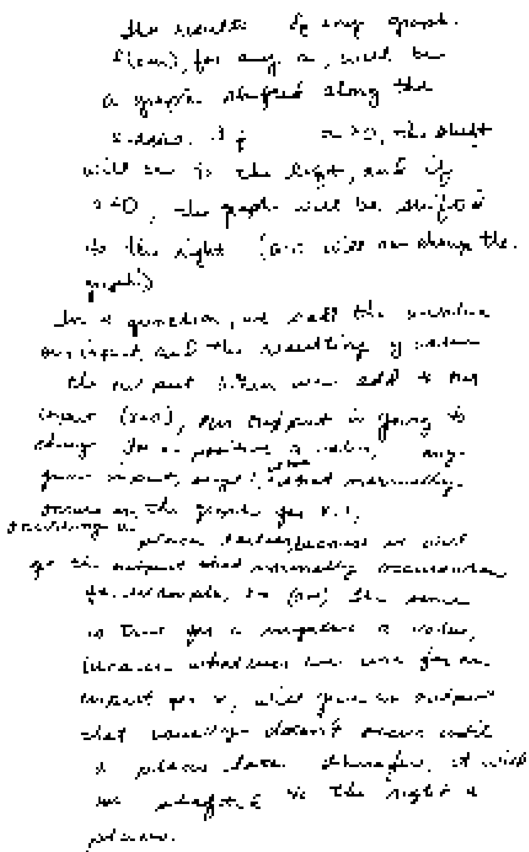

Fig. 6
Description and explanation of the translation $y = f(x + a)$ composed by a student

The suggestions for possible designs gave students a starting point

and is appropriate across function families. The students were better able to offer arguments to explain graphical transformations of functions in connection with changes in the algebraic representations. In particular, the students' explanations tended to address the relationships among the *x*- and *y*-values of the transformed functions.

GREETING CARDS WITH POLAR OR PARAMETRIC EQUATIONS

When investigating polar and parametric representations of functions, some students suggested using these representations to create greeting cards. The use of polar and parametric equations allowed students to easily incorporate such mathematical shapes as circles, ellipses, parabolas, stars, and other geometric figures in their graphical designs.

CONCLUDING REMARKS

Calculator greeting cards are a creative way for students to apply transformations of functions. This context encourages students to make and test conjectures about algebraic equations and the corresponding graphs. This activity is central to developing students' reasoning processes (NCTM 1989, 2000). After making and testing their conjectures, students are better able to develop generalizations about transformations of functions and to explain their generalizations. We should not stop at students' syntactic descriptions that are based on limited observations. We need to question our students to engage them in developing explanations for their generalizations and thus further enhance their reasoning processes and their sense of functions.

BIBLIOGRAPHY

Borba, Marcelo C., and Jere Confrey. "A Student's Construction of Transformations of Functions in a Multiple Representational Environment." *Educational Studies in Mathematics* 31 (October 1996): 319–37.

Confrey, Jere. "Six Approaches to Transformation of Functions Using Multiple Representational Software." In *Proceedings of the XVIII Conference of the International Group for the Psychology of Mathematics Education,* vol. 1, pp. 47–74. Lisbon: Lisbon University, 1994.

Eisenberg, Theodore. "On the Development of a Sense for Functions." In *The Concept of Function: Aspects of Epistemology and Pedagogy,* edited by Ed Dubinsky and Guershon Harel, pp. 153–174. Washington D.C.: Mathematical Association of America, 1992.

Lowther, Martha. "Reader Reflections: Polar valentine." *Mathematics Teacher* 92 (February 1999): 86.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: NCTM, 1989.

———. *Principles and Standards for School Mathematics*. Reston, Va.: NCTM, 2000.

Ruthven, Kenneth. "The Influence of Graphic Calculator Use on Translation from Graphic to Symbolic Forms." *Educational Studies in Mathematics* 21 (October 1990): 431–50.

Vonder Embse, Charles. "Exploring Parametric Transformations of Functions." *Mathematics Teacher* 89 (February 1996): 232–40.

Maria L. Fernández
mariaf@u.arizona.edu
University of Arizona
Tucson, AZ 85721