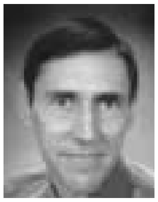

Michael A. Stueben

Soundoff!

A Way of Teaching

THE 1999–2000 YEAR WAS MY BEST TEACHING year ever for student academic achievement. I attribute this success mainly to a teaching philosophy that I have developed during the last five years.

In the first precalculus test of the school year, which dealt with vectors, 11 of 61 students failed the test. I cannot say what happened to each individual student, but I can speak statistically about the group that failed. They did not make the effort to get their individual questions answered each day, and they chose not to review deeply for the test. Also, this test included a new type of question that the students had not seen before. The question seemed to offer two reasonable methods of solution. One method produced the correct answer; the other method produced a preposterous result.

Problem

The path of a motorboat is held perpendicular to the parallel banks of a river. The motorboat at full throttle can move in still water at a constant speed of 9 yps. The stream is 100 yards across and has a constant current of 5 yps. How much time does the motorboat at full throttle take to cross the stream?

People learn from experience

Answer

13.3 seconds

Solution

Consider the right triangle with a hypotenuse of 9 yps and one leg of 5 yps. Then the other leg is $\sqrt{9^2 - 5^2}$ yps, or 7.48 yps. Crossing time is 100 yds/7.48 yps, or 13.3 seconds.

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Incorrect answer

If the 9 yps and the 5 yps are both used as legs of a right triangle, the hypotenuse is 10.2 yps. Some students wrote the answer as 100 yds/10.2 yps, or 9.8 seconds. But this answer is preposterous, because with no current to fight, the boat would need 100 yds/9 yps, or 11.1 seconds to cross.

About a third of the students chose the wrong approach and did not check the reasonableness of their solutions. So how does a teacher encourage students to check their work? Not by asking the students to check and not by giving students an argument showing the usefulness of checking. People do not learn well by listening to advice. People learn from experience. Many of my students will eventually forget almost every mathematics fact that I teach them, but they will likely remember that they were unsuccessful when they did not review for tests and as in the example, when they did not estimate the reasonableness of their "solutions." This benefit is one that our students gain from being challenged.

A claimed benefit of mathematical study is the mental gymnastics that students are required to attempt. Depending on individual experiences, these gymnastics can include—

- analyzing, synthesizing, and drawing inferences;
- applying general rules to specific cases;
- reasoning backward and reasoning by analogy;
- simplifying expressions, expressing data in different forms, and describing relationships with equations;
- recognizing attributes, properties, implications, fallacies, shifting domains, necessary and sufficient conditions, ambiguous or indeterminate conditions, extraneous information, equivalent forms, tautologies, and contradictions; and
- making conjectures and constructing proofs.

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Mathematics education can also be described by the psychological habits that students are encouraged to form, which can be the following habits of the mind:

- Attention to details and subtle distinctions
- Concentration over long periods
- Checking for errors, misinterpretations, and bogus solutions
- Commitment to mastering technique
- Self-reflection about patterns of error

Practicing these habits is, in my opinion, the greatest value of mathematics education. These habits will be useful to any student, no matter which field he or she pursues. But these habits do not come naturally to most students. They must be instilled by the teacher.

If skill and memorization are not demanded early, then later conceptual treatments will become superficial, because the students are often confused about the details. Most lectures are strategy talks. If the teacher writes every necessary detail on the board, then the strategies are obscured. Weak students have difficulty following a mathematics lecture because their skills are too weak to supply details. And when a teacher or classmate sits down with a weak student on a one-to-one basis, discussing details becomes so much of the explanation that the student loses track of the guiding strategy. Consequently, teacher emphasis on skill and memorization are vital to save the weak students.

Many committed mathematics teachers have their students do library research work, give group presentations, take collaborative tests, master mathematical software, experience discovery lessons, perform lab experiments, and write in their journals about problem solving. These progressive teachers point out that years from now, their students may not remember the laws of logarithms, but they will probably remember some of these experiences. These activities are more personally meaningful than working drill problems. These activities are more engaging to students than the traditional experience.

I say, yes and no. I offer only a few of those kinds of experiences in my top classes, because they seem to give the student back too little for the time that they take. The following is an example of the kind of experience that, in my opinion, more efficiently promotes intellectual and emotional maturity.

Last year, the day after I taught my students the rules for taking the derivatives of ln x and sin x, I gave the following two questions on a quiz:

1. Given $y = \sin x°$, find y'.
2. Given $y = \log x$, find y'.

I told the class the following: “I taught you how take the derivative of the sine when it is in radians, but this problem is in degrees. I taught you how to take the derivative of the natural logarithm, but this problem contains a log to base ten. You do not know how to solve problems like this. Good luck.” Then I sat down and waited. I gave no help on this quiz except to say, “Your answer is wrong, go back and try again,” or “Your method looks right; you probably made an arithmetic error—go back and try again.” Within twenty minutes, most of the students turned in both problems correctly solved, and only a few students received a zero or half-credit. A benefit of giving students a second and third chance on quiz questions is that students come to see their teacher as a coach rather than as an adversary. The reality that the teacher is also the cause of their difficulties and is forcing them to grind out one hard problem after another seems to fade into the background.

The grade is a great motivator

The solutions are as follows:

1.
$$y = \sin x° = \sin \frac{px}{180}.$$

Then by the chain rule,

$$y' = \frac{p \cos\left(\frac{p}{180}x\right)}{180} = \frac{p \cos(x°)}{180}.$$

2.
$$y = \log x = \frac{\ln x}{\ln 10}.$$

Thus,

$$y' = \frac{1}{\ln(10)x} = \frac{\log_{10} e}{x}.$$

Note the psychological trap: To beginners, ln 10 looks like a variable function, but ln 10 is constant. Some students evoke the quotient rule and incorrectly write $(\ln 10)' = 1/10$. This quiz problem is a good one for a student who has just learned to differentiate ln x.

These problems are difficult and scary when students encounter them on a quiz. I had expected that the students would change degrees into radians and recall the change-of-base formula for logarithms. The students’ difficulty occurred because they had not used those two operations recently. So why did they eventually solve these problems? First, the solutions were reasonably close at hand. The students just needed to think hard to recall the important ideas. Second, the students concentrated deeply on these problems because the problems were on a quiz. The grade is a great motivator. Third, the students had been placed in this kind of situation more than once a week since the beginning of the course. They had become decent problem solvers through practice.

In no other subject but mathematics can students be given such a challenge and such an emotional experience on a daily basis. This kind of experience is of life benefit. Remove the stress, remove the high standards, remove the unforgiving challenges—and mathematics loses much of its educational value.

The cost of pushing a class to fulfill its potential is that a few students cannot or will not keep up

George Pólya was one of the world's most famous mathematicians in the 1930s and 1940s. In his later years at Stanford University, he taught mathematics pedagogy to high school teachers. Pólya, I believe, was the first to articulate the value of "struggle" in mathematics education:

> Teaching to solve problems is education of the will. Solving problems which are not too easy for him, the student learns to persevere through unsuccess, to appreciate small advances, to wait for the essential idea, to concentrate with all his might when it appears. If the student had no opportunity in school to familiarize himself with the varying emotions of the struggle for the solution, then his mathematical education failed in the most vital point. (Pólya 1957, p. 94)

Learning mathematics requires heavy concentration. Most students have neither the time nor the motivation to concentrate deeply—unless a grade is involved. That is both a reality of adolescent psychology and a key to effective high school pedagogy: A teacher can keep students focused on challenging problems for long periods of time if the problems come in the form of a quiz. In my opinion, the daily quiz is the most powerful teaching tool ever invented.

I have had two difficulties with my implementation of Pólya's philosophy. The first difficulty is the lack of good quiz problems. These problems must differ from the textbook's problems yet must be solved by the same principles and techniques. I have obtained most of my quiz problems from foreign textbooks (Indian, Russian, and Japanese precalculus textbooks translated into English), from mathematics team competitions, and from "Calendar" problems in the *Mathematics Teacher*. Nevertheless, I have needed five years to build my collection of quiz problems.

My second difficulty has been with student conflict and parent conflict. Many students are afraid to be challenged, and a few students resent challenges. Some students and their parents complained about the challenging problems in my class. The result was that some of my students transferred to other teachers. The remaining students, however, came to accept this method of teaching, and some even preferred it. I was successful for two reasons: Most students naturally try to meet their teacher's expectations, and the students eventually took pride in their accomplishments.

At the end of each school year, I always ask my students to give me a letter grade as their mathematics teacher. All year long, we had followed the same routine: warm-up, review homework, interactive lecture, begin homework, and then a fifteen- to thirty-minute quiz almost every class period. Students occasionally took five or six one-question quizzes in a ninety-minute period, and a few students failed more than half these quizzes. (The questions that they did solve were impressive for these students.) The final examination consisted of twelve original problems to be solved in ninety minutes. The highest score that any student achieved was ten out of twelve. A few students received zeros.

The cost of pushing a class to fulfill its potential is that a few students cannot or will not keep up. But to conduct the class at a less-demanding level would be at a sacrifice to the majority. One of my students who earned a zero on the final exam received a B for the year and was later accepted to Duke University. That student never gave up trying, although she once became tearful in class about her C+ third-quarter grade. In her senior year, she told me that passing AB calculus with a B+ had become possible because of my class. That type of remark is one that I often hear from former students. It also convinces me that this teaching method fosters maturity and perspective.

The only reason that an administrator did not pay me a visit or that parents did not complain about students' grades was that I scaled the quarter grades to approximate the grades given by other precalculus teachers. Students did not know about this scale or were not sure of my consistency until each quarter ended. Students who might have received a high A from another precalculus teacher received a low A from me but only after heroic efforts. The result: In the 1999–2000 year, I received my highest marks ever, B+, from my precalculus students. Education is full of paradoxes. Evidently, students do appreciate being challenged—but only after it is over.

REFERENCE

Pólya, George. *How to Solve It.* 2nd ed. Garden City, N.Y.: Doubleday, 1957.