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# Reducing the Sum of Two Fractions

Sometimes when we add fractions, the sum can be reduced even though we have used the least common denominator. Other times, the sum cannot be reduced. For example, if we add 1/3 + 1/6, the sum 3/6 can be reduced to 1/2. However, 2/5 + 1/6 = 17/30 cannot be reduced.

Let's look at another example:

$$\frac{4}{21}+\frac{7}{15}=\frac{20}{105}+\frac{49}{105} = \frac{69}{105}$$

This fraction can be reduced to 23/35. However, when we add 2/21 + 4/15, we obtain 10/105 + 28/105 = 38/105, which cannot be reduced.

There are pairs of denominators for which it seems we never can reduce the sum. For example,

$$\frac{7}{10}+\frac{11}{12}=\frac{42}{60}+\frac{55}{60} = \frac{97}{60}$$

cannot be reduced. If we try 3/10 and 5/12, we obtain 18/60 + 25/60 = 43/60, which still cannot be reduced. By trying other numerator values, the reader will discover that whenever you add $a/10 + b/12$, where each of the two addends is in lowest terms, the resulting sum having denominator equal to 60 (the least common denominator) cannot be reduced.

On the other hand, there are pairs of denominators for which it seems we always can reduce the sum. Choose an $a$ and $b$ for which $a/20$ and $b/12$ are reduced and add them using the least common denominator 60. Every choice seems to produce an answer that can be reduced. You should now be wondering if this is true for all choices of $a$ and $b$.

Several years ago, while discussing adding and reducing fractions in a workshop, we observed that certain sums of two fractions can be reduced, while others cannot be reduced, and we wondered why. If there was a pattern, it was not immediately obvious; and searches of the literature did not provide us with any background information. This observation led us to pose the following general question:

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> Given a pair of natural numbers, $c$ and $d$, when does there exist a pair of natural numbers $a$ and $b$ for which, when $a/c$ and $b/d$ are added to form a single fraction, that fraction can be reduced?

It is assumed that the denominator of the sum is the least common multiple of $c$ and $d$ and that the addends $a/c$ and $b/d$ are in lowest terms.

As is common practice, we shall say that $u$ and $v$ are *relatively prime* if they have no common factor. For example, 15 and 14 are relatively prime, since $15 = (3)(5)$ and $14 = (2)(7)$. If $a/c$ is reduced to lowest terms, then $a$ and $c$ must be relatively prime.

Adding 7/60 to 11/36, we write $60 = (2^2)(5)(3)$ and $36 = (2^2)(3^2)$, find the least common multiple to be $(2^2)(5)(3^2)$, and multiply the numerator and denominator of the first fraction by $u = 3$ and the second by $v = 5$. Notice that $u$ and $v$ are relatively prime, which will always be true, as we shall see shortly. That is, if $m$ is the least common multiple of numbers $c$ and $d$, then there are relatively prime integers, $u$ and $v$, where $m = cu = dv$.

To proceed from here, we need two useful facts:

Fact 1. If $r$, $s$, $t$ are given integers with $r + s = t$ and if the number $n$ divides any two of the numbers $r$, $s$, and $t$, then $n$ divides the third number.

For example, consider 18 + 49. We know that 7 divides 49 but not 18. Therefore, by fact 1, we know—without even doing the arithmetic—that 7 cannot divide 18 + 49.

Fact 2. $r$ and $s$ are relatively prime if and only if there exist integers $x$ and $y$ for which $rx + sy = 1$.

For example, $15x + 14y = 1$ when $x = 1$ and $y = -1$. It should be noted that, in this result, $x$ and $y$ also have to be relatively prime, as are $y$ and $r$ and $s$ and $x$. The proof of fact 2 is based on the result in number theory stating that if $r$ and $s$ are any nonzero integers, then there exist $x$ and $y$ such that $rx + sy$ is equal to the greatest common factor of $r$ and $s$. Proof of this can be found in Burton (1998).

## THE GENERAL CASE

In number theory, we often need to know the exact power of a prime that divides an integer. This power for a prime $p$ and an integer $n$ is called the "$p$-order of $n$," written $\text{ord}_p(n)$. So, for example, since $150 = (2)(3)(5^2)$, $\text{ord}_5(150) = 2$, $\text{ord}_3(150) = 1$, and $\text{ord}_7(150) = 0$. It turns out that the answer to our question about whether or not the sum of two fractions can be reduced depends on the existence of a common prime factor of the denominators that shows up with exactly the same exponent in their prime factorizations. In other words, everything depends on the primes $p$ with the property that $\text{ord}_p(c) = \text{ord}_p(d)$. As a shorthand, we call such a prime *balanced* for $c$ and $d$. Primes for which $\text{ord}_p(c) \neq \text{ord}_p(d)$ will be called *unbalanced* for $c$ and $d$.

**Example**

Suppose
$$\frac{a}{c} = \frac{a}{(2)(5^2)(7^5)(11^3)(13^2)}$$
and
$$\frac{b}{d} = \frac{b}{(3^2)(5^3)(7^3)(11^3)(13^2)}$$

are reduced. Adding these fractions, we obtain

$$\frac{au + bv}{(2)(3^2)(5^3)(7^5)(11^3)(13^2)},$$

where $u = (3^2)(5)$ and $v = (2)(7^2)$. Notice that 11 and 13 are balanced prime factors for the denominators, while 2, 3, 5, and 7 are unbalanced prime factors of the denominators. We make the following observations in this example:

- $u$ and $v$ are relatively prime
- the balanced primes 11 and 13 divide neither $u$ nor $v$
- the unbalanced prime factors 3 and 5 divide $u$ but not $b$ (because $b/d$ is reduced)
- the unbalanced prime factors 2 and 7 divide $v$ but not $a$ (because $a/c$ is reduced)

Generalizing from this example provides an understanding of lemmas 1 and 2.

Throughout what follows, we will always use $m = cu = dv$ to mean the least common multiple of $c$ and $d$.

Lemma 1. *$u$ and $v$ are relatively prime.*

Lemma 2. *Suppose that $a/c$ and $b/d$ are reduced. If $p$ is an unbalanced prime factor of $c$ or $d$, then $p$ divides $u$ and not $b$, or $p$ divides $v$ and not $a$. If $p$ is a balanced prime for $c$ and $d$, then it does not divide $u$, $v$, $a$, or $b$.*

Lemma 3. *Suppose that $a/c$ and $b/d$ are reduced. If $p$ is an unbalanced prime factor for $c$ or $d$, then $p$ does not divide $au + bv$.*

Proof. By lemma 2, either $p$ divides $u$ and not $b$, or $p$ divides $v$ and not $a$. Without loss of generality, suppose $p$ divides $u$ and not $b$. Since $u$ and $v$ are relatively prime, $p$ does not divide $v$. Therefore, $p$ does not divide $bv$. Since $p$ divides $u$, and therefore $au$, and since $p$ does not divide $bv$, it follows from fact 1 that $p$ does not divide $au + bv$.

THEOREM 1. *There exist natural numbers a and b for which a/c and b/d are reduced and $(au + bv)/m$* can *be reduced if and only if c and d have a balanced prime factor.*

PROOF. Assume that $a/c$ and $b/d$ are reduced and write

$$\frac{a}{c}+\frac{b}{d}=\frac{au+bv}{m}.$$

By lemma 3, an unbalanced prime factor of $c$ or $d$ cannot be a divisor of $au + bv$. So, if there exist natural numbers $a$ and $b$ for which $a/c$ and $b/d$ are reduced and $(au + bv)/m$ can be reduced, then $c$ and $d$ must have a balanced prime factor. Notice that if $(au + bv)/m$ can be reduced, the only possible common prime factors of the numerator and denominator are the balanced prime factors of $c$ and $d$.

Conversely, suppose $c$ and $d$ have a balanced prime factor. If $P$ denotes the product of all the balanced primes for $c$ and $d$, then $P$ is a divisor of $m$. Since $u$ and $v$ are relatively prime, $u^2$ and $v^2$ are relatively prime. So, by fact 2, there exist integers $x$ and $y$ such that

$$xu^2 + yv^2 = 1,$$

so

$$Pxu^2 + Pyv^2 = P,$$

and

$$(Pxu - v + gPuv^2)u + (Pyv + u + gPvu^2)v = P(1 + 2gu^2v^2),$$

where $g$ is a natural number large enough that $Pxu - v + gPuv^2$ and $Pyv + u + gPvu^2$ are both positive. If we define $a = Pxu - v + gPuv^2$ and $b = Pyv + u + gPvu^2$, then the fraction $(au + bv)/m$ can be reduced by $P$, since $P$ is a divisor of $au + bv = P(1 + 2gu^2v^2)$ and of $m$.

Also, it turns out that $a$ is relatively prime to $c$ and $b$ is relatively prime to $d$. Here is why:

To see that $a$ and $c$ are relatively prime, take any prime divisor $q$ of $c$. If $q$ is a balanced prime, then $q$ divides $P$ so it divides $Pxu + gPuv^2$. But $q$ does not divide $v$ so it does not divide $a$. If $q$ is an unbalanced prime factor of $c$ or $d$, then $q$ divides $u$ or $v$ but not both. So $q$ divides $-v + gPuv^2$ but not $Pxu$. Hence $q$ does not divide $a$, or $q$ divides $Pxu + gPuv^2$ but not $v$, so $q$ does not divide $a$. Thus $a$ and $c$ are relatively prime. We may show that $b$ and $d$ are relatively prime in the same way.

Let us look at two of our earlier examples in the context of theorem 1. We saw that when we add $4/21 + 7/15$, the sum $69/105$ can be reduced. Since the denominators 21 and 15 have a balanced prime factor, namely, 3, the theorem assures us that this was no accident and that there do exist natural numbers $a$ and $b$ for which $a/21$ and $b/15$ are reduced and $(5a + 7b)/105$ is not reduced. It also seemed that each time we added $a/10 + b/12$, for any pair of integers $a$ and $b$, where each of the two addends is in lowest terms, the resulting sum having denominator equal to 60 (the least common denominator) could not be reduced. Since the denominators 10 and 12 do not have a balanced prime factor, the theorem assures us that what seemed to be true is indeed true.

In the proof of theorem 1, under the assumption that $c$ and $d$ have a balanced prime factor, we constructed a sum $a/c + b/d$ that could be reduced by the product $P$ of all the balanced primes for $c$ and $d$. This is summarized by the following:

COROLLARY. *If c and d have at least one balanced prime factor, then there exist natural numbers a and b for which a/c and b/d are reduced and $au + bv$ is divisible by every balanced prime factor of c and d.*

THEOREM 2. *Let c and d be given natural numbers. The fraction*

$$\frac{a}{c}+\frac{b}{d}=\frac{au+bv}{m}$$

*can be reduced for all natural numbers a and b where a/c and b/d are reduced if and only if 2 is a balanced prime factor for c and d.*

PROOF. If 2 is a balanced prime for $c$ and $d$, then $u$ and $v$ are both odd. If $a$ and $b$ are natural numbers for which $a/c$ and $b/d$ are reduced, then $a$ and $b$ are odd. Therefore, $au + bv$ is even and, consequently,

$$\frac{a}{c}+\frac{b}{d}=\frac{au+bv}{m}$$

can be reduced.

Suppose 2 is not a balanced prime of $c$ and $d$. We shall show that there exists $a$ and $b$ for which $a/c$, $b/d$, and $(au + bv)/m$ are all reduced.

If there are no balanced prime factors of $c$ and $d$, then by theorem 1 $(au + bv)/m$ is always reduced when $a/c$ and $b/d$ are reduced, and we are done. So assume $c$ and $d$ have at least one balanced prime. By the corollary, there exist natural numbers, $A$ and $B$, for which $A/c$ and $B/d$ are reduced and $Au + Bv$ is divisible by every balanced prime factor of $c$ and $d$.

Let $g$ be the maximum of all prime factors of

$c$ and $d$ and define $q$ to be the product of all odd primes less than or equal to $g$. Then $q + 2$ has no prime factors less than or equal to $g$. So, $c + q + 2$ is relatively prime to $c$.

Let $a = (c + q + 2)A$ and $b = B$ to get:

$$\begin{aligned}\frac{a}{c}+\frac{b}{d} &= \frac{(c+q+2)A}{c}+\frac{B}{d}\\ &= \frac{(c+q+2)Au+Bv}{m}\\ &= \frac{(c+q+1)Au+(Au+Bv)}{m}\end{aligned}$$

Suppose $(c+q+1)Au+(Au+Bv)$ and $m$ have a common prime factor $p$. By lemma 3, $p$ must be a balanced prime for $c$ and $d$. Therefore, $p$ is odd and $p$ divides $c$. Also, since $Au + Bv$ is divisible by every balanced prime factor of $c$ and $d$, $p$ divides $Au + Bv$ as well. Recall that $A$ and $c$, and $u$ and $c$ are relatively prime. Then $p$ divides neither $A$ nor $u$, since $p$ divides $c$. Also, since $p$ divides $c$ and $p$ divides $q$, $p$ does not divide $c + q + 1$. Therefore, $p$ does not divide $(c+q+1)Au$. Consequently, by fact 1, $p$ does not divide $(c+q+1)Au+(Au+Bv)$, contradicting our assumption. Therefore, $a/c$, $b/d$, and $(au+bv)/m$ are all reduced.

Let us look at an earlier example in the context of theorem 2. We observed that for every choice of $a$ and $b$, when you add $a/20 + b/12$ using the least common denominator 60, where each of the two addends is reduced, the resulting sum can be reduced, and we wondered if this is always true. Since 2 is a balanced prime for the denominators 20 and 12, the theorem does in fact assure us that this is always true.

## RATIONAL FUNCTIONS

The questions we have examined also arise in the addition of ratios of polynomials, called rational functions. For example, the sum

$$\frac{a}{x-3}+\frac{b}{x+2}=\frac{(a+b)x+(2a-3b)}{(x-3)(x+2)},$$

where $a$ and $b$ are nonzero numbers, cannot be reduced. To verify this, observe that there do not exist nonzero numbers $a$ and $b$ such that $(a + b)x + 2a - 3b$ is a constant multiple of either $x - 3$ or $x + 2$. Likewise, we can show that if

$$\frac{a}{(x-3)(x+1)}+\frac{b}{x(2x-5)(x+3)},$$

where $a$ and $b$ are nonzero numbers, is to be written as a single fraction having denominator $(x - 3)(x + 1)(x)(2x - 5)(x + 3)$, this rational expression cannot be reduced.

However, there do exist nonzero constants $a$ and $b$ such that the sum

$$\frac{a}{(x-2)(x+1)}+\frac{b}{(x-2)(x-3)}=\frac{(a+b)x+(-3a+b)}{(x-2)(x+1)(x-3)}$$

can be reduced. For example, let $a = 3$ and $b = 1$, then

$$\begin{aligned}\frac{3}{(x-2)(x+1)}+\frac{1}{(x-2)(x-3)} &= \frac{4(x-2)}{(x-2)(x+1)(x-3)}\\ &= \frac{4}{(x+1)(x-3)}.\end{aligned}$$

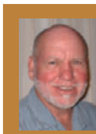
In this last example, the factor $x - 2$ appears with multiplicity 1 in each denominator. It appears to play a role analogous to that of a balanced prime factor in our previous work. There is much known about the way that polynomials factor and divide each other and even the way some of them behave like primes. It would be interesting to know how much of the facts about reducing numerical fractions carry over to the class of rational functions.

## REFERENCE

Burton, David M. *Elementary Number Theory*, 4th ed. New York: McGraw-Hill, 1998.

**Editors' notes:** Harris S. Shultz and Ray C. Shiflett provide yet another example of "mathematics for teaching": mathematical investigations connected to the teaching profession. The question When can you reduce the sum of two fractions? is certainly more likely to come up in the teachers' lounge than in an engineering firm. And while Shultz and Shiflett carry out the investigation for its own sake, the resulting theorems will be useful to teachers as they design activities and problem sets around rational number arithmetic. We suspect that more than a few areas of mathematics were inspired by mathematical problems teachers face as they design activities for their students. One that we recently encountered is how to devise two linear equations in two unknowns with small integer coefficients so that students will not be able to get exact coordinates for the intersection of the graphs simply by zooming with their calculators. Articles that address such questions make ideal submissions to this department.

Shultz and Shiflett end the article with an intriguing question: How much of their result carries over to rational functions in, say, one variable with rational coefficients? This is again a question closely related to our profession, because two basic algebraic systems of precollege mathematics—the integers and polynomials in one variable—have deep structural similarities that allow many results in one system to be transported to the other with minor modification. For example, fact 2 (and the more general result about greatest common divisor) holds for polynomials, too, and for the same reasons. Thus, both systems have a fundamental theorem of arithmetic: the unique prime decomposition property that is used throughout this article. We would be interested in future articles that investigate the problem Shultz and Shiflett pose and, more generally, in articles that investigate the carryover to polynomials of results in arithmetic. ∞

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**CELEBRATE PI DAY**

Activities and information about $\pi$ abound on the Web. The following sites are just a sampling of what's available. Additional sites can be found by using any search engine.

- "Slices of Pi: Rounding up Ideas for Celebrating Pi Day," by Larry Lesser, in *Texas Mathematics Teacher* 51, no. 2 (Fall 2004), www.tenet.edu/tctm / downloads/TMT_Fall_04.pdf
- "The $\pi$ Pages," www.cecm.sft.ca/pi/pi.html
- "Pi Pages on the Internet," www.joyofpi .com/pilinks.html
- "A History of Pi," www-groups.dcs.st-and.ac.uk/~history/HistTopics/Pi_through_the_ages.html