APRIL

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

1 John and a group of his friends took a bus trip. Each person paid the bus driver with the same combination of 9 coins. If the bus driver received \$8.41 from the group, how many dimes did he receive?

2 A man walks 3 miles east, then 3 miles north, and then 2 miles northeast. How far is he from his starting point?

3 A culture of bacteria doubles in size twice every day. A dish with 1 million bacteria is full after 15 days. How long will it take for a dish with 2 million bacteria to fill?

4 A book is to have 250 pages that will be numbered with Arabic numerals. How many times will the digit 2 be used in numbering the pages?

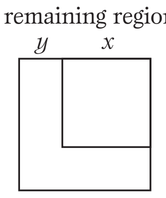
5 Two clocks now indicate the true time. One gains a second every hour, and the other gains 3 seconds every 2 hours. In how many days will both clocks again indicate the true time?

6 In his last game, Joe bowled a 199, raising his average by one pin (one point in bowling) to 178. What must he bowl in his next game to raise his average to 179?

7 It takes 1140 digits to number the pages of a book. How many pages are there in the book?

8 The owner of a bicycle shop took inventory of his bicycles and tricycles. He counted 153 wheels and 136 pedals. How many bicycles and tricycles did he have?

9 A large square has a smaller square cut from its corner in such a way that the area of the square removed equals the area of the remaining region. If x represents the length of a side of the removed square, and y represents the remaining length, find the ratio x/y.



10 A large circular piece of plywood has an area of 9π square feet. A carpenter plans to cut four congruent, maximum-sized circular pieces from the plywood. What is the length of the radius of each of the four congruent circular pieces?

11 If a rectangular region having an area of 500 square units has its length increased by 20 percent and its width decreased by 10 percent, what is the area of the newly formed region?

12 Two positive numbers may be inserted between 3 and 9 such that the first three numbers in the sequence form a geometric progression and the last three form an arithmetic progression. Find the sum of these two positive numbers.

13 A bag of 3 apples, 7 oranges, and 11 pears costs \$6.04, while a bag of 2 apples, 5 oranges, and 8 pears costs \$4.31. What is the cost of a bag of fruit consisting of 1 apple, 1 orange, and 1 pear?

14 Given is a square with each side 2 units long. Arcs are drawn tangent to each other at the midpoints of the sides of the square, as shown. What is the exact area of the shaded region?

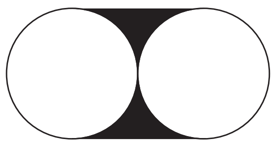
15 If $x + y = 5$, and $xy = 3$, what is the value of $x^2 + y^2$?

16 A number leaves a remainder of 3 when divided by each of the following: 8, 7, 6, 5, and 4. What is the smallest positive integer greater than 3 that satisfies these conditions?

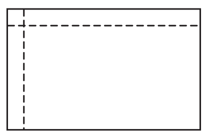
17 Solve for x:

$$\sqrt{1+\sqrt{2+\sqrt{x}}}=3$$

18 The radius of each circle is r. Find the area of the shaded region.



19 Strips of width w are cut off the length and width of a rectangle 10" by 12". How wide must each strip be in order to reduce the rectangle to one having an area of 80 square inches?



20 During a vacation it rained 13 days; but when it rained in the morning the afternoon was sunny, and every rainy afternoon was preceded by a sunny morning. There were 11 sunny mornings and 12 sunny afternoons. How long was the vacation?

21 If $a*b = a^b - b$, find $(2*3)*4$.

22 Given the expression xy^2, if the values of x and y are each decreased by 50 percent, find the percent of decrease in the value of the original.

23 A car travels a distance of 1742 miles at an average rate of 40 mph and returns at an average rate of 55 mph. Find the average rate for the round trip to the nearest tenth.

24 A man invests some money at 8 percent interest and twice as much at 6 percent. If the total income from the two investments is \$250, how much is invested at the lower rate?

25 The product of five prime positive integers is a six-digit number, where all of the digits are the same. Find the six-digit number described above.

26 Solve for x:

$$4^x - 4^{x-1} = 12$$

27 What is the maximum number of points of intersection when two circles and five straight lines intersect each other?

28 Mary sold 13 books, each for the same price, and used part of the money to buy a sweater that cost \$17 (tax included). She had \$22 left. What was the price of each book?

29 Find the coordinates, both positive, of the vertex of the isosceles triangle having as its base the line segment determined by the points (8, 3) and (–3, 3) if the area of the triangle is 66 square units.

30 Find two numbers a and b such that the sum, product, and quotient of the two numbers are all equal.

SOLUTIONS to calendar

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Problems 1–8 for the April "Calendar" were selected from *101 Mathematical Puzzles* by Reinfeld and Rice (1966). Problems 9–30 were selected from the Wisconsin Mathematics Council state mathematics meets archives (1972–1987).

For information on sources for problems in the February 2005 "Calendar," see p. 576.

The Editorial Panel of the *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly "Calendar" during the 2005–2006 academic year. Please write to the *Mathematics Teacher* editor, 1906 Association Drive, Reston, VA 20191-1502, for guidelines.

Three other sources of problems in calendar form are available from NCTM: *Calendar Problems from the* Mathematics Teacher (a book featuring more than 400 problems, organized by topic, order number 12509, $22.95), "Calendars for the Calculating," vol. 2 (a set of nine monthly calendars that originally appeared from September 1987 to May 1988, order number 496, $13.50), and "A Year of Mathematics" (one annual calendar that originally appeared in September 1982, order number 311, $4.00; set of five, order number 312, $8.00). Individual members receive a 20 percent discount off these prices. Write to NCTM for the catalog of educational materials, which includes a listing for the publication *Exploratory Problems in Mathematics*. An online version of the catalog is available at www.nctm.org.–*Eds.*

1. None. Since $841 = 29^2$, John and 28 of his friends could only have been paid $8.41 in equal amounts by spending $0.29 each. The only combination of 9 coins that yields $0.29 is 4 pennies and 5 nickels. So the bus driver did not receive any dimes.

2. $2 + 3\sqrt{2}$. As indicated in the first figure, the man is walking along a path from point A to point D, followed by point C and point B. The distance we are looking for is represented by AB. As shown in the second figure, $\triangle ABE$ and $\triangle CBF$ are isosceles right triangles. Since $BC = 2$,

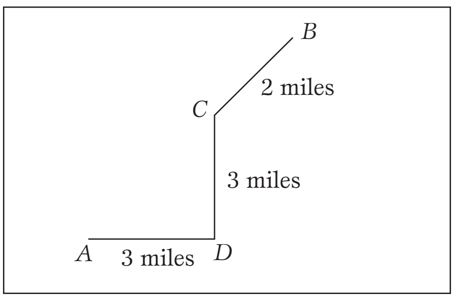
$$CF = \frac{1}{\sqrt{2}}.$$

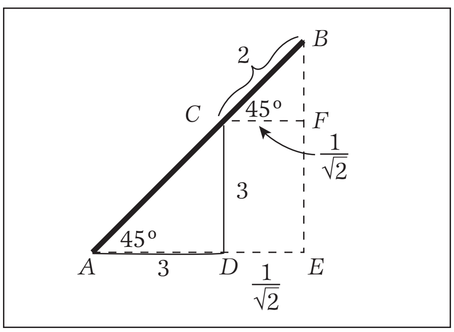
$DE = CF$, so

$$DE = \frac{1}{\sqrt{2}}.$$

Then

$$AE = 3 + \frac{1}{\sqrt{2}}.$$





Since $AE = BE$,

$$BE = 3 + \frac{1}{\sqrt{2}}$$

also. Therefore,

$$\begin{aligned} AB &= \sqrt{2} \bullet AE \\ &= \sqrt{2} \bullet \left(3 + \frac{1}{\sqrt{2}}\right) \\ &= 2 + 3\sqrt{2}. \end{aligned}$$

3. 14.5 days. After half a day, the first dish will contain 2,000,000 bacteria, since it started with 1,000,000 and doubles twice a day. It will then take another 14.5 days to fill. The dish that begins with 2,000,000 bacteria is simply one-half day behind the original and will take 14.5 days to fill.

Alternate Solution: The first dish takes 30 "doublings" to fill and can be represented by $1{,}000{,}000 \bullet 2^{30}$. The second dish will take $2x$ "doublings" to fill, where x = the number of days that the bacteria is in the dish. So,

$$\begin{aligned} 1{,}000{,}000 \bullet 2^{30} &= 2{,}000{,}000 \bullet 2^{2x} \\ &= 1{,}000{,}000 \bullet 2 \bullet 2^{2x} \\ &= 1{,}000{,}000 \bullet 2^{2x+1}. \end{aligned}$$

We see that $30 = 2x + 1$, so $14.5 = x$.

4. 106. A 2 will be used as the units digit once every ten numbers from 1 to 250, so there will be twenty-five 2s in the units place. The tens place will be a 2 ten times every one hundred numbers (for the twenties). There are three groups of twenties from 1 to 250, so there will be thirty uses of the 2 in the tens place. The hundreds place will have a 2 only in the 200s. Each number from 200 to 250 will satisfy this, so there are fifty-one numbers with a 2 in the hundreds place. This gives a total of $25 + 30 + 51 = 106$.

5. 3,600. In order to show the correct time again, each clock must gain an extra 12 hours; and 12 hours = 60 • 60 • 12 = 43,200 seconds. Since the first clock gains 1 second every hour, it gains 24 seconds each day. Thus the first clock gains 12 hours every (43,200 seconds)/(24 seconds per day), or 1,800 days. Since the second clock gains 3 seconds every 2 hours, it gains 36 seconds each day. So, it gains 12 hours every (43,200 seconds)/(36 seconds per day), or 1,200 days. The least common multiple of 1,800 days and 1,200 days is 3,600 days (almost 10 years), when the two clocks will agree in time again.

6. 201. Joe's old average was 177. Since he bowled a 199 (22 points higher than his previous average) he needed 22 points to raise his average 1 whole point. In order to raise the average another point, he needs to score 23 points over his current average of 178 in his 23rd game. 178 + 23 = 201.

Alternate Solution: Joe's average equals the total number of pins in all his games divided by the number of games he played. If he played n games before the game in which he bowled a 199, then he scored $177n$ pins before that game. After the 199th game, he played $n + 1$ games and had a total of $177n + 199$ pins. So his new average is

$$\frac{177n+199}{n+1}=178.$$

Solving this equation will result in $n = 21$. In order to have a new average of 179, he will need to score x pins in his 23rd game. So the new average would be represented by

$$\frac{178(n+1)+x}{n+2}=\frac{178\bullet 22+x}{23}=179.$$

Solving, $x = 201$.

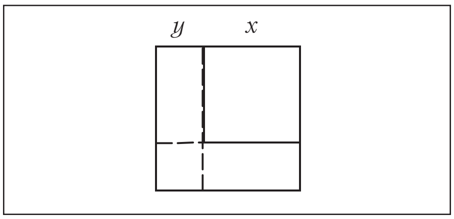
7. 416. Each page from 1 through 9 uses 1 digit, each page from 10 through 99 uses 2 digits, and each page from 100 through 999 uses 3 digits. Pages 1 through 99 will have 9 pages of 1 digit and 90 pages of 2 digits for a total of 9 + 180 = 189 digits. Therefore, there are 1140 – 189 = 951 digits left. All of the remaining pages will each have 3 digits, since there are not enough digits left to reach 1000. Now, 951 divided by 3 is 317. So 317 pages after 99 will be used, which will bring the total number of pages to 99 + 317 = 416.

8. 51 bicycles and 17 tricylces. Let b = number of bicycles and let t = number of tricycles. Since $2b + 2t = 136$, $b + t = 68$. Also, $2b + 3t = 153$. Solving these equations will result in $b = 51$ and $t = 17$.

Alternate Solution: The only difference between the bicycles and tricycles is that a tricycle has 1 more wheel than it has pedals. So, the difference between the number of wheels and pedals, 153 – 136 = 17, indicates that there are 17 more wheels than pedals. These 17 must all be on tricycles, so there are 17 tricycles and consequently 51 bicycles.

9. $1 + \sqrt{2}$. In the figure drawn, it can be seen that the original square has an area of $(x + y)^2$ so that we want

$$\frac{1}{2}(x+y)^2=x^2.$$



Simplifying, we have $2xy + y^2 = x^2$. Since y cannot equal zero, divide both sides of the equation by y^2:

$$\frac{2xy}{y^2}+\frac{y^2}{y^2}=\frac{x^2}{y^2}\to\left(\frac{x}{y}\right)^2-\frac{2x}{y}-1=0$$

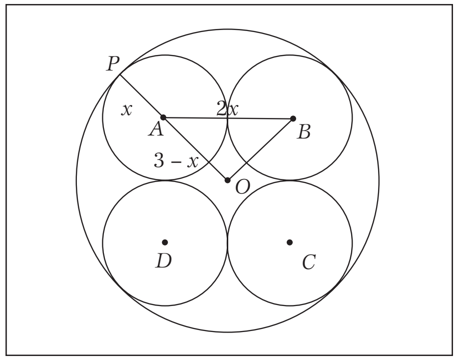
The ratio of x/y now can be solved using the quadratic formula:

$$\frac{x}{y}=1\pm\sqrt{2}$$

Since $(x/y) > 0$ and $1 - \sqrt{2} < 0$, we know that the only solution is:

$$\frac{x}{y}=1+\sqrt{2}$$

10. $-3 + 3\sqrt{2}$. In the figure shown, O is the center of the plywood circle, and A, B, C, and D are the centers of the four

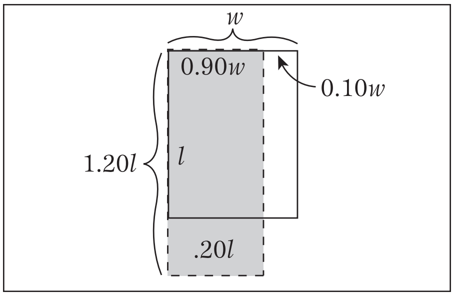


congruent cutouts. To maximize the area used, circles A, B, C, and D must be tangent to circle O and to each other. Let x be the length of the radius of circle A (and, consequently, circles B, C, and D). Since the area of circle O is 9π square feet, the radius of circle O is 3 feet. Both AO and $BO = 3 - x$, and $AB = 2x$, because $\overline{AB}$ is a segment made up of a radius of circle A and a radius of circle B. Since $\angle AOB$ subtends one-fourth of the circle and is therefore 90°, $\triangle AOB$ is an isosceles right triangle. From this fact we know that $(3 - x)^2 + (3 - x)^2 = (2x)^2$. Simplifying this equation leads to the quadratic equation $x^2 + 6x - 9 = 0$. Solving, $x = -3 + 3\sqrt{2}$ or $-3 - 3\sqrt{2}$. Since $x > 0$, the latter solution can be eliminated from consideration and so $x = -3 + 3\sqrt{2}$.

Alternate Solution: Since $\triangle AOB$ is an isosceles right triangle, we can apply the special triangle relationship to yield

$$\sqrt{2}(3-x)=2x\to$$
$$3\sqrt{2}-x\sqrt{2}=2x\to$$
$$3\sqrt{2}=2x+x\sqrt{2}\to$$
$$3\sqrt{2}=x\left(2+\sqrt{2}\right)\to$$
$$\frac{3\sqrt{2}}{\left(2+\sqrt{2}\right)}=x\to$$
$$\frac{3\sqrt{2}}{\left(2+\sqrt{2}\right)}\bullet\frac{\left(2-\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}=x\to$$
$$\frac{6\sqrt{2}-6}{2}=x\to$$
$$3+3\sqrt{2}=x$$

11. 540 square units. The area of the original rectangle with length l and



width w is $lw = 500$ square units. The new rectangle will have a length 20 percent larger, or $1.20l$, and a width 10 percent smaller, $0.90w$. These new dimensions will result in an area

$$\begin{aligned}(1.20l)(0.90w) &= 1.08lw\\ &= 1.08(500)\\ &= 540 \text{ square units.}\end{aligned}$$

12. 11.25, or 45/4. Allow the sequence of numbers to be 3, a, b, 9. If 3, a, and b form a geometric progression, then $3k = a$ and $ak = b$, for some real $k \geq 0$. Solving these equations yields $3b = a^2$, since $k = (b/a)$. If a, b, and 9 form an arithmetic progression, then $a + m = b$ and $b + m = 9$, for some real $m \geq 0$. These equations imply that $9 - b = m$ and $b - a = m$. So $9 = 2b - a$. Substituting $b = (a^2/3)$ reduces the equation to solving the quadratic $2a^2 - 3a - 27 = 0$. Solving, $a = 4.5, -3$. Since we are looking for only positive values, $a = 4.5$ and by substitution, $b = 6.75$. Therefore the sum equals 11.25, or 45/4.

13. \$0.88. The problem indicates tha $3A + 7O + 11P = \$6.04$ and $2A + 5O + 8P = \$4.31$. Solving simultaneously, multiply the first equation by 3 and the second equation by 4. This results in $9A + 21O + 33P = \$18.12$ and $8A + 20O + 32P = \$17.24$. By subtracting the second equation from the first: $A + O + P = \$0.88$.

14. $4 - \pi$. The square has a side length of 2, so the area of the square is 4. The shaded area can be thought of as taking four quarter-circles away from the area of the square. Four quarter-circles of radius 1 are equivalent in area to one full circle of radius 1, which has an area of $\pi(1)^2 = \pi$. Therefore the shaded region is $4 - \pi$.

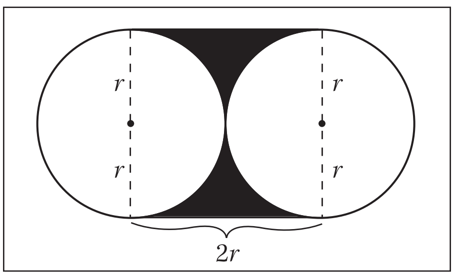
15. 19. $x + y = 5$, so $(x + y)^2 = 5^2$. So $x^2 + 2xy + y^2 = 25$. Since $xy = 3$, we can use substitution to get $x^2 + 2(3) + y^2 = 25$. So now, $x^2 + y^2 = 25 - 6$, or 19.

16. 843. If a remainder of 3 occurs when the number is divided by 8, 7, 6, 5, and 4, then it will be 3 more than a multiple of each. Since we want the smallest number, we are looking for the least common multiple of these five numbers. The least common multiple of 8, 7, 6, 5, and 4 is the product $2^3 \bullet 7 \bullet 3 \bullet 5 = 840$. In order to have a remainder of 3, we need $840 + 3 = 843$.

17. 3844.

$$\begin{aligned}\sqrt{1+\sqrt{2+\sqrt{x}}} &= 3 \rightarrow\\ 1+\sqrt{2+\sqrt{x}} &= 9 \rightarrow\\ \sqrt{2+\sqrt{x}} &= 8 \rightarrow\\ 2+\sqrt{x} &= 64 \rightarrow\\ \sqrt{x} &= 62 \rightarrow\\ x &= 3844\end{aligned}$$

18. $r^2(4 - \pi)$. The shaded region is the area of a square with side length $2r$ minus the area of two semi-circles of radius r. Two semi-circles of radius r have the same area as one circle of radius r. So the area of the shaded region is $(2r)^2 - \pi r^2 = r^2(4 - \pi)$.



19. 2 inches. If w represents the width of each strip to be cut off the rectangle, then the dimensions of the new rectangle will be $12 - w$ and $10 - w$. The new area: $(12 - w)(10 - w) = 80$. So, $w = 2$, or $w = 20$. Since a strip of 20 cannot be cut from side lengths of 12 and 10, the width of the strip must be 2 inches.

20. 18. Only three types of days occurred on this vacation: Rain-Sun, Sun-Sun, and Sun-Rain. Since rainy mornings were followed by sunny afternoons, there were no Rain-Rain days. Let SM = number of sunny mornings, SA = number of sunny afternoons, RM = number of rainy mornings, RA = number of rainy afternoons. $RM + RA = 13$, $SM = 11$, $SA = 12$. We know that $SM + RM = SA + RA \rightarrow 11 + RM = 12 + RA$ (by substitution) $\rightarrow RM = 1 + RA$. However, $RM = 13 - RA$, so again by substitution $13 - RA = 1 + RA \rightarrow RA = 6$, and $RM = 7$. So there were 7 days of rain followed by sun, and 6 days of sun followed by rain. Since there were 12 days of afternoon sun, there must have been 5 days of total sun. The total number of days on vacation was $7 + 6 + 5 = 18$ days.

21. 621. $(2*3)*4 = (2^3 - 3)*4 = (8 - 3)*4 = 5*4 = 5^4 - 4 = 625 - 4 = 621$.

22. 87.5 percent. Replace x with $0.5x$ and y with $0.5y$. Therefore, $xy^2 = (0.50x)(0.50y)^2 = 0.125xy^2$. This value indicates that 12.5 percent of the original remains; therefore, there is a decrease of (100 percent – 12.5 percent) = 87.5 percent.

23. 46.3 mph. The trip took 1742/40 hours at a rate of 40 mph, and 1742/55 hours at a rate of 55 mph. The average rate on the trip would be found by taking total distance divided by total time:

$$\begin{aligned}\frac{1742+1742}{\left(\frac{1742}{40}+\frac{1742}{55}\right)} &= \frac{3484}{\left(\frac{16549}{220}\right)}\\ &= \frac{880}{19}\\ &\approx 46.3 \text{ mph}\end{aligned}$$

24. 2500. Let I_1 = interest earned at 8 percent and I_2 = interest earned at 6 percent. If x = amount of money invested at 8 percent, then $2x$ represents the amount of money invested at 6 percent. So, $I_1 + I_2 = x(0.08) + 2x(0.06) = 250$. Solving, $0.20x = 250$ and $x = 1250$. So $2x = 2500$.

25. 111,111. All six-digit numbers of the form ddd,ddd are divisible by 111,111. So begin by examining the prime factorization of $111{,}111 = 3 \bullet 7 \bullet 11 \bullet 13 \bullet 37$. Since it has five prime positive integer factors, any multiple of it would contain an additional factor, and 111,111 must be the desired result.

26. 2. By factoring and solving, we find that $4^x - 4^{x-1} = 12 \to 4^{x-1}(4-1) = 12 \to 4^{x-1} = 4 \to x - 1 = 1 \to x = 2$.

27. 32. The circles will intersect in, at most, 2 points. Each line will intersect the circles in, at most, 4 points. Since there are five lines, this will yield, at most, 20 new points of intersection. Each line will intersect four other lines once each. Since there are five lines, this would yield, at most, 20 new points of intersection. However, each linear intersection point is counted twice, so we can only count 10 unique points of intersection. This gives a maximum total of $2 + 20 + 10 = 32$.

Alternate Solution: The circles intersect in, at most, 2 points. The first line will intersect the circles in, at most, 4 points. The second line will intersect the circles in 4 points and the first line, at most, once, for a total of 5 points. The third line will intersect the circles in 4 points and each of the first two lines, at most, once each for a total of 6 points. Similarly, the fourth line will create 7 new points of intersection, and the fifth line will create 8 new intersection points. Thus the total number of points of intersection is $2 + 4 + 5 + 6 + 7 + 8 = 32$.

28. \$3.00. If b = number of books that Mary sold, then $13b - 17 = 22$. Solving, $b = 3$.

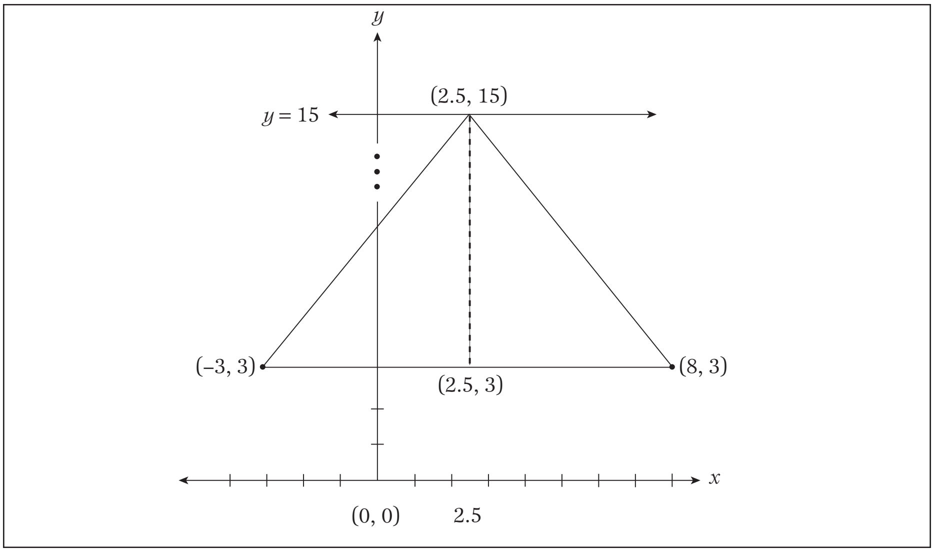
29. (5/2,15). Since the points lie on a horizontal segment, the line of symmetry that contains the third vertex will be at

$$x = \frac{8+(-3)}{2} = \frac{5}{2}.$$

Since the area of the triangle is 66, and the length of the base is $8 - (-3) = 11$, we know that

$$66 = \frac{1}{2}(11)(h).$$

Solving, the height of the triangle is 12. The base of the triangle is on the line $y = 3$, so the new vertex (with only positive coordinates) will have a y-coordinate of $3 + 12 = 15$. Together this gives the ordered pair for the vertex: (5/2, 15).



30. $\left(\frac{1}{2}, -1\right)$.

Write the problem as $a + b = ab = (a/b)$. Multiplying both sides of the equation $ab = (a/b)$ by b results in

$$\begin{aligned} ab^2 - a &= 0 \to \\ a(b^2 - 1) &= 0 \to \\ a(b-1)(b+1) &= 0. \end{aligned}$$

Solving, $a = 0$, $b = 1$, or $b = -1$ are the possible solutions. However, if $a = 0$, then $ab = (a/b) = 0$, then $a + b = 0 \to 0 + b = 0 \to b = 0$. This will result in division by 0 in (a/b), so $a = 0$ is not possible. Suppose that $b = 1$. Then

$$\begin{aligned} a + b &= ab \to \\ a + 1 &= a \to \\ 1 &= 0, \end{aligned}$$

so $b = 1$ is not a possible solution. Considering $b = -1$, $a + b = ab \to a + (-1) = -a \to 2a = 1 \to a = (1/2)$. Checking,

$$\begin{aligned} a + b &= \frac{1}{2} + -1 \\ &= -\frac{1}{2}; \\ ab &= \frac{1}{2}(-1) \\ &= -\frac{1}{2}; \\ \frac{a}{b} &= \frac{\frac{1}{2}}{-1} \\ &= -\frac{1}{2}. \end{aligned}$$

So the solution is

$$a = \frac{1}{2},\ b = -1.$$ ∞