AUGUST

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Arrange ten dots in such a way that there are five rows of dots with four dots in each row.

1

Alicia earns \$20 per hour, of which 1.45 percent is deducted to pay local taxes. How many cents per hour of Alicia's wages are used to pay local taxes?

2

Jenny attends basketball practice every day after school. At each practice last week, she made twice as many free throws as she had made at the previous practice. At her fifth practice she made 48 free throws. How many total free throws did she make during the week?

3

Bertha has 6 daughters and no sons. Some of her daughters have 6 daughters, and the rest have none. Bertha has a total of 30 daughters and granddaughters, but she has no great-granddaughters. How many of Bertha's daughters and granddaughters have no daughters?

4

An equilateral triangle and a regular hexagon have the same perimeter. Which area is greater and by how much?

5

Given that $-4 \le x \le -2$ and $2 \le y \le 4$, what is the largest possible value of

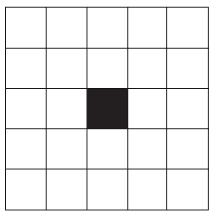
$$\frac{x+y}{x}?$$

6

Two 8-sided dice each have faces numbered 1 through 8. When the dice are rolled, each face has an equal probability of appearing on the top. What is the probability that the product of the two numbers on the top of the dice is greater than their sum?

7

The 5×5 grid contains squares with dimensions of 1×1 to 5×5. How many of these squares include the black square?



8

What quadrilateral can be divided into three equilateral triangles?

9

Henry's Hamburger Heaven offers its hamburgers with ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties and any assortment of condiments. How many different kinds of hamburgers can be ordered?

10

Coins in the United States have the following thicknesses: penny, 1.55 mm; nickel, 1.95 mm; dime, 1.35 mm; quarter, 1.75 mm. If a stack of coins is exactly 14 mm high, how many coins are in the stack?

11

A function f is defined by $f(z) = i\bar{z}$, where $i = \sqrt{-1}$ and $\bar{z}$ is the complex conjugate of z. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

12

In an equilateral triangle, $\triangle ABC$, P is the midpoint of $\overline{AB}$, $\overline{PR} \perp \overline{BC}$, and $\overline{PQ} \perp \overline{AC}$, where points R and Q are on $\triangle ABC$. If the area of $\triangle ABC = 32$ cm^2, what is the area of $\triangle PRQ$?

13

Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters?

14

Points A and B are on the parabola $y = 4x^2 + 7x - 1$, and the origin is the midpoint of $\overline{AB}$. What is the length of $\overline{AB}$?

15

A standard six-sided die is rolled, and P is the product of the five numbers that are visible. What is the largest number that is certain to divide P?

16

The sum of the interior angles of a polygon is less than 2005°. What is the largest possible number of sides of the polygon?

17

The values of a, b, c, and d are 0, 1, 2, and 3, although not necessarily in that order. What is the maximum value of the expression $c \cdot a^b - d$?

18

If $f(x) = ax + b$ and $f^{-1}(x) = bx + a$ with a and b real, what is the value of $a + b$?

19

How many two-digit positive integers have at least one 7 as a digit?

20

Given that they are made of the same material, which is heavier: a ball with a radius of 10 inches or 10 balls each with a radius of 1 inch?

21

A grocer makes a display of cans in which the top row has one can and each subsequent row has two more cans than the row above it. If the display is made up of 100 cans, how many rows does it contain?

22

Minneapolis–St. Paul International Airport is 8 miles southwest of downtown St. Paul and 10 miles southeast of downtown Minneapolis. To the nearest integer, what is the number of miles between downtown St. Paul and downtown Minneapolis?

23

For how many pairs of positive integers (x, y) is $x + 2y = 100$?

24

The volume of a cube is 8 times less than the volume of another cube. What is the relationship of their surface areas?

25

The point $A(-3, 2)$ is rotated 90° clockwise around the origin to point B. Point B is then reflected over the line $y = x$ to point C. What are the coordinates of C?

26

All the students in an algebra class took a 100-point test. Five students scored 100, each student scored at least 60, and the mean score was 76. What is the smallest possible number of students in the class?

27

If x and y are positive integers for which $2^x3^y = 1296$, what is the value of $x + y$?

28

In $\triangle ABC$, $4 \cdot m\angle A > 9 \cdot m\angle B$ and $3 \cdot m\angle C < 2 \cdot m\angle B$. What type of triangle is $\triangle ABC$?

29

The two digits in Jack's age are the same as the digits in Bill's age, but in reverse order. In five years Jack will be twice as old as Bill will be then. What is the difference in their current ages?

30

Let $A = (0, 9)$ and $B = (0, 12)$. Points A' and B' are on the line $y = x$, and $\overline{AA'}$ and $\overline{BB'}$ intersect at $C = (2, 8)$. What is the length of $\overline{A'B'}$?

31

SOLUTIONS to calendar

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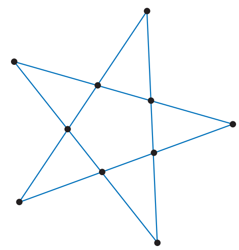
Three other sources of problems in calendar form are available from NCTM: *Calendar Problems from the* Mathematics Teacher (a book featuring more than 400 problems, organized by topic, order number 12509, \$22.95), "Calendars for the Calculating," vol. 2 (a set of nine monthly calendars that originally appeared from September 1987 to May 1988, order number 496, \$13.50), and "A Year of Mathematics" (one annual calendar that originally appeared in September 1982, order number 311, \$4.00; set of five, order number 312, \$8.00). Individual members receive a 20 percent discount off these prices. Write to NCTM for the catalog of educational materials, which includes a listing for the publication *Exploratory Problems in Mathematics.* An online version of the catalog is available at www.nctm.org.—Eds.

1. Draw a 5-point star and place a dot at each point of intersection, as shown.



2. \$0.29. Since \$20 is 2000 cents, she pays (0.0145)(2000) = 29 cents per hour in local taxes.

3. 93. At Jenny's fourth practice she made (1/2)(48) = 24 free throws. She made 12 free throws at her third practice, 6 at her second practice, and 3 at her first practice. During the week, she had a total of 48 + 24 + 12 + 6 + 3 = 93 free throws.

4. 26. Bertha has 30 – 6 = 24 granddaughters, none of whom have any daughters. The granddaughters are the children of 24/6 = 4 of Bertha's daughters. So 4 of Bertha's 6 daughters have daughters themselves. Thus the number of women who have no daughters is 30 – 4 = 26. Alternately, there are 24 granddaughters without daughters and 2 daughters without daughters for a total of 24 + 2 = 26.

5. The area of the regular hexagon is 1.5 times the area of the equilateral triangle, or 50 percent larger.

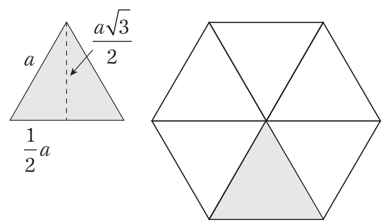
Let a represent the length of each side of the triangle and b represent the length of each side of the hexagon. Since the perimeters are equal, $3a = 6b \rightarrow a = 2b$. The area of the equilateral triangle with side length a is $(\sqrt{3}/4)a^2$. By substitution,

$$\begin{aligned}\frac{\sqrt{3}}{4}a^2 &= \frac{\sqrt{3}}{4}(2b)^2\\ &= \frac{\sqrt{3}}{4}4b^2\\ &= \sqrt{3}b^2.\end{aligned}$$

On the other hand, since a regular hexagon can be divided into 6 equilateral triangles, the area of a regular hexagon with side b is

$$6\left(\frac{\sqrt{3}}{4}b^2\right) = \frac{3}{2}(\sqrt{3}b^2).$$

So the area of the hexagon is 3/2 = 1.5 times the area of the triangle, or 50 percent larger.



6. The largest possible value is 1/2. We know that

$$\frac{x+y}{x} = 1 + \frac{y}{x}.$$

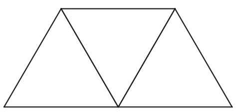
Since $x < 0$ and $y > 0$, $(y/x) < 0$. Therefore, the value is maximized when $|y/x|$ is minimized, that is, when $|y|$ is minimized and $|x|$ is maximized. So $y = 2$ and $x = -4$ give the largest value, which is $1 + (-1/2) = 1/2$.

7. The probability is 3/4. There are $8 \cdot 8 = 64$ ordered pairs that can represent the top numbers on the two dice. Let m and n represent the top numbers on the dice. Then $mn > m + n$ implies that $mn - m - n > 0$, that is, $mn - m - n + 1 > 1$. Since $mn - m - n + 1 = (m - 1) \cdot (n - 1)$, then $(m - 1) \cdot (n - 1) > 1$. Since $m > 0$, and $n > 0$, this inequality is satisfied except when $m = 1$, $n = 1$, or $m = n = 2$. Sixteen total ordered pairs (m, n) are excluded by these conditions (when $m = 1$: 8 pairs; $n = 1$: 7 additional pairs; and $m = n = 2$: 1 more pair), so the probability that the product is greater than the sum is

$$\frac{64-16}{64} = \frac{48}{64} = \frac{3}{4}.$$

8. 19. All squares of size 5×5, 4×4, and 3×3 contain the black square, and there are $1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$ of them. In addition, 4 of the 2×2 squares and 1 of the 1×1 squares contain the black square, for a total of $14 + 4 + 1 = 19$.

9. An isosceles trapezoid with a small base that is equal to the length of a leg and 1/2 the length of the large base, as shown.



10. 768. A customer makes one of two choices for each condiment, to include it or not to include it. The choices are made independently, so there are $2^8 = 256$ possible combinations of condiments. For each of those combinations there are three choices regarding the number of meat patties, or $(3)(256) = 768$ different kinds of hamburgers.

11. 8. The height in millimeters of any stack with an odd number of coins would have a 5 in the hundredths place. The height of any two coins would have an odd digit in the tenth place and a zero in the hundredth place. Therefore any stack with zeros in both its tenth and hundredth places must consist of a number of coins that is a multiple of 4. The highest stack of 4 coins would only have a height of $4(1.95) = 7.8$ mm, which is too short. The shortest stack of 12 coins would have a height of $12(1.35) = 16.2$ mm, which is too tall. This indicates that the only possible multiple of 4 that will work is a stack of 8 coins. Note that a stack of 8 quarters has height of $8(1.75) = 14$ mm.

12. 2. Let $z = x + iy$ and $\bar{z} = x - iy$. By definition of f,

$$\begin{aligned} f(z) &= f(x+iy) \\ &= i(x-iy) \\ &= ix - i^2y \\ &= ix + y \end{aligned}$$

for all real numbers x and y. Since $f(z) = z$, $ix + y = x + iy \rightarrow x = y$. So the numbers that satisfy $f(z) = z$ are of the form $x + ix$. The set of all such numbers is a line through the origin in the complex plane. The set of all numbers that satisfy $|z| = 5$ is a circle centered at the origin of the complex plane. The numbers satisfying both equations correspond to the points of intersection of the line and circle, of which there are two.

13. 6 cm^2. Let a represent the side length of $\triangle ABC$. We know that the area of $\triangle ABC = 32$ cm^2, so

$$\begin{aligned} 32\text{cm}^2 &= \frac{\sqrt{3}}{4}a^2 \rightarrow \\ 128 &= \sqrt{3}a^2 \rightarrow \\ \frac{128}{\sqrt{3}} &= a^2. \end{aligned}$$

Because P is the midpoint of $\overline{AB}$, $|AP| = |PB| = (a/2)$. Since $\triangle PBR$ is the right triangle with $m\angle B = 60^\circ$, $BR = (a/4)$, $PR = (\sqrt{3}/4)a$, and $m\angle BPR = 30^\circ$. Because $\triangle ABC$ is equilateral and $\overline{CP}$ is a median, $\overline{CP}$ is also a height of $\triangle ABC$. So $m\angle CPR = m\angle CPB - m\angle BPR = 90^\circ - 30^\circ = 60^\circ$. Let S be the point of intersection of segments CP and QR. $\triangle PSR$ is then a 30-60-90 triangle, with $PR = (\sqrt{3}/4)a$. Consequently,

$$PS = \frac{\frac{\sqrt{3}}{4}a}{2} = \frac{\sqrt{3}}{8}a$$

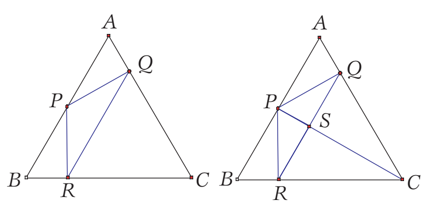
and $SR = (3/8)a$.

The area of $\triangle PRQ = 2(\triangle PSR)$

$$\begin{aligned} &= 2\left(\frac{1}{2} \cdot \frac{3}{8}a \cdot \frac{\sqrt{3}}{8}a\right) \\ &= \frac{3\sqrt{3}}{64}a^2. \end{aligned}$$

By substitution, the area of $\triangle PSQ$

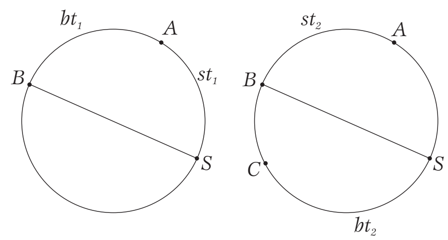
$$\begin{aligned} &= \frac{3\sqrt{3}}{64} \cdot \frac{128}{\sqrt{3}} \\ &= 6. \end{aligned}$$



14. 350 meters. When Brenda and Sally first meet, they have run a combined distance equal to half the length of the track. Between their first and second meetings, they run a combined distance equal to the full length of the track. Because both Brenda and Sally run at constant speeds, and Brenda runs 100 meters before their first meeting, she runs twice as far, or $2(100) = 200$ meters between their first and second meetings. The total distance around the track is the sum of the distances they have each run from their first meeting. Therefore the length of the track is $200 + 150 = 350$ meters.

Alternate solution. In the illustration:

B represents the starting point for Brenda
S represents the starting point for Sally
A represents the first meeting point
C represents the second meeting point
b = Brenda's constant running speed
s = Sally's constant speed
t_1 = time between start and first meeting
t_2 = time between the first and second meetings
T = total length of the track



When they first meet, Sally has run counterclockwise from point S to point A and Brenda has run clockwise from point B to point A. Since they have run a combined distance equal to half the length of the track, $bt_1 + st_1 = (1/2)T$.

Between the first and second meetings, Sally travels from point A to point C counterclockwise, while Brenda travels from point A to point C clockwise. Since they have run a combined distance equal to the entire length of the track, $bt_2 + st_2 = T$.

Using substitution, $bt_1 + st_1 = (1/2)\ (bt_2 + st_2) \rightarrow bt_1 + st_1 = (1/2)bt_2 + (1/2)st_2 \rightarrow t_1(b + s) = (1/2)t_2(b + s)$. Since $(b + s)$ equals a nonzero constant, this implies that $t_1 = (1/2)t_2$.

Brenda runs 100 meters in t_1, so she will run twice as far, 200 meters, in t_2.

Sally ran 150 meters in t_2. The sum of the distances run in t_2 is the total length of the track, and this is 150 + 200 = 350 meters.

15. $5\sqrt{2}$. Let $B = (a, b)$ and $A = (-a, -b)$. Then $4a^2 + 7a - 1 = b$ and $4(-a)^2 + 7(-a) - 1 = (-b) \to 4a^2 - 7a - 1 = -b$. Subtracting the second equation from the first equation gives $2b = 14a \to b = 7a$, so by substitution, $4a^2 + 7a - 1 = 7a$. Thus $a^2 = (1/4)$ and $b^2 = (7a)^2 = (49/4)$, so

$$\begin{aligned} AB &= \sqrt{(2a)^2 + (2b)^2} \\ &= \sqrt{4a^2 + 4b^2} \\ &= 2\sqrt{a^2 + b^2} \\ &= 2\sqrt{\frac{50}{4}} \\ &= 5\sqrt{2}. \end{aligned}$$

16. 12. Since $6! = 720 = 2^4 \cdot 3^2 \cdot 5$, the prime factors of P can consist of at most 2s, 3s, and 5s. The least possible number of 2s is two, which occurs when 4 is not visible. The least possible number of 3s is one, which occurs when either 3 or 6 is not visible, and the least number of 5s is zero, when 5 is not visible. Thus P must be divisible by $2^2 \cdot 3 = 12$, but not necessarily by any larger number.

17. 13. The sum of the interior angles of a polygon $= 180^\circ(n - 2) < 2005^\circ \to 180^\circ n < 2365^\circ \to n < 13.139$. Since n must be an integer, the largest value for n is 13.

18. 9. If $d \neq 0$, then the value of the expression can be increased by interchanging 0 with the value of d. Therefore the maximum value must occur when $d = 0$. Now consider the value of the expression when a, b, or c equal 1. If $a = 1$, the value of the expression is c, which is 2 or 3. If $b = 1$, the value of the expression is $c \cdot a = 6$. If $c = 1$, the value of the expression is a^b, which is $2^3 = 8$ or $3^2 = 9$. Thus the maximum value is 9.

19. −2. Since $f(f^{-1}(x)) = x$, it follows that $a(bx + a) + b = x$. So $abx + a^2 + b = x$, $ab = 1$, and $a^2 + b = 0$. By substitution,

$$a^2 + \frac{1}{a} = 0 \to \frac{1}{a}\left(a^3 + 1\right) = 0.$$

Hence $a = -1$, so $b = -1$, and $a + b = -2$.

20. 18. There are 10 two-digit numbers with a 7 as the 10s digit, and 9 two-digit numbers with 7 as the units digit. Because 77 satisfies both of these properties, the answer is 10 + 9 − 1 = 18.

21. The ball with a radius of 10 inches is heavier. Since the volume of a sphere is $(4/3)\pi r^3$, one ball with a radius of 10 inches would have a volume of $(4/3)\pi \cdot 10^3 = (4000/3)\pi$ in^3. Thus 10 balls of radius 1 inch would have an accumulated volume of only

$$10 \cdot \left(\frac{4}{3}\pi \cdot 1^3\right) = \frac{40}{3}\pi \text{ in}^3.$$

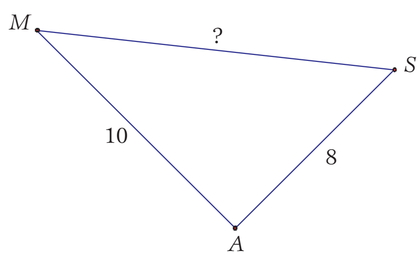
22. 10. If there are n rows in the display, the bottom row contains $2n - 1$ cans. The total number of cans is therefore the sum of the arithmetic series $1 + 3 + 5 + L + (2n - 1)$, which is

$$\frac{n}{2}\left[\left(2n - 1\right) + 1\right] = n^2.$$

Thus $n^2 = 100$, so $n = 10$.

23. 13 miles. Let downtown St. Paul, downtown Minneapolis, and the airport be located at S, M, and A, respectively. Then $\triangle MAS$ has a right angle at A, so by the Pythagorean theorem,

$$\begin{aligned} MS &= \sqrt{10^2 + 8^2} \\ &= \sqrt{164} \\ &\approx \sqrt{169}, \text{ or } 13. \end{aligned}$$



24. 49 pairs. The value of $x = 100 - 2y$ is a positive integer when $100 - 2y > 0$. This is true for each positive integer y with $1 \leq y \leq 49$. Thus there are 49 pairs.

25. One surface area is 24 times the surface area of the other cube. Let x represent the length of a side of the first cube and y represent the length of the side of a second cube. The volume of the first cube, x^3, is 8 times less than the volume of the second cube, y^3. So

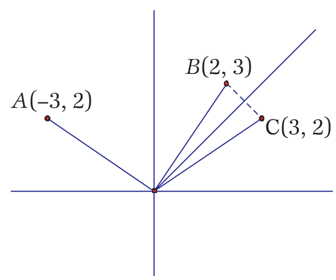
$$x^3 = \frac{1}{8}y^3 \to \sqrt[3]{8}x = y.$$

The surface area of the first cube is $6x^2$, while the surface area of the second cube is

$$\begin{aligned} 6y^2 &= 6\left(\sqrt[3]{8}x\right)^2 \\ &= 6\left(4\right)x^2 \\ &= 24x^2. \end{aligned}$$

So the surface area of the second cube is 24 times the surface area of the first cube.

26. (3, 2). The rotation takes $A(-3, 2)$ to $B(2, 3)$, and the reflection takes B to $C(3, 2)$, as shown.



27. 13. Each score of 100 is 24 points above the mean, so the five scores of 100 represent a total of (5)(24) = 120 points above the mean. Those scores must be balanced by scores totaling 120 points below the mean. Since no student scored more than 76 − 60 = 16 points below the mean, the number of other students in the class must be an integer no less than 120/16. The smallest such integer is 8, so the number of students in the class is at least 13. Note that the conditions of the problem are met if 5 students score 100 and 8 score 61.

Alternate solution. If k students are in the class, the sum of their scores is $76k$. If the five scores of 100 are excluded, the sum of the remaining scores is $76k - 500$. Since each student scored at least 60, the sum is at least $60(k - 5)$. Thus $76k - 500 \geq 60(k - 5)$, so $k \geq 12.5$. Since k must be an integer, $k \geq 13$.

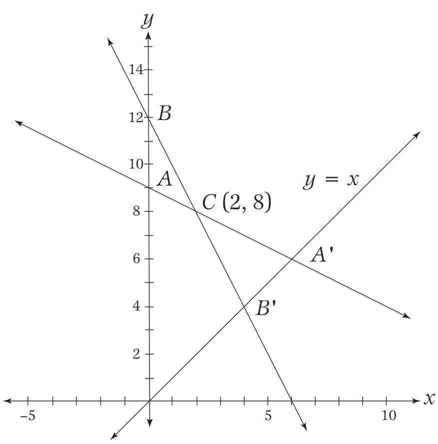
28. 8. Note that $1296 = 6^4 = 2^4 \cdot 3^4$, so $x = y = 4$ and $x + y = 8$.

29. $\triangle ABC$ is obtuse. Since $4 \cdot m\angle A > 9 \cdot m\angle B$, then $m\angle B < (4/9) \cdot m\angle A$. Since $3 \cdot m\angle C < 2 \cdot m\angle B$, then $m\angle C < (2/3) \cdot m\angle B$. This implies $m\angle C < (2/3) \cdot ((4/9) \cdot m\angle A\ ) = (8/27) \cdot m\angle A$. $180^\circ = m\angle A + m\angle B + m\angle C \rightarrow$ (by substitution) $180^\circ < m\angle A + (4/9) \cdot m\angle A + (8/27) \cdot m\angle A = (47/27) \cdot m\angle A$. $180^\circ < (47/27) \cdot m\angle A \rightarrow 103.4^\circ < m\angle A$, so $\angle A$ must be obtuse, and $\triangle ABC$ is obtuse.

30. 18. Let Jack's age be $10x + y$ and Bill's age be $10y + x$. In five years, Jack will be twice as old as Bill. Therefore, $10x + y + 5 = 2(10y + x + 5)$, so $8x = 19y + 5$. This implies that $19y + 5$ is a multiple of 8. In order for the expression $19y + 5$ to be a multiple of 8, notice that $19y + 5 = 16y + 8 + 3(y - 1)$. Since $16y$ and 8 are both multiples of 8, the expression as a whole is a multiple of 8 if and only if $(y - 1)$ is a multiple of 8. Since both x and y are 9 or less, the only solution is $y = 1$ and $x = 3$, or $y = 9$ and $x = 22$. The latter does not satisfy the condition on x. Thus, $y = 1$ and $x = 3$, so Jack is 31 and Bill is 13. The difference between their ages is $31 - 13 = 18$ years.

31. $2\sqrt{2}$. Line AC has a slope of $-1/2$ and y-intercept $(0, 9)$, so its equation is $y = (-1/2)x + 9$. Since the coordinates of A' satisfy both this equation and $y = x$, it follows that $A' = (6, 6)$. Similarly, the line BC has the equation $y = -2x + 12$, and $B' = (4, 4)$. Thus,

$$A'B' = \sqrt{(6-4)^2 + (6-4)^2} = 2\sqrt{2}.$$



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