PRESIDENT'S choice

Johnny W. Lott

In 1972, the *Mathematics Teacher* published a series of three articles in "The Forum," a section of the journal devoted to diverging opinions with respect to the role of geometry and the best approach to it. The February issue addressed the question "What should become of the high school geometry course?" The articles in "The Forum" in February were written by Howard F. Fehr, Frank M. Eccles, and Bruce E. Meserve. I chose the article by Fehr as one that has had an effect on high school curricula and still poses some answers to the original question today. The question was the subject of a panel discussion at the August 2006 MathFest of the Mathematical Association of America sponsored by the NCTM/MAA Joint Committee on Mutual Concerns.

Consider the effect that "The Forum" has had simply from the titles of the three articles in the issue. Meserve wrote about an improved year of geometry, and Eccles wrote about transformations in high school geometry. In some sense, both of those have been attempted with many reforms in geometry, including a shift from a year of geometrical proofs to a more informal approach and the inclusion of transformations in virtually every secondary textbook since that time. However, Fehr's is the article with the most direction for geometry. In particular, he discussed geometrical thinking in what was an unconventional manner that foreshadowed the integrated mathematics textbooks now commonly known as the "NSF reform curricula." His vision of school geometry being woven into the whole of mathematics foreshadowed the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) and *Principles and Standards for School Mathematics* (2000). The geometry study proposed was eclectic, but he claimed that it was more in line with the rest of mathematics. He cited as an example the study of mathematics in European countries and questioned the yearlong study of geometry in this country. His article is still a model for the morphing of geometry in the United States into more mainstream mathematics.

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The Present Year-Long Course in Euclidean Geometry Must Go

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It is assumed that *the* geometry course refers to one that is commonly taught in the tenth school year. It is traditional Euclidean synthetic geometry, of 2- and 3-space, modified by an introduction of ruler and protractor axioms into the usual synthetic axioms. A unit of coordinate geometry of the plane is usually appended. It is a course that is reflected in textbooks prepared by the School Mathematics Study Group and in most commercial textbooks.

Goals for Geometric Instruction

Before any pedagogically sound reply can be given to the question posed, one must examine the goals of geometric instruction. Many such goals have been stated during the past one hundred years, but in the light of all we know today, I propose three objectives that I believe would be generally accepted.

The first goal, an objective of all mathematics instruction, is to foster *intellectual formation.*[1] We should like our students to come to know what geometrical thinking is—that is, what geometry is, what it studies, and how it devises its method to do this study. We should not identify geometrical thinking with logical thinking, for the latter is the domain of all mathematics.

A second goal is to transmit important *information about space* that has been provided in the past and appears to be necessary in the years to come. The necessity applies not only to preparing for future study of mathematics but for applying geometric knowledge to specific everyday affairs.

The third goal is to develop *skill in geometric problem solving,* that is, techniques by which one may find answers to unknown situations through building geometrical models of physical and behavioral theories or by using geometry as a means of explanation.

While the present geometry course contributes to each of these goals, it does so in a very trivial manner. The survival of Euclid's geometry rests on the assumption that it is the only subject available at the secondary school level to introduce students

A word on the editorial approach to reprinted articles: *Obvious typographical errors have been silently corrected. Additions to the text for purposes of clarification appear in square brackets. No effort has been made to reproduce the layouts or designs of the original articles, although the subheads are those that first appeared with the text. The use of words and phrases now considered outmoded, even slightly jarring to modern sensibilities, has likewise been maintained in an effort to give the reader a better feel for the era in which the articles were written.—Ed.*

to an axiomatic development of mathematics. This was true a century ago. But recent advances in algebra, probability theory, and analysis have made it possible to use these topics, in an elementary and simple manner, to introduce axiomatic structure. In fact, geometrical thinking today is vastly different from that used in the narrow synthetic approach.

Geometric instruction should be included in every year of study from grade seven through grade twelve

The important information about geometrical objects and their measure is now taught in elementary school. The useful conclusions of Euclid's synthetic geometry are learned intuitively by the end of grade eight. To spend a year putting these facts in a semirigorous system is a waste of valuable learning time. In addition, much important, useful geometrical knowledge has come to light in the last century, knowledge that is not reflected in current geometry instruction.

As for applications of Euclid's geometry, they are almost entirely out of the mainstream of mathematics activity. Almost everything that is presented has "as much relevance to what mathematicians—pure and applied—are doing today as magic squares or chess problems" (OECD 1961). So I must agree with Dieudonné's point of view:

The present year-long course in Euclidean geometry must go.

So, likewise, must any other year-long course in geometry be eliminated from our curriculum (and from our thinking). Mathematics is no longer conceived of as a set of disjoint branches, each evolving in its own way. The present conception is that of a unified whole in which all the branches contribute to the development of each other. The program in geometry must be built anew and integrated into a unified body of secondary school study of mathematics from grade seven through grade twelve.

Development of Geometry

To arrive at a contemporary conception of geometry and to see how this conception should dictate the nature of geometric instruction in the schools, it will be useful to review briefly some recent historical developments.

From 325 B.C. to A.D. 1827 only one geometry, that of Euclid, existed as a means to study space. During all this time, the only controversial question was the possibility of proof of the parallel postulate, and it occupied the energies of great mathematicians—Wallis, Saccheri, Lambert, Legendre, Gauss, Bolyai, Lobachevski. The work of these men paid tribute to Euclid's genius, and through their efforts, non-Euclidean geometries were invented. For the first time we have the obvious implication that *there is more than one geometry.*

The mathematical world did not at first accept the conclusions of these men. But in 1854 Bernhard Riemann generalized the concept of space by considering new kinds of geometry. The immediate result of Riemann's paper (published posthumously in 1868) was a burst of activity in the development of different types of geometry. A new kind of geometric interpretation was presented by Felix Klein in 1872 in his famous Erlanger Program (Klein 1893), where he showed that one geometry may be distinguished from another by its group of transformations. A geometry may determine a group, and a group determines a geometry. For example, the group of similitudes and the group of isometries lead respectively to affine and Euclidean spaces. However, there are geometries that do not possess a group structure.

The Perfection of Euclid

Riemann, in his paper, also pointed out some of the flaws of Euclid's axioms, thereby initiating a spate of activity among outstanding mathematicians to clear Euclid of all blemishes. This task was first completed by Pasch (1882) and subsequently by Peano, Pieri, Hilbert (1899), and Veblen. With the problem of perfecting Euclid solved, outside of the possible discovery of a few more exceptional points, lines, or circles, the study reached a dead end. However, the solution resulted in a set of axioms considered far too complicated and abstract to be used as a secondary school approach. There then followed a sixty-year period of sporadic efforts to do something about the subject as a secondary school subject—Euclid must be saved. The first modification of Hilbert's axioms was given in 1929 by G. D. Birkhoff, who introduced the order and the completeness properties of the real number. While many other similar modifications were given, Birkhoff's properties were translated into the "real ruler" and the "real protractor" axioms that are the base for the geometry course today.

Geometry Today

Riemann also extended the growing subject of differential geometry from a study of curves and surfaces in three-dimensional Euclidean space to a study of quadratic forms with n coordinates. The story of the advance from Riemann to the present-day "global differential geometry" and differential topology is well known to researchers in this field (Willmore 1970). Today, the development of geometry and its counterpart, topology, is going on in all directions. Its pervasiveness in mathematics and science may be shown by a partial list of geometries—affine, projective, Euclidean, hyperbolic, elliptic, combinatorial, absolute, analytic, differential, algebraic, Minkowskian, integral, transformation, vector, linear, topological, conformal, relativistic, optical, and so forth—involving infinite dimensional spaces, convex spaces, metric space, finite n-dimensional spaces, and the like. It is thus quite evident that geometry today has quite a different aspect from that prevailing in the contemporary high school program.

The Geometry for Secondary Schools

Today, geometry must be conceived of as a *study of spaces.* Each geometry is a (set, structure) where the elements of the set are called points and the structure is a set of axioms, including definitions, which relate the points and their important subsets. With this conception, instruction in geometry must be brought more and more into relation with algebra and its structures, and thus it must be developed so as to permit and exhibit the use of algebraic structures and techniques. This is the spirit of the times. In this respect, *a very important objective should be to develop geometry so that it leads to a basic understanding of vector spaces and linear algebra.*

There are a number of ways, all valid, to study spaces. One can use intuition alone and study physical objects in 2- and 3-space and, by abstracting shape, position, and metric properties where they exist, develop a practical geometry or at least a useful set of geometrical relations. One can proceed synthetically, as Euclid did, choosing a convenient (but small) set of axioms. One can coordinatize space and make use of the real numbers, as Descartes indicated could be done. Perpendicularity and a distance function can then be used to obtain the Euclidean coordinatized plane. One can also follow the Erlanger Program of Klein, studying mappings, transformations, and groups and the resulting geometries. One can also use vectors, first as sensed line segments, and then as points in a space with a fixed origin, leading to an algebra of points (n-tuples). Then one can go from affine to Euclidean vector space by way of an inner product of vectors. Since in the secondary school we are not, or do not need to be, concerned with teaching future professional mathematicians, none of these approaches should be used to the exclusion of the others. It appears that a contemporary view of geometry for the educated layman is best achieved by a study that contains all these approaches.

With the educational goals stated at the start of this article and the mathematical content feasible for secondary school instruction, the following objectives should guide the development of the secondary school geometry instruction:

1. Develop the concept of space as a set with special subsets, having structures that are linked to others—especially vector, affine, and Euclidean space.
2. Develop the knowledge of precise relationships between the line and the set of real numbers. This leads to coordinatized space.
3. Develop an understanding of the principal transformations, groups of transformations, and their application, especially in a coordinatized space.
4. Develop an understanding of an axiomatic structure by this sequence of study: the affine line, the affine plane, the affine space, metric space, Euclidean space as a vector space.
5. Develop skill in applying the several methods of geometric development to the solution of original problems—both mathematical and applied.
6. Unify the mathematical study of algebra and geometry in the concept and application of vector spaces and linear algebra.

Geometric instruction should be included in every year of study beginning in grade seven and continuing through grade twelve. It should grow in complexity and abstraction and at all times be related to those algebraic methods that enable it to become embedded in a vector-space structure. At all times it should be applied so that it becomes a way of thinking. As Willmore (1970) has said, "What is important is a geometrical way of looking at a mathematical situation; geometry is essentially a way of life."

There are many sequences in which the geometric instruction outlined above can be organized to achieve desired objectives. One need only study the official syllabi of European countries to recognize how many different approaches, with different emphases, reach the same goal (SSMCIS 1971). The following sequence, integrated into a six-year unified study, is one proposal:

The official syllabi of European countries show how many different approaches, with different emphases, reach the same goal

1. Start with a physical, informal study, using drawings, paper folding, measuring, and physical objects to gain an intuitive feeling for figures in Euclidean 2- and 3-space, especially for lines, rays, segments, and angles.
2. Develop the number line as a mapping of real numbers into the set of points on a line, preserving order. Scale the line many ways to develop the linearity of

relations to the scales $x' = ax + b$. Compare yard with meter, different temperature scales, and so forth.
3. Develop lattice points as intersection points of two directions. Use the coordinatized affine plane (parallelism only). Introduce perpendicularity and develop transformations of the plane that are isometries. Use translations and groups of translations in connection with vectors, equipollence of vectors, and composition of vectors. Use both transformations and vectors to prove relations in the plane.
4. Introduce dilations with a fixed point and elementary ideas of similitudes.
5. Introduce axiomatic affine plane geometry with a minimum (3) of axioms. Develop ideas of proof and prove theorems, apply them to finite models, then to lattice points, and finally to the continuous plane.
6. Using further axioms (or better, informally), introduce the coordinated affine plane.
7. Introduce perpendicularity and distance to obtain the Euclidean plane. Examine the group of transformations constituting isometries; treat congruence by isometry. Do linear equations and inequalities with respect to the intersection of lines and half planes. Relate them to matrices. Relate 2×2 matrices to transformations in the plane.
8. Introduce 3-space, both affine and Euclidean, informally. Study relations of lines and planes in space. Consider the measure of length, area, and volume.
9. Introduce (informally or with axioms) coordinated affine 3-space.
10. Do the algebra of points in an affine plane. Develop the notion of a localized vector and the equation of a vector line and an affine line. Apply this to geometric properties in a plane.
11. Develop the vector-space structure and its linear algebra; apply this to the plane using the concepts of basis, linearity, dependence, and independence. Give many other illustrations and applications of vector space.
12. Introduce the concept of inner product; develop affine 3-space as a vector space, define perpendicularity and Euclidean 3-space, and develop theorems in Euclidean 3-space.
13. Develop the conic sections, either by vectors or by rectangular coordinate geometry. Generalize transformations in the affine and the Euclidean plane.
14. Use matrices, transformations, and complex numbers to develop and relate all mathematics in developing trigonometric analysis.

The geometry program suggested in the brief outline above is an eclectic one, to be sure. But it shows what geometry is today; it gives important geometrical knowledge; and above all, it shows how the subject gives clarity and understanding to all other branches. Further, it develops a tool for genuine use for all those who continue their study of mathematics and science. It is a program already in existence in most European countries. If the reader has not done so already, he certainly should now seriously question a geometry program that consists of a year-long sequence of a modified form of Euclid's synthetic geometry. The United States of America is the only nation of all the developed countries of the world that retains such a study.

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ENDNOTE

1. The word "formation" is here used in the sense of the French meaning of the word, which can be interpreted as the "making" or "shaping" or "creating" of the intellect. ∞