READER reflections

COMER'S RULE (I)

The February 2006 issue of the *Mathematics Teacher* includes an article about an alternative way of evaluating a 3×3 determinant: "You've Heard of Cramer's Rule, Now Try Comer's: An Alternative Approach to Finding Determinants." Readers might be interested to know that the method Kristina Comer found is a special case of a result for $n \times n$ determinants found by J. J. Sylvester in 1851. To read about it, search the Web for Sylvester's determinant identity. It is nice when students discover something they had not known, and exciting when the teacher also does not know it. It is exciting in a different way when one learns that a famous mathematician had found the result earlier. Some of the students who are rediscovering results now will be the ones whose work (often in special cases) will be rediscovered by a student in a future generation.

Richard Askey
askey@math.wisc.edu
University of Wisconsin—Madison
Madison, WI 53706

COMER'S RULE (II)

While student teaching, I had an exchange student from Brazil who used something like Comer's method. He called it Chió, though no one could determine what the translation of that word was. The Comer article prompted me to search for the Chió method on the Internet. Information at mathworld.wolfram.com/ChioPivotalCondensation.html indicates basically that Comer's method is a particular case of the Chió method. Despite its being something that some use around the world and something already discovered, I share Michelle Genovese's elation that a student discovered this method for herself.

Steven Willott
Steven_Willott@fhsd.k12.mo.us
National Board Certified Teacher
Francis Howell North High School
St. Charles, MO 63303

COMER'S RULE (III)

I believe Michelle Genovese's student Kristina Comer rediscovered a rule that Charles Dodgson, better known as Lewis Carroll, wrote about in his book *Condensation of Determinants* (1866). I have used this rule with some of my algebra 2 students and find that it works well. Students enjoy the connection with Lewis Carroll.

Since Comer also saw how to correct the method when the center element is zero, she might also see how to apply Cramer's rule more easily than is shown in the textbooks. In my class, students were initially confused by Cramer's rule because the matrices in the numerator for x and y aren't symmetrical. But just as switching rows changes the sign of determinants, so does switching columns. If students can remember to change the sign, they don't have to memorize Cramer's rule and can extend the method to the 3×3 case.

$$\begin{aligned}\text{Example: } 2x + 2y &= 10\\ x + 4y &= 11\end{aligned}$$

Put a pencil over the 10 and 11 and find the determinant of the coefficient matrix $2*4 - 1*2 = 6$. Since this is not zero, there is a unique solution, and we write 4 under the constants (on the right). Now move the pencil over the $2y$ and $4y$ and find the next determinant: $2*11 - 1*10 = 12$. Put this under the y column. Move the pencil and find the last determinant: $2*11 - 4*10 = -18$. Here we must change the sign and write 18 under the x column.

Now we read off the answer: $x = 18/6 = 3$ and $y = 12/6 = 2$.

Jeffrey Jones
jkentjones@yahoo.com
Detroit Lakes Area Learning Center
Detroit Lakes, MN 56502

DISTANCE FROM A POINT TO A LINE

On pages 4, 5, 61, and 62 of the August 2005 issue of the *Mathematics Teacher* we find various proofs of the formula for the distance from a given point $P(x_0, y_0)$ to a given straight line ℓ: $ax + by + c = 0$. Each proof is interesting in its own way but makes explicit use of square roots, trigonometric functions, and a fair amount of algebra. All of these difficulties can be avoided by using just a modicum of elementary vector algebra. See **figure 1 (Kandall)**.

We denote by F the foot of the perpendicular from P to ℓ, and we use the vector notation $X = [x, y]$, $N = [a, b]$. The vector N is perpendicular to ℓ. (The reason is that if X_1 and X_2 are any two distinct points of ℓ, then $N \cdot X_1 = N \cdot X_2 = -c$; consequently, $N \cdot (X_2 - X_1) = 0$, that is, N is perpendicular to $\overrightarrow{X_1X_2}$.)

Thus, $\overrightarrow{PF} = tN$ for some scalar t; we are seeking a formula for $d = |tN| = |t|\,|N|$, where d represents the distance from point P to line ℓ. Since $F = P + tN$ lies on ℓ, we have $N \cdot (P + N) = -c$, that is, $N \cdot P + t|N|^2 = -c$. Therefore,

$$\frac{N \cdot P + c}{|N|} = -t|N|,$$

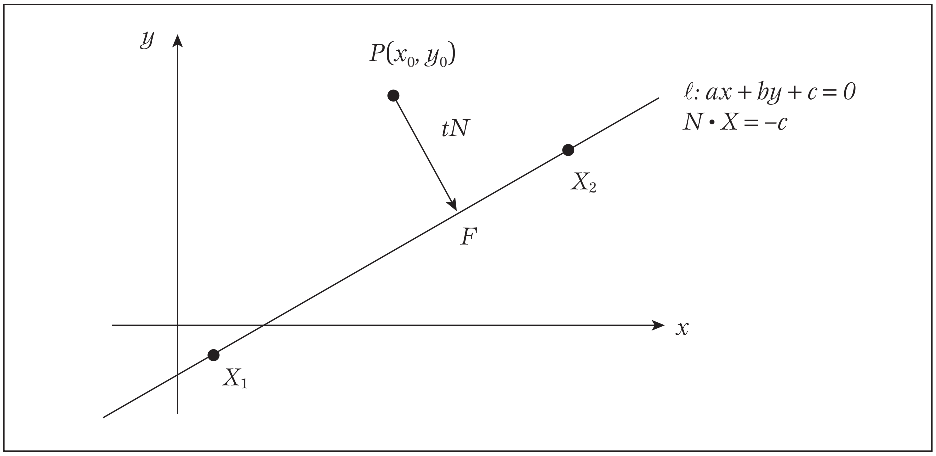

Fig. 1 (Kandall)

hence,

$$d = |t||N| = \frac{|N \cdot P + c|}{|N|} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Geoffrey A. Kandall
gkandall@snet.net
Hamden, CT 06514

MORE ON MULTIPLYING POLYNOMIALS

In the articles "Multiplying Polynomials," Michael O'Neil (*Mathematics Teacher,* March 2006) shared a teaching strategy that I, too, have found successful in my algebra classes. This method, which I like to call the "box method," has been an invaluable tool for my inclusion and regular education classes alike. Besides displaying beautiful symmetry, this method is also a wonderful organizational tool that makes visual sense to many of my students.

In my classroom, there are actually two procedures that my students and I call the box method. The second procedure, which I was taught by my trigonometry students, involves factoring quadratic trinomial expressions.

This method was primarily developed for factoring quadratic expressions of the form $ax^2 + bx + c$. Students begin by setting up a 2×2 box as shown in **figure 1 (Becker)**. To fill in the two empty boxes, students must find factors of $a(c)$ whose sum is equal to b.

Example 1. $6x^2 + 19x + 10$

Traditionally, when factoring an expression like that in example 1, students are required to find all of the factors of 6 and 10 and, by trial and error, discover the correct combination that results in the sought-after binomial factors. The box method eliminates the trial-and-error work and requires students to sift through the factors of only one number. In example 1, students need to know the two factors of 60 that, when added together, equal 19. See **figure 2 (Becker**). With a little effort, they find that the missing numbers are 15 and 4. See **figure 3 (Becker)**. Since the middle term is $19x$, we fill the empty blocks of our box with $15x$ and $4x$, in no particular order. See **figure 4 (Becker)**.

To find the binomials that were multiplied together to generate the original problem, students must find the greatest common factor of each row and column. We can see from **figure 5 (Becker)** that the greatest common factors for the rows are $2x$ and 5. From **figure 6 (Becker)** we see that the greatest common factors for the columns are $3x$ and 2.

The factors of $6x^2 + 19x + 10$ are $2x + 5$ and $3x + 2$. Students can check their answer by multiplying the binomials and verifying that they produce the original expression.

Example 2. $2x^2 + 5x - 12$

In example 2, we will see how the box method handles subtraction within the quadratic expression. To begin, students insert the first and last terms into the appropriate blocks, including the negative sign with the 12. See **figure 7 (Becker)**. Next, students find the factors of -24 whose difference is $+5$. They place the $8x$ and $-3x$ into the blocks, remembering that order does not matter. We determine the greatest common factors for the rows to be x and 4. The greatest common factors of the columns are $2x$ and -3. Our final answer is $2x^2 + 5x - 12 = (2x - 3)(x + 4)$.

Michael A. Becker
Spring Grove Area High School
Spring Grove, PA 17362

ax^2		$\_\_\_ + \_\_\_ = bx$
	c	$\_\_\_ \times \_\_\_ = a(c)$

Fig. 1 (Becker)

$6x^2$		$\_\_\_ + \_\_\_ = 19x$
	10	$\_\_\_ \times \_\_\_ = 60$

Fig. 2 (Becker) Initial setup

$6x^2$		$\underline{15x} + \underline{4x} = 19x$
	10	$\underline{15} \times \underline{4} = 60$

Fig. 3 (Becker) Find the right pair

$6x^2$	$4x$	$\underline{15x} + \underline{4x} = 19x$
$15x$	10	$\underline{15} \times \underline{4} = 60$

Fig. 4 (Becker) Complete the box

GCF = $2x$	$6x^2$	$4x$
GCF = 5	$15x$	10

Fig. 5 (Becker) GCF of rows

GCF = $3x$	GCF = 2
$6x^2$	$4x$
$15x$	10

Fig. 6 (Becker) GCF of columns

	GCF = $2x$	GCF = -3	
GCF = x	$2x^2$	$-3x$	$\underline{8x} + \underline{-3x} = 5x$
GCF = 4	$8x$	-12	$\underline{8} \times \underline{-3} = -24$

Fig. 7 (Becker) Complete solution to example 2

READER reflections

VERTICES AND x-INTERCEPTS OF PARABOLAS (I)

I enjoyed reading James Metz's letter in the "Reader Reflections" of February 2006. I would like to propose a generalization of his method of finding the x = coordinate of the vertex of a parabola and use the result to find the zeroes of the parabola.

> For $f(x) = ax^2 + bx + c$, then x is the vertex of the parabola if $f(x + d) = f(x - d)$ for any d.
>
> Let's find x. Substitution leads to:
>
> $$a(x+d)^2 + b(x+d) + c = a(x-d)^2 + b(x-d) + c \to 4axd = -2bd$$

As $d = 0$ is of no interest, we divide by d to get

$$x = -\frac{b}{2a}.$$

This result leads to the quadratic formula without completing the square. Let's see how.

It is easy to find the zeroes of the parabola $f(x) = ax^2 + c$ whose axis of symmetry is the y axis.

Every parabola $f(x) = ax^2 + bx + c$ can be translated so that its axis of symmetry is the y axis:

$$f\left(x - \frac{b}{2a}\right)$$

is the translated parabola. Solving

$$f\left(x - \frac{b}{2a}\right) = 0$$

will give us the two zeroes of the translated parabola. We shall find the zeroes:

$$a\left(x - \frac{b}{2a}\right)^2 + b\left(x - \frac{b}{2a}\right) + c = 0$$

$$ax^2 - \frac{2axb}{2a} + \frac{ab^2}{4a^2} + bx - \frac{b^2}{2a} + c = 0$$

$$ax^2 + \frac{b^2}{4a} - \frac{b^2}{2a} + c = 0$$

$$ax^2 = \frac{b^2}{2a} - \frac{b^2}{4a} - c$$

$$= \frac{2b^2 - b^2 - 4ac}{4a} = \frac{b^2 - 4ac}{4a}$$

$$x = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

These are the zeroes of the translated parabola, and it is easy to see the zeroes of the original one.

Many students fail to see the connection between the solutions of the quadratic equation and the zeroes of the parabola. Proving the famous formula again in the setting of functions can be interesting and productive. This method also provides us with an application of translations.

Ayana Touval
Ayana.Touval@montgomerycollege.edu
Montgomery College
Rockville, MD 20850-1733

VERTICES AND x-INTERCEPTS OF PARABOLAS (II)

In "Finding the x-Coordinate of the Vertex of a Parabola" ("Reader Reflections," *Mathematics Teacher* 99 [February 2006]: 390), James Metz gives another, simpler, noncalculus way of finding the vertex of a parabola. It should be noted that the vertex can be found more directly using calculus.

Consider the equation of a parabola, the function $f(x) = ax^2 + bx + c$. Its derivative, the slope of the curve, is $f'(x) = 2ax + b$. The x-coordinate of the vertex is found from $f'(x_0) = 0$, the position of zero slope. It follows immediately that

$$x_0 = \frac{-b}{2a}.$$

Of course, the y-coordinate also follows immediately:

$$y_0 = \frac{-\left(b^2 - 4ac\right)}{4a}$$

John F. Goehl Jr.
jgoehl@mail.barry.edu
Barry University
Miami Shores, FL 31161

PROBLEM 29, OCTOBER 2005

"Calendar" problem 29, from October 2005, reads:

> Suppose θ is an angle between 0° and 90° for which cos (θ) cos (2θ) = 1/4. What is the value of θ?

The solution presented is obtained by multiplying both sides of the equation by 4 sin θ and using a trigonometric identity to obtain sin (4θ) = sin (θ). Relying on the fact that $0° \le \theta \le 90°$, it is shown that $\theta = 36°$. The extraneous root $\theta = 0°$ must be rejected.

In the spirit of one of the excellent departments of the *Mathematics Teacher*, let us engage in "delving deeper." Specifically, we will drop the restriction on θ, and find the general solution of cos (θ) cos (2θ) = 1/4. Let $\mathbb{Z}$ be the set of integers. By the periodic nature of the cosine function, θ is a solution precisely if $\theta + 360°n$ is a solution, for $n \in \mathbb{Z}$. Since cos θ is an even function, θ is a solution if and only if $-\theta$ is one. Therefore, it suffices to restrict the consideration to the interval $\theta \in [0°, 180°]$.

As in the presented solution, the restriction of θ to the first quadrant forces 4θ to be in the second quadrant, and $\theta = 36°$. When $\theta \in [90°, 180°]$, $4\theta \in [360°, 720°]$. However, for sin (4θ) = sin (θ) to be true, sin (4θ) must be positive, which reduces the interval to $4\theta \in [360°, 540°]$, angles in the first and second quadrant. When 4θ is in the first quadrant, it is 360° greater than reference angle $180° - \theta$:

$$\begin{aligned} 4\theta &= 360° + 180° - \theta \\ 5\theta &= 540° \\ \theta &= 108° \end{aligned}$$

If 4θ is in the second quadrant,

$$\begin{aligned} 4\theta &= 360° + \theta \\ 3\theta &= 360° \\ \theta &= 120°. \end{aligned}$$

Therefore the general solution is

$$\left\{\begin{array}{l} 36^\circ + 360^\circ n_1,\ 108^\circ + 360^\circ n_2,\ 120^\circ + 360^\circ n_3, \\ -36^\circ + 360^\circ n_4,\ -108^\circ + 360^\circ n_5, \\ \hfill -120^\circ + 360^\circ n_6; \\ n_1, n_2, n_3, n_4, n_5, n_6 \in \mathbb{Z} \end{array}\right\}.$$

An alternative, although more difficult, approach is instructive. In the given equation $\cos(\theta)\cos(2\theta) = 1/4$, replace $\cos(2\theta)$ by $2\cos^2\theta - 1$:

$$\cos(\theta)\cos(2\theta) = 1/4$$
$$\cos(\theta)(2\cos^2\theta - 1) = 1/4$$
$$4\cos(\theta)(2\cos^2\theta - 1) = 1$$
$$8\cos^3\theta - 4\cos\theta - 1 = 0$$

Letting $z = \cos\theta$ yields the cubic equation $8z^3 - 4z - 1 = 0$. The rational root theorem finds the root $z = (-1/2)$, and the quadratic formula completes the solution:

$$z = \frac{1 \pm \sqrt{5}}{4}$$

Restricting θ to $[0°, 180°]$ as above, we immediately find the root $\theta = 120°$ corresponding to $z = (-1/2)$. A computer algebra system shows that

$$\arccos\left(\frac{1+\sqrt{5}}{4}\right) = 36^\circ$$

and

$$\arccos\left(\frac{1-\sqrt{5}}{4}\right) = 108^\circ.$$

This approach produces no extraneous roots.

It is instructive to view the graphs of the functions in question: $y = \cos(x)\cos(2x)$, in blue, and $y = 1/4$, in red, are shown on the interval $[-180°, 180°]$ in **figure 1 (Stanton)**. The six roots in this interval are represented by the intersections of the two graphs. Note the symmetry with respect to the y-axis.

Figure 2 (Stanton) shows $y = \sin(x)$ in red and $y = \sin(4x)$ in blue. Aside from the extraneous roots when $y = 0$, the intersections of the two graphs correspond to the solutions of the original equation. As expected with sine functions, the graphs display symmetry about the origin.

Robert O. Stanton
stantonr@stjohns.edu
St. John's University
Jamaica, NY 11439 ∞

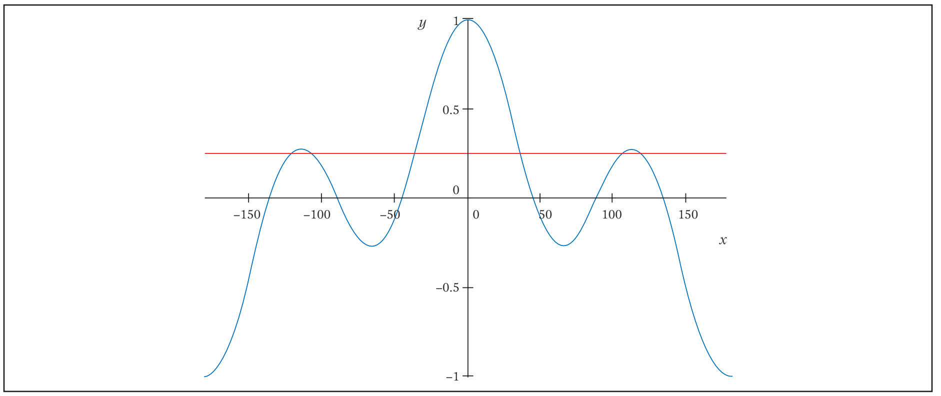

Fig. 1 (Stanton)

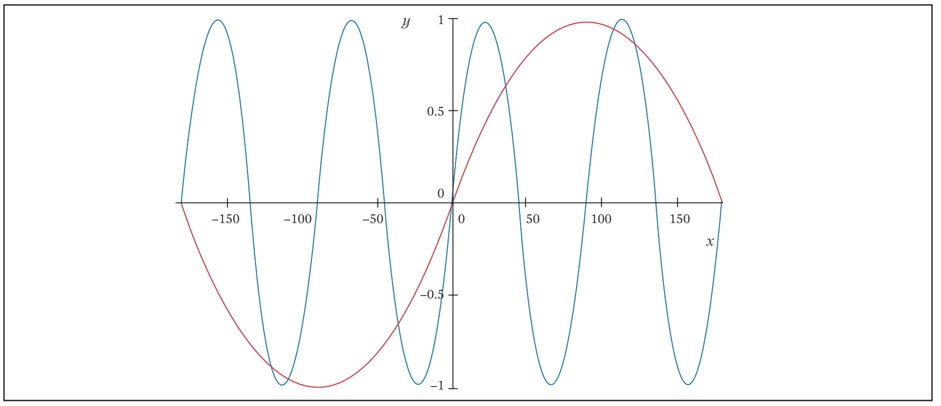

Fig. 2 (Stanton)