MARCH

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

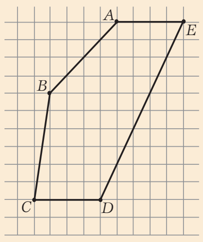
NCTM

1
What is the value of $1 - 2 + 3 - 4 + \cdots + 99 - 100$ (alternating signs for each term)?

2
A used car salesman sold two cars and received \$840 for each car. One of these transactions yielded a 40 percent profit for the dealer (compared to his purchase price), whereas the other amounted to a 25 percent loss. What is the dealer's net profit (or loss) on the two transactions?

3
Johnny was ill and had to take a test a day late. His 96 raised the class average from 71 to 72. How many students, including Johnny, took the test?

4
What is the number of square units enclosed by the polygon *ABCDE*?



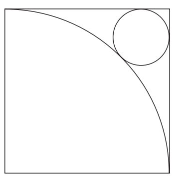
5
The square shown here can be filled in so that each row and each column contains each of the numbers 1, 2, 3, and 4 exactly once. What does x equal?

			1
	2		
		x	
1			4

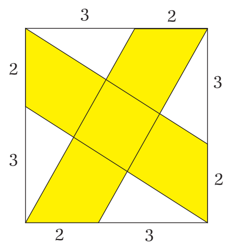
6
Joan holds a 26-inch-long piece of lightweight string with a heavy bead on it. One end is in her right hand; the other is in her left hand. Initially, her hands are together. How many inches apart must she pull her hands, keeping them at the same height, so the bead moves upward by 8 inches?

7
A bowl contains 50 colored balls: 13 green, 10 red, 9 blue, 8 yellow, 6 black, and 4 white. If you are blindfolded, what is the smallest number of balls you must pick to guarantee that you have at least 7 balls of the same color?

8
A circular arc centered at one vertex of a square of side length 4 passes through two other vertices. A small circle is tangent to the large circle and to two sides of the square. What is the radius of the small circle?



9
In the diagram, two sets of parallel lines intersect to form a square. What is the area of the shaded region?



10
How many ordered pairs (x, y) of integers satisfy

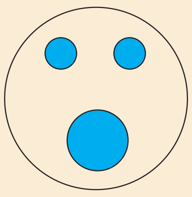
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{2}?$$

11
A square is partitioned into 30 nonoverlapping triangles so that each of the four sides of the square is also a side of one of the 30 triangles. The intersection of any two triangles is empty, a common vertex, or a common edge. How many points in the interior of the square serve as vertices of one or more triangles?

12
Two fair coins are flipped simultaneously. This is done repeatedly until at least one of the coins comes up heads, at which point the process stops. What is the probability that both coins came up heads on the last flip?

13
George's car gets 3 more miles per gallon during highway driving than it does during city driving. Recently he drove 112 miles on a highway and 150 in the city and used exactly 10 gallons of gasoline. How many miles per gallon does his car get during city driving?

14
The radius of each small circle is 1/6 of the radius of the large surrounding circle. The radius of the middle-sized circle is twice that of either small circle. What fraction of the area of the large circle is not shaded?



15
An equal number of juniors and seniors responded to the question "Do you like math?" Each respondent said yes or no. If 70 percent of those who said yes were seniors and 80 percent of those who said no were juniors, what percentage of the seniors polled said yes?

16
Two matching decks have 52 cards each. One deck is shuffled and put on top of the second deck. For each card of the top deck, we count the number of cards between it and the corresponding card of the bottom deck (not including the cards themselves). What is the sum of these numbers?

17
Chicken nuggets come in packets of 6, 9, or 20. What is the largest number of nuggets that you cannot buy when combining various packets?

18
Which is the largest: 1^{48}, 2^{42}, 3^{36}, 4^{30}, 5^{24}, 6^{18}, 7^{12}, 8^{6}, or 9^{0}?

19
Two bicyclists 2 miles apart start pedaling toward each other traveling at 9 mph and 10 mph, respectively. A fly flies at 12 mph from one bicycle to the other, turns around instantly, and flies back to the other bicycle. If the fly continues this until the bicyclists meet, how many miles does the fly travel?

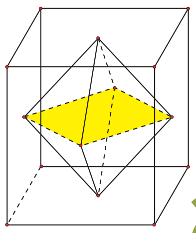
20
What is the number of sides of a regular polygon for which the number of diagonals is 44 or 45?

21
The length of each leg of an isosceles triangle is $x + 1$, and the length of the base is $3x - 2$. Determine all possible real values of x.

22
What is the number of different 7-digit numbers that can be made by rearranging the digits of 3053354?

23
In what positive base b does the equation $4 \cdot 12 = 103$ for multiplication of base b hold?

24
An octahedron is formed by connecting the centers of the faces of a cube. What is the ratio of the volume of the cube to that of the contained octahedron?



25
What is the largest number of pieces into which a circular pizza can be cut with 9 straight cuts?

26
Suppose the odd numbers are grouped in the following way: $\{1\}$, $\{3, 5\}$, $\{7, 9, 11\}$, $\{13, 15, 17, 19\}$, ... What is the sum of the numbers in the tenth grouping?

27
Two students attempted to solve a quadratic equation $x^2 + bx + c = 0$. Although both students did the work correctly, one miscopied the middle term and obtained the solution set $\{2, 3\}$, while the other miscopied the constant term and obtained a solution set $\{2, 5\}$. What is the correct solution set?

28
Compute

$$S=\frac{1}{5}+\frac{1}{25}+\frac{2}{125}+\frac{3}{625}+\frac{5}{3125}+\cdots,$$

where each numerator after the second is the sum of the two preceding numerators and each denominator is 5 times the preceding one.

29
Let $ABCD$ be a rectangle with $BC = 2AB$, and let BCE be an equilateral triangle, $\overline{BE}$ and $\overline{EC}$ intersecting $\overline{AD}$ at F and G, respectively. If M is the midpoint of $\overline{EC}$, how many degrees are in angle CMD?

30
In triangle ABC, $AC = 6$ and $BC = 5$. Point D on $\overline{AB}$ divides it into segments of length $AD = 1$ and $DB = 3$. What is the length of segment CD?

31
How many ordered triples (x, y, z) of positive integers satisfy $xyz = 4000$?

SOLUTIONS to calendar

1. –50. There are 50 pairs, each summing to –1. So $50(-1) = -50$.

2. \$40 loss. If the purchase price on the first car is x, then $1.4x = 840$. Thus $x = 600$. Similarly, his purchase price on the second car must have been

$$\frac{840}{0.75} = 1120.$$

Thus, he paid \$1720 and only recouped \$1680.

3. 25. If x is the desired number of students, then $71(x - 1) + 96 = 72x$, and so $x = 96 - 71 = 25$.

4. 50. The region can be divided into two trapezoids by a horizontal line through B. The area of the upper trapezoid is

$$\frac{4(4+6)}{2} = 20,$$

while the area of the lower trapezoid is

$$\frac{6(4+6)}{2} = 30.$$

The sum of these two regions is 50.

5. 4. Consider the fourth row first. The entry in column 2 must be 3, forcing a 2 into the third column. Now consider the second column. The first row of that column must be a 4, and the third row must be a 1. Now the 1 for the third column must go in the second row, and the 4 for that column cannot go in the first row, so it must go in the third row.

			1
	2		
		x	
1	3		4

	4		1
	2		
	1	x	
1	3	2	4

2	4	3	1
	2		
	1	x	
1	3	2	4

6. 24. Initially, the bead is 13 inches below the level of her hands. When it moves up 8 inches, it is 5 inches below the level of her hands and 13 inches from each hand. Thus each hand will have moved outward

$$\sqrt{13^2 - 5^2} = \sqrt{144} = 12 \text{ inches.}$$

7. 35. You might choose 6 greens, 6 reds, 6 blues, 6 yellows, 6 blacks, and 4 whites. The next one will certainly give you 7 of some color.

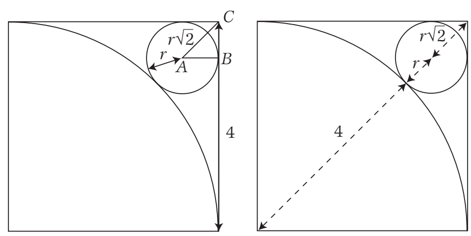
8. $12 - 8\sqrt{2}$. Let r denote the desired radius. The small 45-45-90° triangle ABC will consequently have legs of length r and a hypotenuse of length $r\sqrt{2}$. We also know that the diagonal of the square will have a length of $4\sqrt{2}$. Since the radius of the large circle is the length of the side of the square, the diagonal of the square can be represented as $4 + r + r\sqrt{2}$, as shown. This indicates that

$$4 + r + r\sqrt{2} = 4\sqrt{2} \to$$
$$r\sqrt{2} + r = 4\sqrt{2} - 4.$$

Thus $r\left(\sqrt{2}+1\right) = 4\left(\sqrt{2}-1\right)$,

and $r = \dfrac{4\left(\sqrt{2}-1\right)}{\sqrt{2}+1} = \dfrac{4\left(2+1-2\sqrt{2}\right)}{2-1}$

$$= 12 - 8\sqrt{2}.$$



9. $\dfrac{290}{17}$ or $17\dfrac{1}{17}$.

Let h denote the width of each strip, y denote the length of the leg of a nonshaded triangle emanating from a vertex of the square, and x denote the length of the shorter leg of a nonshaded triangle. As shown in the second diagram, we can make use of two sets of similar right triangles: $\triangle ACD \sim \triangle AEB$ and $\triangle GFA \sim \triangle AEB$. The proportional sides of these

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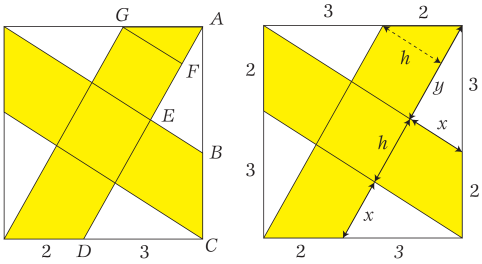
Problems in this month's calendar are from the Lehigh University High School Annual Math Contest, 2003, 2004, and 2005.

Three other sources of problems in calendar form are available from NCTM: *Calendar Problems from the* Mathematics Teacher (a book featuring more than 400 problems, organized by topic, order number 12509, \$22.95), "Calendars for the Calculating," vol. 2 (a set of nine monthly calendars that originally appeared from September 1987 to May 1988, order number 496, \$13.50), and "A Year of Mathematics" (one annual calendar that originally appeared in September 1982, order number 311, \$4.00; set of five, order number 312, \$8.00). Individual members receive a 20 percent discount off these prices. A catalog of educational materials, which includes a listing for the publication *Exploratory Problems in Mathematics*, is available at www.nctm.org.—Eds.

sets of triangles yield

$$\frac{y}{x}=\frac{5}{3},\ \frac{y}{3}=\frac{h}{2},\text{ and }\frac{y}{3}=\frac{5}{(x+y+h)}.$$



From this we obtain $y=\frac{5}{3}x.$

Then $h=\frac{2}{3}y=\frac{2}{3}\left(\frac{5}{3}x\right)=\frac{10}{9}x.$

By substitution,

$$\frac{\left(\frac{5}{3}x\right)}{3}=\frac{5}{\left(x+\frac{5}{3}x+\frac{10}{9}x\right)}\ \rightarrow$$

$$\frac{5x}{9}=\frac{5}{\left(\frac{34}{9}x\right)}\ \rightarrow$$

$$45=5x\bullet\left(\frac{34}{9}x\right)\ \rightarrow$$

$$\frac{81}{34}=x^2.$$

So, $x=\frac{9}{\sqrt{34}},\ y=\frac{15}{\sqrt{34}},\text{ and }h=\frac{10}{\sqrt{34}}.$

This results in the shaded area of

$$\begin{aligned}&2\bullet(x+y+h)\bullet h-h^2\\&=2\bullet\left(\frac{9}{\sqrt{34}}+\frac{15}{\sqrt{34}}+\frac{10}{\sqrt{34}}\right)\\&\bullet\frac{10}{\sqrt{34}}-\left(\frac{10}{\sqrt{34}}\right)^2\ \rightarrow\\&2\bullet\left(\frac{34}{\sqrt{34}}\right)\bullet\frac{10}{\sqrt{34}}-\frac{100}{34}\\&=20-\frac{100}{34}=\frac{290}{17}.\end{aligned}$$

10. 5. For $x, y \neq 0$, the equation reduces to $2(x+y)=xy$ or

$$x=\frac{2y}{y-2}.$$

Note that if (x, y) satisfies the original equation, so does (y, x). Therefore, the only y values that make x an integer are 1, 3, and 4. When $y = 4$, so does x. Hence, there are a total of 5 solutions.

11. 14. If there are v interior vertices, then the total number of degrees in all the angles (including the interior vertices and the four vertices of the square) is, on the one hand, $30 \bullet 180$ and, on the other hand, $360v + 4 \bullet 90$. Thus, $30 = 2v + 2$, and $v = 14$.

12. $\frac{1}{3}.$

The previous flips are irrelevant. Of the four equally likely outcomes of a double flip, HH, HT, TH, and TT, the first three have at least one head, and just one of them has the other coin also a head.

13. 25. If x denotes the requested number, then

$$\frac{112}{x+3}+\frac{150}{x}=10.$$

Factoring out a 2, this reduces to $0 = 5x^2 - 116x - 225$. This can be solved by the quadratic formula, or it can be factored as $0 = (5x + 9)(x - 25)$. Since only positive solutions make sense for the problem, the car must get 25 mpg in the city.

14. $\frac{5}{6}.$

Let the radius of the large circle be 1 unit. Then the sum of the areas of the three shaded circles is

$$\pi\left(\frac{1}{36}+\frac{1}{36}+\frac{1}{9}\right)=\frac{\pi}{6}\ \text{ square units.}$$

Since the area of the large circle is π square units, the fraction of the area that is not shaded is

$$\frac{5}{6}.$$

15. 84. Let j denote the fraction of juniors and s denote the fraction of seniors that answer yes in each case. Then

$$j=\frac{3}{7}s,$$

while $\quad 1-j=4(1-s).$

Thus, $\quad 1-\frac{3}{7}s=4-4s,$

and $\quad \frac{25}{7}s=3,$

so that

$$s=\frac{21}{25}=84\%.$$

16. 2652. Label the pairs of matching cards 1, 2, . . . , 52. Let a_i be the location from the bottom of the pile of the top card I, and b_i be the location of the corresponding bottom card, where I goes from 1 to 104. We wish to find

$$\sum_{i=1}^{52}(a_i-b_i-1).$$

This equals

$$\begin{aligned}\sum_{i=1}^{52}a_i-\sum_{i=1}^{52}b_i-52&=(53+\cdots+104)\\&\quad-(1+\cdots+52)-52\\&=52^2-52=2652.\end{aligned}$$

Alternately, consider decks consisting of 3, 4, 5, and 6 cards each, which respectively yield sums of 6, 12, 20, and 30. Using finite differences, it can be determined that the required sum is $n^2 - n$, where n is the number of cards in the deck.

17. 43. Any multiple of 3 that is greater than 3 can be obtained from packets of 6 and 9 nuggets. Since 36 = 9 + 9 + 9 + 9, 38 = 20 + 9 + 9, and 40 = 20 + 20, any even number ≥ 36 can be achieved by adding 6s to each of these. Similarly, by adding another 9, any odd number ≥ 45 can be achieved. But 43 is not yet guaranteed, so we need to examine the possibility of combinations that yield 43 nuggets. Since 43 is not a multiple of 3, likewise, 43 – 20 = 23 is not a multiple of 3, and 43 – 2 • 20 = 3 is too small to achieve. Consequently, 43 cannot be obtained.

18. 4^{30}. Raise each to the 1/6 power. This yields $1^8, 2^7, 3^6, 4^5, 5^4, 6^3, 7^2, 8^1, 9^0$. Now compare $3^6 = 729$, $4^5 = 2^{10} = 1024$, and $5^4 = 625$. The others are all clearly smaller than these, so 4^{30} is the largest.

19. $\frac{24}{19}$.

The bicyclists are approaching each other at 19 mph, so they will meet in 2/19 hour. Thus the fly will travel

$$12 \bullet \frac{2}{19} = \frac{24}{19} \text{ miles.}$$

20. 11. Setting the number of diagonals of a regular n-gon

$$\frac{n(n-3)}{2}$$

equal to 44, obtain $n^2 - 3n - 88 = 0 \rightarrow (n - 11)(n + 8) = 0 \rightarrow n = 11$. Using 45 does not yield integral answers.

21. $\frac{2}{3} < x < 4$.

We must have $2(x + 1) > 3x - 2$ and hence $x < 4$. The base must have a positive length and $(3x - 2) + (x + 1) > (x + 1)$, which implies

$$x > \frac{2}{3}.$$

22. 360. The 0 cannot be in the left-most position. The three 3s can be in any of

$$\binom{6}{3} = 20 \text{ positions.}$$

Then the two 5s can be in any of

$$\binom{3}{2} = 3 \text{ positions.}$$

Finally, the position of the 4 is forced by the preceding choices. So the total number of possibilities is $6 \bullet 20 \bullet 3$.

Alternately, there are

$$\frac{7!}{3!2!}$$

permutations of the 7 digits. Subtract

$$\frac{6!}{3!2!}$$

for those permutations beginning with 0. Or, there are 6 choices for the left-most digit and 6! ways to arrange the other digits. Divide $6 \bullet 6!$ by $3!2!$ for repeating digits.

23. 5. We must have $4(b + 2) = b^2 + 0b + 3$. Hence, $b^2 - 4b - 5 = 0$. Since b is positive, $b = 5$.

24. 6 or 6 : 1. Let the side length of the cube equal 2 units. The octahedron is the union of two pyramids of height 1 on a base that is a square of side length $\sqrt{2}$. The volume of each pyramid is

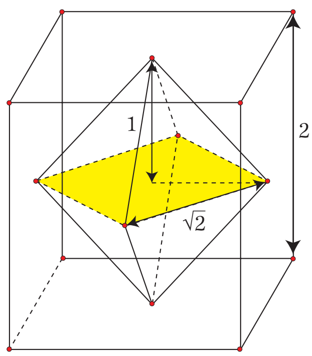
$$\frac{hA}{3} = 1 \bullet 2 \bullet \frac{1}{3} = \frac{2}{3},$$

where h is the height of the pyramid and A the area of the base. So the combined volume of the pyramids is

$$2 \bullet \frac{2}{3} = \frac{4}{3} \text{ cubic units,}$$

while the volume of the cube is 8 cubic units. Thus the ratio of the cube to the octahedron is

$$\frac{8}{\left(\frac{4}{3}\right)} = 6.$$



25. 46. The maximum number of pieces that can be added by a cut is 1 greater than the number of lines that the new cut intersects. Hence, the answer is

$$1 + 1 + 2 + \cdots + 9 = 1 + \frac{9 \bullet 10}{2} = 1 + 45 = 46.$$

26. 1000. If you compute the first four sums 1, 8, 27, 64, you can perhaps guess that the sum of the i^{th} group is i^3. One way to prove it is to note that the i^{th} group has i numbers and that their average is i^2. To see this when i is odd, note that there will be

$$\frac{i(i-1)}{2}$$

odd numbers preceding the group, and the middle entry will be the

$$\left(\frac{i+1}{2}\right)^{\text{th}}$$

in the group. It will thus equal

$$-1 + \frac{2i(i-1)}{2} + \frac{i+1}{2} = i^2.$$

A similar argument will work if i is even.

27. $\{1, 6\}$ or just 1, 6. Because the constant term is the product of the roots, we know from the first student that $c = 2 \bullet 3 = 6$. Since $-b$ equals the sum of the roots, we can use the solution for the second student to find that $-b = 7$ and the polynomial is $x^2 - 7x + 6$.

28. $\frac{5}{19}$.

$$5S = 1 + \frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \frac{5}{625} + \cdots$$

Subtract the series for S from the series for $5S$:

$$\begin{aligned} 5S &= 1 + \frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \frac{5}{625} + \cdots \\ &- \left(S = \frac{1}{5} + \frac{1}{25} + \frac{2}{125} + \frac{3}{625} + \frac{5}{3125} + \cdots\right) \\ 4S &= 1 + \frac{1}{25} + \frac{2}{625} + \cdots \\ &= 1 + \frac{1}{5}S \\ \frac{19}{5}S &= 1 \\ S &= \frac{5}{19} \end{aligned}$$

29. 75. Both CM and CD equal half the length of the side of the triangle. Hence triangle CMD is isosceles. Since angle MCD is $90 - 60 = 30°$, the other angles of the triangle are each

$$\frac{180-30}{2} = 75°.$$

30. $\frac{11}{2}$.

If x is the desired length and $\theta = m\angle BDC$, then we have $x^2 + 1 + 2x \cos \theta = 36$ and $x^2 + 9 - 6x \cos \theta = 25$. Adding 3 times the first equation to the second yields

$$4x^2 + 12 = 133 \rightarrow$$
$$4x^2 = 121 \rightarrow x^2 = \frac{121}{4},$$

and so $x = \frac{11}{2}$.

31. 210. Since $4000 = 2^5 \cdot 5^3$, we must have $x = 2^a \cdot 5^d$, $y = 2^b \cdot 5^e$, and $z = 2^c \cdot 5^f$. Then our answer will be AB, where A is the number of ordered triples (a, b, c) of nonnegative integers such that $a + b + c = 5$ and B is the number of ordered triples (d, e, f) of nonnegative integers such that $d + e + f = 3$. Note: $a = 0$ has 6 possibilities for (b, c), namely, $0 \le b \le 5$, $a = 1$ has 5 possibilities for (b, c), etc., down to $a = 5$ having only one possible for (b, c). Thus the number of possible triples (a, b, c) is $6 + 5 + 4 + 3 + 2 + 1 = 21$. Similarly, the number of possible triples (d, e, f) is $4 + 3 + 2 + 1 = 10$. So, the total number of possibilities is $21 \cdot 10 = 210$. ∞