# MATHEMATICAL lens


**Photograph 1**
Metulla, Israel

**Photograph 2**
Bangkok, Thailand



"Mathematical Lens" uses photographs as a springboard for mathematical inquiry. The goal of this department is to encourage readers to see patterns and relationships that they can think about and extend in a mathematically playful way.

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The photographs, questions, and answers for this month's Mathematical Lens were submitted by Ron Lancaster, one of the editors of this department and a world traveler.

1. Because of the geometrical representation shown in **photograph 1**, the number 3 is commonly called a triangular number. Find examples of triangular numbers in **photographs 2** and **3**.

2. The number 36 (**photograph 3**) is both a triangular and a square number. The first seven numbers having this property are 1, 36, 1225, 41616, 1413721, 48024900, and 1631432881.

(*a*) Given that the *n*th triangular number is equal to

$$\frac{n(n+1)}{2},$$

verify that the seven numbers above are both triangular and square.


**Photograph 3**
Montreal, Quebec



(*b*) Develop a recursive formula for the sequence of numbers, $t_n$, that are both triangular and square. Use this formula to determine the eighth number that is both triangular and square.

If you experience difficulty in obtaining this formula, try working with the square root of the numbers that make up this sequence, namely, 1, 6, 35, 204, 1189, 6930, and 40391. Starting with the third term, there is a recursive formula that can be used to generate this sequence.

(*c*) Let $u_n$ be the $n$th term in the sequence 1, 6, 35, 204, 1189, 6930, 40391, ... (the square roots of the sequence of numbers that are both triangular and square). It can be shown that

$$u_n = \frac{\left(3+2\sqrt{2}\right)^n - \left(3-2\sqrt{2}\right)^n}{4\sqrt{2}}.$$

Without using any technology, show that this formula provides correct results for the first four terms (1, 6, 35, and 204). Use a computer algebra system (CAS), such as Mathematica, MAPLE, or the version of CAS available on the TI-89 or TI-*n*spire, to show that the formula provides correct results for the next three terms.

3. Read the following excerpt from *Recreations in the Theory of Numbers,* by Albert H. Beiler (1964), and use the test to check the number that appears in **photograph 1**. Why does this test work?

   A very simple negative test for a triangular number is to add together the digits of the unknown number [the number to be tested], then the digits of the sum, and so on until one digit remains. If this digit is 2, 4, 5, 7 or 8, the number is not triangular; otherwise it may or may not be. (p. 190)

4. A given number $x$ can also be tested for being triangular by calculating

$$n = \frac{\sqrt{8x+1}-1}{2}.$$

   If $n$ is a positive integer, then $x$ is the $n$th triangular number. If $n$ is not a positive integer, then $x$ is not a triangular number. Use this test to check the number that appears in **photograph 1**. Why does this test work?

# **MATHEMATICAL LENS** solutions

1. In **photograph 2**, the triangular number 15 appears in the tablecloth in the white floral design, and the triangular number 10 appears in the gray spaces in between. By focusing on a smaller portion of the tablecloth, one can also see the triangular numbers 1, 3, 6, and 10. Similarly, the triangular numbers 1, 3, 6, 10, 15, 21, 28, and 36 can be seen in **photograph 3**.

2. (*a*) A number can be shown to be a square by taking its square root and checking that the result is a positive integer. One way to check that these numbers are triangular is to create a table of such numbers by using the general formula and a spreadsheet program such as Excel or a graphing calculator such as the TI-84 Plus (see **fig. 1**). If a given number appears in the table, it is triangular; otherwise, it is not. Another way to check is to double the number to be checked and determine the square root of the result. Then choose the two integers that surround the square root. If their product is twice the number in question, the number is triangular. Or a given number (say, 1225) can be checked simply by solving the equation

$$\frac{n(n+1)}{2} = 1225.$$

If one of the solutions is a positive integer, then the given number is triangular.

(*b*) Let

$$u_n = \sqrt{t_n}$$

where $t_n$ represents the $n$th number in our list of perfect square triangular numbers. Note that $u_1 = 1$ and $u_2 = 6$. For $n \geq 3$, $u_n = 6u_{n-1} - u_{n-2}$. The sequence $t_n$ can be determined using $t_n = u_n^2$. The eighth triangular number can be shown to be 55420693056, as follows:

$$\begin{aligned}
u_6 &= 6930\\
u_7 &= 40391\\
u_8 &= 6u_7 - u_6\\
u_8 &= 6 \times 40391 - 6930\\
u_8 &= 235416\\
t_8 &= u_8^2\\
t_8 &= 55420693056
\end{aligned}$$

(*c*) Using $n = 3$ as a typical example, $u_3 = 35$ can be shown to be true, as follows:

$$\begin{aligned}
u_3 &= \frac{\left(3+2\sqrt{2}\right)^3 - \left(3-2\sqrt{2}\right)^3}{4\sqrt{2}}\\
&= \frac{\left(3+2\sqrt{2}\right)^2\left(3+2\sqrt{2}\right) - \left(3-2\sqrt{2}\right)^2\left(3-2\sqrt{2}\right)}{4\sqrt{2}}\\
&= \frac{\left(17+12\sqrt{2}\right)\left(3+2\sqrt{2}\right) - \left(17-12\sqrt{2}\right)\left(3-2\sqrt{2}\right)}{4\sqrt{2}}\\
&= \frac{99+70\sqrt{2} - \left(99-70\sqrt{2}\right)}{4\sqrt{2}}\\
&= \frac{140\sqrt{2}}{4\sqrt{2}}\\
&= 35
\end{aligned}$$

3. The single digit obtained at the end of the summing process is called the digital root of the given number. Starting with


(a) (b) (c)

**Fig. 1** The equation, table setup, and table to check for triangular perfect squares

7669635, the first sum is 42, followed by 6. Because the digital root of 7669635 is 6, the test fails to determine that this number is triangular.

4.

$$\begin{aligned} n &= \frac{\sqrt{8 \times 7669635 + 1} - 1}{2} \\ &= \frac{\sqrt{61357081} - 1}{2} \\ &\simeq 3916.04 \end{aligned}$$

Thus, 7669635 is not triangular.

Readers are encouraged to send explanations of why these two tests work to "Reader Reflections," *Mathematics Teacher*.

## REFERENCE

Beiler, Albert H. *Recreations in the Theory of Numbers*. 2nd ed. New York: Dover, 1964. ∞

Members who wish to use this month's photographs in a classroom setting can download the images from NCTM's Web site, www.nctm.org. Follow links to *Mathematics Teacher*, and choose Current Issue. Then select Mathematical Lens from the Departments, and look for the link to the image.